PERIODICITY OF THE SPECTRUM OF A FINITE UNION OF INTERVALS

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Abstract. A set Ω, of Lebesgue measure 1, in the real line is called spectral if there is a set Λ of real numbers such that the exponential functions $e_\lambda(x) = \exp(2\pi i\lambda x)$, $\lambda \in \Lambda$, form a complete orthonormal system on $L^2(\Omega)$. Such a set Λ is called a spectrum of Ω. In this note we present a simplified proof of the fact that any spectrum Λ of a set Ω which is finite union of intervals must be periodic. The original proof is due to Bose and Madan.

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1. Introduction and statement of the result

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set of Lebesgue measure 1. A set $\Lambda \subseteq \mathbb{R}^d$ is called a spectrum of Ω (and Ω is said to be a spectral set) if the set of exponentials

$$E(\Lambda) = \left\{ e_\lambda(x) = e^{2\pi i\lambda\cdot x} : \ \lambda \in \Lambda \right\}$$

is a complete orthonormal set in $L^2(\Omega)$. (The inner product in $L^2(\Omega)$ is $\langle f, g\rangle = \int_\Omega f\overline{g}$.)

It is easy to see (see, for instance, [5]) that the orthogonality of $E(\Lambda)$ is equivalent to the *packing condition*

$$\sum_{\lambda\in\Lambda} \left|\widehat{\chi_\Omega}\right|^2 (x-\lambda) \le 1, \quad \text{a.e. } (x), \tag{1}$$

as well as to the condition

$$\Lambda - \Lambda \subseteq \{0\} \cup \{\widehat{\chi_\Omega} = 0\}. \tag{2}$$

The completeness of $E(\Lambda)$ is in turn equivalent to the *tiling condition*

$$\sum_{\lambda\in\Lambda} \left|\widehat{\chi_\Omega}\right|^2 (x-\lambda) = 1, \quad \text{a.e. } (x). \tag{3}$$

These equivalent conditions follow from the identity

$$\langle e_\lambda, e_\mu\rangle = \int_\Omega e_\lambda \overline{e_\mu} = \widehat{\chi_\Omega}(\lambda-\mu) \tag{4}$$

and from the completeness of all the expontials in $L^2(\Omega)$.

Example: If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $\mathbb{Z}^d$ is a spectrum of Q_d.

In the one dimensional case, which will concern us in this paper, condition (2) implies that the set Λ has gaps bounded below by a positive number, the smallest zero of $\widehat{\chi_\Omega}$.

Research on spectral sets has been driven for many years by a conjecture of Fuglede [4] which stated that a set Ω is spectral if and only if it is a translational tile. A set Ω is a translational tile if we can translate copies of Ω around and fill space without overlaps. More precisely there exists a set $S \subseteq \mathbb{R}^d$ such that

$$\sum_{s\in S} \chi_\Omega(x-s) = 1, \quad \text{a.e. } (x). \tag{5}$$

This conjecture is now known to be false in both directions if $d \ge 3$ [12, 11, 7, 8, 2, 3] and both directions are still open in dimensions $d = 1, 2$.

In this paper we present a new proof of the periodicity of the spectrum, which is a considerable simplification of that in [1].

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Theorem 1 (Bose and Madan [1]). *If $\Omega = \bigcup_{j=1}^n (a_j, b_j) \subseteq \mathbb{R}$ is a finite union of intervals of total length 1 and $\Lambda \subseteq \mathbb{R}$ is a spectrum of Ω then there exists a positive integer T such that $\Lambda + T = \Lambda$.*

This is the spectral analogue of a result [10, 9, 6] which states that all translational tilings by a bounded measurable set (or by a compactly supported function) are necessarily periodic. The proof of Theorem 1 is given in the next section.

2. Proof of the periodicity of the spectrum

Let us observe first, as in [1], that the spectrum $\Lambda = \{\lambda_j :\ j \in \mathbb{Z}\}$, $\lambda_j < \lambda_{j+1}$, of any bounded set $\Omega \subseteq \mathbb{R}$ has "finite complexity", in the sense that all gaps $\lambda_{j+1} - \lambda_j$ are drawn from the discrete set ($\widehat{\chi_\Omega}$ is analytic as χ_Ω has bounded support) $\{\widehat{\chi_\Omega} = 0\}$. This implies that if we consider all intesections of Λ with a sliding window of width h

$$[\lambda, \lambda + h] \cap \Lambda, \quad (\text{where } \lambda \in \Lambda),$$

then we only see finitely many different sets.

If $\Omega = \bigcup_{j=1}^n (a_j, b_j) \subseteq \mathbb{R}$ it follows by a simple calculation that

$$\widehat{\chi_\Omega}(\xi) = \frac{1}{2\pi i \xi} \sum_{j=1}^n \left(e^{-2\pi i a_j \xi} - e^{-2\pi i b_j \xi} \right). \tag{6}$$

The important ingredient of the approach in [1] that we keep in our approach is the view of the spectrum as a linear space via the map $\phi = \phi_\Omega : \mathbb{R} \to \mathbb{C}^{2n}$ given by

$$x \to (e^{-2\pi i a_1 x}, \ldots, e^{-2\pi i a_n x}, e^{-2\pi i b_1 x}, \ldots, e^{-2\pi i b_n x}).$$

Define the bilinear form A on $\mathbb{C}^{2n}$ by (writing $z = (z_1, z_2)$, $z_1, z_2 \in \mathbb{C}^n$)

$$A(z, w) = \langle z_1, w_1 \rangle - \langle z_2, w_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{C}^n$. Using (6) we see that if $\lambda \neq \mu$ then

$$e_\lambda \perp e_\mu \ \text{ if and only if } \ A(\phi(\lambda), \phi(\mu)) = 0.$$

Write

$$V(\Lambda) = \operatorname{span} \phi(\Lambda)$$

for the subspace of $\mathbb{C}^{2n}$ generated by the set $\phi(\Lambda) = \{\phi(\lambda) :\ \lambda \in \Lambda\}$.

Suppose now that $B = \{b_1, \ldots, b_m\} \subseteq \Lambda$ is a generating set, i.e., that $V(\Lambda) = \operatorname{span} \phi(B)$. It follows that $x \in \Lambda$ if and only if $A(\phi(x), \phi(b_j))$ for $j = 1, 2, \ldots, m$. Indeed, if the latter condition is true it follows by linearity that $A(\phi(x), \phi(\mu)) = 0$ for all $\mu \in \Lambda$ and hence that $e_x \perp e_\mu$, $\mu \in \Lambda \setminus \{x\}$. This implies that $x \in \Lambda$, otherwise $E(\Lambda)$ would not be a complete set of exponentials for $L^2(\Omega)$. As remarked in [1] this means that Λ is determined by any such generating set B.

Lemma 1. *Let Ω be a finite union of intervals. If $A \subseteq \mathbb{R}$ is a set of positive minimum gap δ then for $R > 0$ we have*

$$\sum_{\substack{a \in A \\ |a| > R}} |\widehat{\chi_\Omega}|^2(a) \le C/R,$$

for some constant $C > 0$ that may depend on Ω and δ only.

Proof. This is immediate from the fact that $|\widehat{\chi_\Omega}|^2(y) \le C/|y|^2$ (see (6)). ☐

Lemma 2. *There is a finite $T > 0$ such that for all $x \in \mathbb{R}$ the set $\Lambda \cap (x, x+T)$ is a generating set.*

Proof. Suppose not, so that there is a sequence $m_k \in \Lambda$, $k = 1, 2, \ldots$, such that

$$\dim \operatorname{span} \phi(\Lambda \cap (m_k - k, m_k + k)) < \dim \operatorname{span} \phi(\Lambda).$$

Consider the sequence of finite sets

$$M_k = [\Lambda \cap (m_k - k, m_k + k)] - m_k,$$

i.e., the sets $\Lambda \cap (m_k - k, m_k + k)$ translated so that they are centered at 0 (therefore they all contain 0). Observe that in any given interval $(-t, t)$ the sets M_k may only take finitely many forms.

For $n = 1, 2, 3, \ldots$ in turn we look at the infinite sequence

$$M_k \cap (-n, n), \quad k = 1, 2, \ldots.$$

There is an infinite sequence of k's such that all sets $M_k \cap (-n, n)$ are the same. Keep only these indices and define L_n to be this common set. In this way we define an increasing infinite sequence of sets L_n, $L_n \subseteq L_{n+1}$, each of which contains 0 and is of the form

$$L_n = \Lambda \cap (c_n - n, c_n + n) - c_n,$$

for some $c_n \in \Lambda$.

Let $L = \bigcup_{n=1}^{\infty} L_n$. Since each finite part of L is a translate of a part of Λ it follows that the elements of $E(L)$ are orthogonal. We now show that $E(L)$ is also complete and is thus also a spectrum of Ω.

For this it suffices to show that $F(x) := \sum_{\ell \in L} |\widehat{\chi_\Omega}|^2(x - \ell) = 1$ for almost every $x \in \mathbb{R}$. Assume for simplicity that $x \geq 0$. We have for $t > 2x$

$$\begin{aligned}
1 &\geq F(x) && \text{(from (1), since } E(L) \text{ is an orthogonal set)}\\
&\geq \sum_{\ell \in (-t,t) \cap L} |\widehat{\chi_\Omega}|^2(x - \ell) \\
&= \sum_{\ell \in (-t,t) \cap L_n} |\widehat{\chi_\Omega}|^2(x - \ell) && \text{(for some } n = n(t) > t)\\
&= \sum_{\ell \in \Lambda - c_n,\ |\ell| < t} |\widehat{\chi_\Omega}|^2(x - \ell) \\
&= 1 - \sum_{\ell \in \Lambda - c_n,\ |\ell| \geq t} |\widehat{\chi_\Omega}|^2(x - \ell) && \text{(by (3) for a.e. } x, \text{ since } \Lambda \text{ is a spectrum)}\\
&\geq 1 - \sum_{\ell \in \Lambda - c_n,\ |x-\ell| \geq t/2} |\widehat{\chi_\Omega}|^2(x - \ell) && \text{(as } |\ell| \geq t > 2x \text{ implies } |x - \ell| \geq t/2)\\
&= 1 - \sum_{a \in x - \Lambda + c_n,\ |a| \geq t/2} |\widehat{\chi_\Omega}|^2(a) && \text{(with } a = x - \ell)\\
&\geq 1 - \frac{C}{t} && \text{(from Lemma 1 applied to the set } x - \Lambda + c_n).
\end{aligned}$$

Letting $t \to \infty$ we obtain that $F(x) = 1$ for almost all $x \in \mathbb{R}$. (Notice that the constant C that appears above does not depend on n.)

Since every finite subset of L is contained in some L_n it follows that

$$\dim \operatorname{span} \phi(L) < \dim \operatorname{span} \phi(\Lambda). \tag{7}$$

To derive a contradiction let the finite set $\Lambda' \subseteq \Lambda$ be such that $\phi(\Lambda')$ is a basis of $\operatorname{span} \phi(\Lambda)$ and also let the finite set $L' \subseteq L$ be such that $\phi(L')$ is a basis of $\operatorname{span} \phi(L)$. Some translate $s + L'$ of the finite set L' is contained in Λ, hence

$$A(\phi(s + \ell'), \phi(\lambda')) = 0, \quad \text{(for all } \ell' \in L' \text{ and } \lambda' \in \Lambda'),$$

which implies

$$A(\phi(\ell'), \phi(-s + \lambda')) = 0, \quad \text{(for all } \ell' \in L' \text{ and } \lambda' \in \Lambda'),$$

and this means that $-s + \Lambda' \subseteq L$ and therefore that

$$\dim \operatorname{span} \phi(L) \geq \dim \operatorname{span} \phi(-s + \Lambda') = \dim \operatorname{span} \phi(\Lambda') = \dim \operatorname{span} \phi(\Lambda),$$

in contradiction with (7). We have used the easy fact that $\dim \operatorname{span} \phi(A + x) = \dim \operatorname{span} \phi(A)$ for any $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$. □

Completion of the proof: The set Λ is periodic.

Let T be as in Lemma 2 and consider all subsets of Λ of the form

$$B_\lambda = \Lambda \cap [\lambda, \lambda + T], \quad \lambda \in \Lambda.$$

It follows from Lemma 2 that B_λ is a generating set for each λ. But there are only finitely many different forms the set $B_\lambda - \lambda$ can take, hence there are $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 > \lambda_2$, such that

$$B_{\lambda_1} - \lambda_1 = B_{\lambda_2} - \lambda_2,$$

or

$$B_{\lambda_1} = B_{\lambda_2} + \lambda_1 - \lambda_2.$$

Since B_{λ_1} and B_{λ_2} are both generating sets for $\phi(\Lambda)$ it follows that

$$\begin{aligned} x \in \Lambda &\Leftrightarrow e_x \perp e_y \quad (y \in B_{\lambda_2}) \\ &\Leftrightarrow e_{x+\lambda_1-\lambda_2} \perp e_y \quad (y \in B_{\lambda_1}) \\ &\Leftrightarrow x + (\lambda_1 - \lambda_2) \in \Lambda. \end{aligned}$$

In other words, $T = \lambda_1 - \lambda_2$ is a period of Λ.

Let us also remark that any period of Λ must be an integer. This is a consequence of the fact that Λ has density 1: if T is a period of Λ this implies that there are exactly T elements of Λ in each interval $[x, x+T)$ hence T is an integer.

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