BOUNDED REMAINDER SETS, BOUNDED DISTANCE EQUIVALENT CUT-AND-PROJECT SETS, AND EQUIDECOMPOSABILITY

MARK MORDECHAI ETKIND, SIGRID GREPSTAD, MIHAIL N. KOLOUNTZAKIS, AND NIR LEV

ABSTRACT. We use the measurable Hall's theorem due to Cieśla and Sabok to prove that (i) if two measurable sets $A, B \subset \mathbb{R}^d$ of the same measure are bounded remainder sets with respect to a given irrational d-dimensional vector α, then A, B are equidecomposable with measurable pieces using translations from $\mathbb{Z}\alpha+\mathbb{Z}^d$; and (ii) given a lattice $\Gamma \subset \mathbb{R}^m \times \mathbb{R}^n$ with projections p_1 and p_2 onto $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, if two cut-and-project sets in $\mathbb{R}^m$ obtained from Riemann measurable windows $W, W' \subset \mathbb{R}^n$ are bounded distance equivalent, then W, W' are equidecomposable with measurable pieces using translations from $p_2(\Gamma)$. We also prove by a different method that for one-dimensional cut-and-project sets the pieces can be chosen Riemann measurable.

CONTENTS

1. Introduction 2
2. Equidecomposability and Hall's condition 3
2.1. Equidecomposability 3
2.2. Equidecomposability up to measure zero 4
2.3. The measurable Hall's theorem 5
3. Bounded remainder sets 6
3.1. 6
3.2. 6
3.3. 7
3.4. 9
3.5. 9
4. Bounded distance equivalent cut-and-project sets 9
4.1. 9
4.2. 10
4.3. 11
4.4. 11
4.5. 12
4.6. 13

Date: November 25, 2025.

2020 *Mathematics Subject Classification.* 52C23, 52B45, 11K38.

Key words and phrases. Bounded remainder sets, cut-and-project sets, bounded distance equivalence, equidecomposability.

Research supported by ISF Grant 854/25 and Grant 334466 of the Research Council of Norway.

5. One-dimensional cut-and-project sets 13
5.1. 13
5.2. Lattices in general position 14
5.3. Lattices in special form 14
5.4. Point counting function 15
5.5. Cut-and-project sets 16
5.6. Bounded distance equivalence and equidecomposability 17
References 17

1. Introduction

Let X be a set endowed with a group of transformations G. Two subsets $A, B \subset X$ are called G-*equidecomposable* if they can be partitioned into the same *finite* number of pieces $A = \bigcup_{i=1}^{n} A_i$, $B = \bigcup_{i=1}^{n} B_i$, which can be pairwise matched via elements of G, i.e. $B_i = g_i \cdot A_i$ for some $g_i \in G$, $i = 1, 2, \dots, n$, where $g\,\cdot$ denotes the group action.

A famous example of equidecomposability is the so-called Tarski circle squaring problem, which was posed by Tarski (1925) [TW16]: is a square of area 1 equidecomposable to a disk of area 1 via plane isometries? This was answered in the affirmative by Laczkovich [Lac90]: the square of unit area can be partitioned into a finite number of pieces which can then be translated to form a partition of a disk of unit area (thus the group of transformations of the plane used is not the whole group of isometries but merely the group of translations). Moreover, it was proved by Grabowski, Máthé and Pikhurko [GMP17] that the pieces in this result can be chosen Lebesgue measurable.

In the present paper we consider the case where G is a finitely generated group of translations of $\mathbb{R}^d$, usually dense in the group of all translations. We also relax the concept of equidecomposability to ignore sets of Lebesgue measure zero: two sets A, B are called G-equidecomposable *up to measure zero* if we can remove from them a set of measure zero such that the remaining sets are G-equidecomposable. This relaxation is particularly natural if one is to impose the requirement of measurability on the pieces of the equidecomposition. This relaxation does not usually cause any problems in applications of equidecomposability, e.g. to tilings [GK25]. Subject to these assumptions and demands, our goal in this paper is generally to achieve equidecomposability with *measurable* pieces.

One can think of the equidecomposability of A and B as a problem of finding a perfect matching in a bipartite graph. Take the bipartite graph with the points of A on one side and the points of B on the other. Then A, B are G-equidecomposable if and only if there exists a finite set $F \subset G$ such that the bipartite graph whose edges are all pairs of the form $(a, f \cdot a)$ with $a \in A$, $f \cdot a \in B$, $f \in F$, has a perfect matching. Recall that a *perfect matching* is a collection of disjoint edges that touch all points of A and B. Let us call such a perfect matching a G-*matching*.

Our main tool in the effort to produce measurable pieces in an equidecomposition is the *measurable Hall's theorem* due to Cieśla and Sabok [CS22] (see Theorem 2.4

below), which uses an appropriately mixing group action on the ambient space in order to deduce the existence of a *measurable* G-matching between two sets A, B from the existence of an arbitrary (not necessarily measurable) G-matching. By a measurable G-matching we mean a G-matching for which the set $A_g = \{a \in A : (a, g \cdot a)$ is part of the matching$\}$ is measurable for each $g \in G$.

The structure of the rest of this paper is as follows.

In the preliminary Section 2 we review the equidecomposability concepts that will be used in the paper and formulate the measurable Hall's theorem due to Cieśla and Sabok [CS22].

In Section 3 we discuss *bounded remainder sets*, and we show that if two measurable sets A, B of the same measure are bounded remainder sets with respect to a given irrational d-dimensional vector α, then A, B are equidecomposable with measurable pieces using translations from $\mathbb{Z}\alpha + \mathbb{Z}^d$.

In Section 4 we show that if two model sets defined by two different Riemann measurable windows W and W' are *bounded distance equivalent* then (and only then, see [FG18, Theorem 6.1]) the two windows are equidecomposable up to measure zero with measurable pieces using translations from $p_2(\Gamma)$, where Γ is the lattice defining the model sets and p_2 is its projection onto the subspace containing the windows W, W'. This bridges a gap that has arisen in the proof of [Gre25a, Theorem 1.1], see [Gre25b].

The results in Sections 3 and 4 rely on the measurable Hall's theorem [CS22]. This is not the case in Section 5, where we prove by a different method that in the special case of one-dimensional model sets, if two model sets are bounded distance equivalent then the corresponding Riemann measurable windows are equidecomposable *with Riemann measurable pieces* using translations from $p_2(\Gamma)$.

2. Equidecomposability and Hall's condition

In this preliminary section we review the connection between equidecomposability and Hall's condition, and state the measurable Hall's theorem due to Cieśla and Sabok [CS22] that will be used later on.

2.1. **Equidecomposability.** Let X be a set endowed with an action of a group G. We use $g \cdot x$ to denote the action of an element $g \in G$ on a point $x \in X$.

We say that two sets $A, B \subset X$ are *G-equidecomposable* if there exist finitely many sets $A_1, \dots, A_n \subset X$ and elements $g_1, \dots, g_n \in G$ such that $\{A_j\}_{j=1}^n$ forms a partition of A, while $\{g_j \cdot A_j\}_{j=1}^n$ forms a partition of B.

We say that $A, B \subset X$ satisfy *Hall's condition* with respect to G, if there exists a finite set $F \subset G$ such that

(i) $|S| \leqslant |(F \cdot S) \cap B|$ for every finite set $S \subset A$;

(ii) $|T| \leqslant |(F^{-1} \cdot T) \cap A|$ for every finite set $T \subset B$.

To motivate this definition, consider A, B as two disjoint vertex sets of a bipartite graph, where two vertices $a \in A$ and $b \in B$ are connected by an edge if and only if $b = g \cdot a$ for some $g \in F$. The conditions (i) and (ii) then say that the size of every finite set of vertices in A or in B does not exceed the size of the set of its neighbors in the graph.

The following proposition clarifies the connection between the notions of equidecomposability and Hall's condition.

Proposition 2.1. *Two sets $A, B \subset X$ are G-equidecomposable if and only if A and B satisfy Hall's condition with respect to G.*

Proof. We first prove the 'if' part. Suppose that there is a finite set $F \subset G$ such that (i) and (ii) hold. By the classical Hall's marriage theorem, the condition (i) implies that for every finite set $S \subset A$ there exists an injective map $\varphi_S : S \to B$ satisfying $\varphi_S(a) \in F \cdot a$ for all $a \in S$. By an application of Tychonoff's theorem, see [HV50], there is an injective map $\varphi : A \to B$ such that $\varphi(a) \in F \cdot a$ for all $a \in A$. In a similar way, we deduce from (ii) that there is an injective map $\psi : B \to A$ such that $\psi(b) \in F^{-1} \cdot b$ for all $b \in B$. In turn, the proof of the Cantor-Schröder-Bernstein theorem (see [TW16, Theorem 3.6]) yields a bijection $\chi : A \to B$ such that $\chi(a) \in F \cdot a$ for all $a \in A$. This implies that A and B are equidecomposable using only actions of the finite set F.

Next we prove the 'only if' part. Suppose that $\{A_j\}_{j=1}^n$ forms a partition of A and that $\{g_j \cdot A_j\}_{j=1}^n$ forms a partition of B, where $g_1, \dots, g_n \in G$. This allows us to define a bijection $\chi : A \to B$ given by $\chi(a) = g_j \cdot a$ if $a \in A_j$. By the necessity part of the classical Hall's marriage theorem, this implies that both conditions (i) and (ii) are satisfied with the finite set $F = \{g_1, \dots, g_n\}$. □

Remarks. 1. The proof shows that if A, B satisfy Hall's condition with a given finite set $F \subset G$, then A, B are equidecomposable using only actions of the same finite set F, and also the converse it true.

2. In the case where the sets A, B are *countable*, the application of Tychonoff's theorem can be replaced by a standard diagonalization argument.

2.2. **Equidecomposability up to measure zero.** Let (X, μ) be a measure space, either finite or infinite, endowed with a measure preserving action of a *countable* group G.

We say that two measurable sets $A, B \subset X$ are G-equidecomposable *up to measure zero*, if there exist finitely many sets $A_1, \dots, A_n \subset X$, elements $g_1, \dots, g_n \in G$ and a full measure subset $X' \subset X$, such that $\{A_j \cap X'\}_{j=1}^n$ forms a partition of $A \cap X'$, while $\{(g_j \cdot A_j) \cap X'\}_{j=1}^n$ forms a partition of $B \cap X'$. If the sets $A_1, \dots, A_n$ can be chosen measurable, then we say that A, B are G-equidecomposable up to measure zero *with measurable pieces*.

Following [CS22, Definition 1] we say that two measurable sets $A, B \subset X$ satisfy Hall's condition *a.e.* with respect to G, if there is a finite set $F \subset G$ and a full measure subset $X' \subset X$, such that for every $x \in X'$ we have

(i') $|S| \leqslant |(F \cdot S) \cap B|$ for every finite set $S \subset A \cap (G \cdot x)$;

(ii') $|T| \leqslant |(F^{-1} \cdot T) \cap A|$ for every finite set $T \subset B \cap (G \cdot x)$.

In other words, for almost every $x \in X$ the two sets $A \cap (G \cdot x)$ and $B \cap (G \cdot x)$ satisfy Hall's condition with the same finite set $F \subset G$.

Proposition 2.2. *Let (X, μ) be a measure space endowed with a measure preserving action of a countable group G. Two measurable sets $A, B \subset X$ are G-*equidecomposable *up to measure zero (with possibly non-measurable pieces) if and only if A, B satisfy* Hall's condition *a.e. with respect to G.*

Proof. We first prove the 'if' part. Assume that there is a finite set $F \subset G$ and a full measure subset $X' \subset X$ such that (i') and (ii') hold for every $x \in X'$. Since the group G is countable, then by replacing X' with $\bigcap_{g \in G}(g \cdot X')$ we may assume that $G \cdot X' = X'$, that is, X' is a G-invariant set. It follows that the two sets $A' = A \cap X'$ and $B' = B \cap X'$ satisfy Hall's condition with the finite set F, hence A', B' are G-equidecomposable by Proposition 2.1. As a consequence, A, B are G-equidecomposable up to measure zero.

To prove the converse 'only if' part, suppose now that $\{A_j \cap X'\}_{j=1}^n$ forms a partition of $A \cap X'$ and $\{(g_j \cdot A_j) \cap X'\}_{j=1}^n$ forms a partition of $B \cap X'$, where $g_1, \dots, g_n \in G$ and X' is a full measure subset of X. Again by replacing X' with $\bigcap_{g \in G}(g \cdot X')$ we may assume that X' is a G-invariant set. This implies that the two sets $A \cap X'$ and $B \cap X'$ are G-equidecomposable considered as subsets of the set X'. Hence by Proposition 2.1 there is a finite set $F \subset G$ such that (i') and (ii') hold for every $x \in X'$. □

2.3. **The measurable Hall's theorem.** Next we state the measurable Hall's theorem proved in [CS22]. The theorem gives conditions guaranteeing that two measurable sets $A, B \subset X$ satisfying Hall's condition are equidecomposable *with measurable pieces*.

Assume now that (X, μ) is a standard Borel probability space, endowed with a *free* pmp (probability measure preserving) action of a *finitely generated abelian* group G. We recall that the action of G on X is called *free* if $g \cdot x \neq x$ for every nontrivial element $g \in G$ and every $x \in X$.

By the structure theorem for finitely generated abelian groups, we may assume that $G = \mathbb{Z}^d \times \Delta$ where d is a nonnegative integer and Δ is a finite abelian group.

Definition 2.3 (see [CS22, Definition 5])**.** A measurable set $A \subset X$ is called *G-uniform* if there exist positive constants c and n_0, such that for almost every $x \in X$ and for every $n > n_0$ we have $|A \cap (F_n \cdot x)| \geqslant cn^d$, where $F_n := \{0, 1, \dots, n-1\}^d \times \Delta$.

The measurable Hall's theorem due to Cieśla and Sabok states the following:

Theorem 2.4 ([CS22, Theorem 2])**.** *Let (X, μ) be a standard Borel probability space, endowed with a free pmp action of a finitely generated abelian group G, and let $A, B \subset X$ be two measurable G-uniform sets. Then the following conditions are equivalent:*

(a) *A and B satisfy Hall's condition a.e. with respect to G;*

(b) *A and B are G-equidecomposable up to measure zero (with possibly non-measurable pieces);*

(c) *A and B are G-equidecomposable up to measure zero with measurable pieces.*

The equivalence of (a) and (b) was given in Proposition 2.2. Theorem 2.4 asserts that these conditions are also equivalent to (c). This result will be used below.

3. Bounded remainder sets

3.1. If $A \subset \mathbb{R}^d$ is a bounded measurable set, we use $\mathbb{1}_A$ to denote its indicator function, and we let

$$\chi_A(x) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_A(x+k), \quad x \in \mathbb{R}^d, \tag{3.1}$$

be the multiplicity function of the projection of A onto $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a fixed real vector such that the numbers $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are linearly independent over the rationals. A bounded measurable set $A \subset \mathbb{R}^d$ is called a *bounded remainder set* (BRS) if there is a constant $C = C(A, \alpha)$ such that

$$\Big| \sum_{k=0}^{n-1} \chi_A(x + k\alpha) - n \operatorname{mes} A \Big| \leqslant C \quad (n = 1, 2, 3, \dots) \quad \text{a.e.} \tag{3.2}$$

Bounded remainder sets form a classical topic in discrepancy theory, see [GL15] for an overview of the subject and a survey of basic results.

3.2. It is easy to show that if two bounded measurable sets $A, B \subset \mathbb{R}^d$ are equidecomposable up to measure zero using only translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$, and if A is a bounded remainder set, then so is B, see [GL15, Proposition 4.1].

A question posed in [GL15, Section 7.2] asks whether a converse statement holds in the following sense: Let $A, B \subset \mathbb{R}^d$ be two bounded remainder sets of the same measure. Is it true that A and B must be equidecomposable (up to measure zero, with measurable pieces) using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ only?

It was proved in [GL15, Theorem 2] that the answer is affirmative if the sets A, B are assumed to be Riemann measurable, and moreover, in this case there exists an equidecomposition with Riemann measurable pieces.

However, the question has remained open in the general case. Our goal here is to answer this question affirmatively.

Theorem 3.1. *Let $A, B \subset \mathbb{R}^d$ be two bounded remainder sets of the same measure. Then A and B are equidecomposable up to measure zero with measurable pieces, using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$.*

It follows that equidecomposability provides a method for constructing all bounded remainder sets. We also note that, as mentioned in [GL15, Section 7.2], this result allows to extend [GL15, Theorem 5] to all bounded remainder sets.

We now turn to the details of the proof. In what follows, we assume that the sets A and B both have positive measure (otherwise we have nothing to prove).

3.3. Since A, B are bounded subsets of $\mathbb{R}^d$, we can choose a sufficiently large positive integer q and vectors $r_1, \dots, r_q \in \mathbb{Z}^d$ such that, if we denote $Q = [0,1)^d$, then the union of cubes $Q+r_1, \dots, Q+r_q$ covers both A and B. This induces a partition of each set A and B into subsets $A_i := A \cap (Q + r_i)$ and $B_i := B \cap (Q + r_i)$, $1 \leqslant i \leqslant q$.

Let $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$ be the cyclic group of order q, endowed with its probability Haar measure assigning the mass $1/q$ to each element.

Now consider the product space $X = \mathbb{T}^d \times \mathbb{Z}_q$ and denote by μ the product probability measure on X. We also consider the finitely generated abelian group $G = \mathbb{Z} \times \mathbb{Z}_q$. It induces a free pmp action on X, where the action of the element $(n, \sigma) \in G$ on the point $(x, \tau) \in X$ is given by $(n, \sigma) \cdot (x, \tau) = (x + n\alpha, \sigma + \tau)$.

Next, we define two measurable sets $A', B' \subset X$ by

$$A' = \bigcup_{i=1}^{q} A_i \times \{i\}, \quad B' = \bigcup_{i=1}^{q} B_i \times \{i\}. \tag{3.3}$$

Here we identify the sets A_i and B_i with their projections on $\mathbb{T}^d$, which we may do since both A_i and B_i are contained in the cube $Q + r_i$.

We claim that the sets A' and B' are G-equidecomposable up to measure zero, with possibly non-measurable pieces. It suffices to show that there is a finite set $F \subset G$ and a full measure subset $X' \subset X$, such that for every point $(x, \tau) \in X'$ there exists a bijection from $A' \cap (G \cdot (x, \tau))$ onto $B' \cap (G \cdot (x, \tau))$ that moves elements using only actions of the set F.

To prove this, we will use a technique similar to [GL18, Section 6.2].

3.3.1. Since A is a bounded remainder set, it follows from [GL15, Proposition 2.3] that there is a constant C and a full measure subset $\Omega \subset \mathbb{T}^d$ such that

$$\sup_{n>0} \sup_{j \in \mathbb{Z}} \Big| \sum_{k=j+1}^{j+n} \chi_A(x + k\alpha) - n \operatorname{mes} A \Big| \leqslant C, \quad x \in \Omega. \tag{3.4}$$

The set $X' = \Omega \times \mathbb{Z}_q$ is a full measure subset of X. We now fix a point $(x, \tau) \in X'$ and consider the set $A' \cap (G \cdot (x, \tau))$. We construct an enumeration of the elements of this set in the following way. Define

$$A^n = A \cap (x + n\alpha + \mathbb{Z}^d), \quad n \in \mathbb{Z}, \tag{3.5}$$

and let $\{s_n\}$, $n \in \mathbb{Z}$, be a sequence of integers such that

$$s_0 = 0, \quad s_{n+1} - s_n = \#A^n \tag{3.6}$$

(we note that each A^n is a finite set, and that some of the sets A^n may be empty). For each $n \in \mathbb{Z}$ we then choose some enumeration $\{a_j\}$, $s_n \leqslant j < s_{n+1}$, of the points in the set A^n. We also observe that, since $A_1, \dots, A_q$ form a partition of A, for each j there is a unique element $\sigma_j \in \{1, \dots, q\}$ such that $a_j \in A_{\sigma_j}$. It is now easy to check that the sequence $\{(a_j, \sigma_j)\}$, $j \in \mathbb{Z}$, forms an enumeration of the set $A' \cap (G \cdot (x, \tau))$.

We now claim that

$$|s_n - n \operatorname{mes} A| \leqslant C, \quad n \in \mathbb{Z}. \tag{3.7}$$

Indeed, by (3.5), (3.6) we have the equality $s_{k+1} - s_k = \chi_A(x + k\alpha)$. If we sum this equality over $0 \leqslant k \leqslant n-1$ and use (3.4), we obtain that (3.7) holds for $n > 0$. In the case $n < 0$ we establish (3.7) similarly, by summing the equality over $n \leqslant k \leqslant -1$.

3.3.2. In a similar way, we define

$$B^m = B \cap (x + m\alpha + \mathbb{Z}^d), \quad m \in \mathbb{Z}, \tag{3.8}$$

and let $\{t_m\}$, $m \in \mathbb{Z}$, be a sequence of integers such that

$$t_0 = 0, \quad t_{m+1} - t_m = \# B^m. \tag{3.9}$$

We choose an enumeration $\{b_j\}$, $t_m \leqslant j < t_{m+1}$, of the points in the set B^m, and let $\tau_j \in \{1, \dots, q\}$ be the unique element such that $b_j \in B_{\tau_j}$. We thus obtain an enumeration $\{(b_j, \tau_j)\}$, $j \in \mathbb{Z}$, of the set $B' \cap (G \cdot (x, \tau))$.

Moreover, since B is a bounded remainder set, we may assume that the constant C and the full measure subset $\Omega \subset \mathbb{T}^d$ have been chosen such that we have

$$|t_m - m \operatorname{mes} B| \leqslant C, \quad m \in \mathbb{Z}. \tag{3.10}$$

3.3.3. We now claim that there exists a finite set $E \subset \mathbb{Z}$, which does not depend on the point (x, τ), such that

$$b_j - a_j \in E\alpha + \mathbb{Z}^d, \quad j \in \mathbb{Z}. \tag{3.11}$$

Indeed, given j there exist n, m such that $a_j \in A^n$ and $b_j \in B^m$. Hence

$$b_j - a_j \in (m - n)\alpha + \mathbb{Z}^d \tag{3.12}$$

which follows from (3.5), (3.8). We now write

$$m - n = \Big(m - \frac{t_m}{\operatorname{mes} B}\Big) + \Big(\frac{t_m}{\operatorname{mes} B} - \frac{s_n}{\operatorname{mes} A}\Big) + \Big(\frac{s_n}{\operatorname{mes} A} - n\Big). \tag{3.13}$$

Due to (3.7) and (3.10), the first and third terms on the right hand side are bounded in modulus by a certain constant $K_1 = K_1(A, B)$. To estimate the second term, note that $s_n \leqslant j < s_{n+1}$ and $s_{n+1} - s_n = \# A^n$ which cannot exceed q, hence $0 \leqslant j - s_n < q$. In a similar way, $0 \leqslant j - t_m < q$. As a consequence, $|t_m - s_n| < q$. Since A and B have the same measure, it then follows that also the second term on the right hand side of (3.13) is bounded in modulus by some constant $K_2 = K_2(A, B)$. We conclude that $m - n$ lies in some finite set $E \subset \mathbb{Z}$ that does not depend on the point (x, τ). Hence, (3.12) implies (3.11).

3.3.4. We now define $F := E \times \mathbb{Z}_q$, which is a finite subset of G. It follows from (3.11) that for each $j \in \mathbb{Z}$, the two points (a_j, σ_j) and (b_j, τ_j) of the space X differ by an element of the set $E\alpha \times \mathbb{Z}_q$. In other words, this means that $(b_j, \tau_j) \in F \cdot (a_j, \sigma_j)$. As the sequence $\{(a_j, \sigma_j)\}$ is an enumeration of $A' \cap (G \cdot (x, \tau))$, while the sequence $\{(b_j, \tau_j)\}$ is an enumeration of $B' \cap (G \cdot (x, \tau))$, this shows that there exists a bijection from $A' \cap (G \cdot (x, \tau))$ onto $B' \cap (G \cdot (x, \tau))$ that moves elements using only actions of the set F. As this holds for every $(x, \tau) \in X' = \Omega \times \mathbb{Z}_q$ which is a full measure subset of X, and since the finite set F does not depend on the point (x, τ), it follows that A', B' are G-equidecomposable up to measure zero, with possibly non-measurable pieces.

3.4. We now wish to invoke Theorem 2.4 in order to conclude that the two sets A' and B' are G-equidecomposable up to measure zero with measurable pieces. To this end, we need to verify that the sets A' and B' are G-uniform.

Let $F_n := \{0, 1, \dots, n-1\} \times \mathbb{Z}_q$. To prove that A' is G-uniform, we need to show that there are positive constants c and n_0, such that for all (x, τ) in some full measure subset of X and for every $n > n_0$, we have

$$|A' \cap (F_n \cdot (x, \tau))| \geqslant cn. \tag{3.14}$$

We check that this holds for all $(x, \tau) \in X' = \Omega \times \mathbb{Z}_q$. Indeed, observe that the elements of the set $A' \cap (F_n \cdot (x, \tau))$ are given in our enumeration as $\{(a_j, \sigma_j)\}$, $s_0 \leqslant j < s_n$, and therefore this set contains exactly s_n elements. In turn, it follows from (3.7) that we have $s_n \geqslant n \operatorname{mes} A - C$. Hence, we can choose $c > 0$ small enough and n_0 large enough, not depending on the point (x, τ), such that (3.14) holds for every $n > n_0$. This shows that A' is a G-uniform set.

In a similar way, it can be shown that also the set B' is G-uniform.

3.5. We can therefore apply Theorem 2.4 and conclude that the two sets A' and B' are G-equidecomposable up to measure zero with measurable pieces. Finally, we need to show that this implies that $A, B \subset \mathbb{R}^d$ are equidecomposable up to measure zero with measurable pieces, using only translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$.

First, by refining the pieces in the equidecomposition if needed, we may assume that each piece of A' is entirely contained in one of the sets $A_i \times \{i\}$, $1 \leqslant i \leqslant q$. Hence, if P' is one of the pieces of A', then $P' = P \times \{i\}$ for some $i \in \{1, \dots, q\}$ and for some measurable set $P \subset A_i = A \cap (Q + r_i)$. The piece P' is carried by some element $(n, \sigma) \in G$ onto a piece R' of the set B'. If we choose $j \in \{1, \dots, q\}$ such that $j = i + \sigma \pmod q$, then $R' = R \times \{j\}$ for some measurable set $R \subset B_j = B \cap (Q + r_j)$. The fact that $(n, \sigma) \cdot P' = R'$ implies that P and R are equidecomposable using translations by vectors from $n\alpha + \mathbb{Z}^d$. It remains to note that as P' goes through all the pieces of A', the corresponding sets $\{P\}$ and $\{R\}$ form partitions of A and B respectively, up to measure zero. It thus follows that A and B are equidecomposable up to measure zero with measurable pieces, using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$.

4. Bounded distance equivalent cut-and-project sets

4.1. Two discrete point sets $\Lambda, \Lambda' \subset \mathbb{R}^m$ are said to be *bounded distance equivalent* with constant $K > 0$ if there exists a bijection $\chi : \Lambda \to \Lambda'$ satisfying

$$|\chi(\lambda) - \lambda| \leqslant K, \quad \lambda \in \Lambda. \tag{4.1}$$

We indicate this using the shorthand notation $\Lambda \overset{\text{bd}}{\sim} \Lambda'$.

Let Γ be a lattice in $\mathbb{R}^m \times \mathbb{R}^n$. Denoting the projections from $\mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$ and $\mathbb{R}^n$ by p_1 and p_2 respectively, we assume that $p_1|_\Gamma$ is injective, and that the image $p_2(\Gamma)$ is dense in $\mathbb{R}^n$. If $W \subset \mathbb{R}^n$ is a bounded set (called a "window") then the set

$$\Lambda(\Gamma, W) = \{p_1(\gamma) : \gamma \in \Gamma,\ p_2(\gamma) \in W\} \tag{4.2}$$

is called the *cut-and-project set*, or the *model set*, in $\mathbb{R}^m$ obtained from the lattice Γ and the window W.

There is an intimate relation between bounded remainder sets and one-dimensional model sets, in the sense that a one-dimensional model set with a Riemann measurable window W is bounded distance equivalent to an arithmetic progression if and only if a linear image of W is a bounded remainder set with respect to a certain irrational vector, see [HK16], [HKK17], [GL18, Section 6], [FG18, Theorem 4.5].

It follows that certain results on bounded remainder sets have natural analogs, or extensions, to model sets. For instance, [GL15, Theorem 1] states that any parallelepiped in $\mathbb{R}^d$ spanned by linearly independent vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d$ is a bounded remainder set; this can be seen as a special case of [DO90, Theorem 3.1] providing a sufficient condition on a parallelepiped window W in order for the corresponding model set to be bounded distance equivalent to a lattice.

The relation between bounded remainder sets and model sets prompts the question as to whether Theorem 3.1 admits (at least, for Riemann measurable sets) an extension to higher-dimensional model sets. The next result provides such an extension.

Theorem 4.1. *Let $W, W' \subset \mathbb{R}^n$ be two bounded Riemann measurable sets of positive measure. If the model sets $\Lambda(\Gamma, W)$ and $\Lambda(\Gamma, W')$ are bounded distance equivalent, then W, W' are equidecomposable up to measure zero with measurable pieces, using only translations by vectors from $p_2(\Gamma)$.*

This result was previously announced in [Gre25a, Theorem 1.1] but the original proof turned out to contain a gap, see [Gre25b]. The remainder of the section is devoted to a new proof of Theorem 4.1 which bridges this gap.

4.2. We now turn to the proof of Theorem 4.1. By assumption, the two model sets $\Lambda(\Gamma, W)$ and $\Lambda(\Gamma, W')$ are bounded distance equivalent. As in the original proof given in [Gre25a, Section 3] this implies, using the assumption that $p_1|_\Gamma$ is injective, that the “lifted” sets

$$\Gamma_W = \{\gamma \in \Gamma : p_2(\gamma) \in W\}, \quad \Gamma_{W'} = \{\gamma \in \Gamma : p_2(\gamma) \in W'\}, \tag{4.3}$$

are also bounded distance equivalent.

Let us denote $N = \{\gamma \in \Gamma : p_2(\gamma) = 0\}$. Then N is a sublattice of Γ (remark that if $p_2|_\Gamma$ is injective, then $N = \{0\}$). In turn, there is a sublattice L of Γ such that we have the direct sum decomposition

$$\Gamma = L \oplus N \tag{4.4}$$

(see [Cas97, I.2.2, Corollary 3]). Then $p_2|_L$ is injective, and $p_2(L) = p_2(\Gamma)$. Define

$$L_W = \{\gamma \in L : p_2(\gamma) \in W\}, \quad L_{W'} = \{\gamma \in L : p_2(\gamma) \in W'\}, \tag{4.5}$$

then it follows that

$$\Gamma_W = L_W \oplus N, \quad \Gamma_{W'} = L_{W'} \oplus N. \tag{4.6}$$

4.3. We wish to prove that L_W and $L_{W'}$ are bounded distance equivalent. We will obtain this as a consequence of the following lemma.

Lemma 4.2. *Let $A, B \subset \mathbb{Z}^r$ and suppose that $A \times \mathbb{Z}^s \overset{bd}{\sim} B \times \mathbb{Z}^s$ with constant K. Then also $A \overset{bd}{\sim} B$ with the same constant K.*

Proof. By assumption there exists a bijection $\chi : A \times \mathbb{Z}^s \to B \times \mathbb{Z}^s$ that moves points by distance at most K. We consider A, B as subsets of $\mathbb{Z}^r$ viewed as a group acting on itself by translations. To prove the claim it suffices to show that A, B are equidecomposable using only actions of the finite set $F = \{j \in \mathbb{Z}^r : |j| \leqslant K\}$. In turn, by Proposition 2.1 it suffices to check that A, B satisfy Hall's condition with the finite set F. That is, we need to show that $|S| \leqslant |(S+F) \cap B|$ for any finite set $S \subset A$, and that $|T| \leqslant |(T-F) \cap A|$ for any finite set $T \subset B$. We will only check that the first condition holds, as the second condition can be established similarly.

Let $S \subset A$ be a finite set. Then for any positive integer R, the bijection χ maps the set $S \times \{0, \dots, R-1\}^s$ injectively into $((S+F) \cap B) \times \{-K, \dots, R+K-1\}^s$. Hence

$$|S| \cdot R^s \leqslant |(S+F) \cap B| \cdot (R+2K)^s, \tag{4.7}$$

and letting $R \to +\infty$ we conclude that $|S| \leqslant |(S+F) \cap B|$, as we had to show. □

Since Γ_W and $\Gamma_{W'}$ are bounded distance equivalent, it follows from (4.6) that after applying a suitable invertible linear transformation, we may use Lemma 4.2 in order to conclude that also L_W and $L_{W'}$ are bounded distance equivalent.

4.4. Let K be the bounded distance equivalence constant of L_W and $L_{W'}$.

Lemma 4.3. *$L_{W-x} \overset{bd}{\sim} L_{W'-x}$ with the same constant K for every $x \in \mathbb{R}^n$ satisfying*

$$(\partial W - x) \cap p_2(\Gamma) = (\partial W' - x) \cap p_2(\Gamma) = \emptyset. \tag{4.8}$$

Proof. Let $F = \{\gamma \in L : |\gamma| \leqslant K\}$ which is a finite subset of L. Since $L_W \overset{\mathrm{bd}}{\sim} L_{W'}$ with constant K, there is a bijection $\chi : L_W \to L_{W'}$ and a function $f : L_W \to F$ such that $\chi(\tau) = \tau + f(\tau)$ for all $\tau \in L_W$. Fix a point $x \in \mathbb{R}^n$ satisfying (4.8), and consider the sets $A = L_{W-x}$ and $B = L_{W'-x}$ as subsets of L viewed as a group acting on itself by translations. It suffices to show that A, B are equidecomposable using only actions of the finite set F.

In turn, by Proposition 2.1 it suffices to check that A, B satisfy Hall's condition with the finite set F. We will do this by showing that given a finite set $S \subset A$ there is an injective map $\varphi : S \to B$ satisfying $\varphi(\gamma) \in \gamma + F$ for all $\gamma \in S$; and given a finite set $T \subset B$ there is an injective map $\psi : T \to A$ satisfying $\psi(\gamma) \in \gamma - F$ for all $\gamma \in T$. We will only prove the first claim, as the second claim can be proved similarly.

Let $S \subset A = L_{W-x}$ be a finite set. Since the image $p_2(L) = p_2(\Gamma)$ is dense in $\mathbb{R}^n$, we may choose a sequence $\gamma_j \in L$ such that $x_j = p_2(\gamma_j) \to x$. The assumption that $\partial W - x$ does not intersect $p_2(\Gamma)$ implies that the elements of the finite set $p_2(S)$ lie in the interior of $W - x$. Hence, there is j_0 such that $p_2(S) \subset W - x_j$ for all $j > j_0$. This means that

$$S \subset L_{W-x_j} = L_W - \gamma_j, \tag{4.9}$$

and therefore for each $\gamma \in S$ there is $\tau_j(\gamma) \in L_W$ such that $\gamma = \tau_j(\gamma) - \gamma_j$. Since both S and F are finite sets, then by passing to a subsequence if needed we may assume that for each $\gamma \in S$ the value $f(\tau_j(\gamma))$ does not depend on j, so there is a function $h : S \to F$ such that $f(\tau_j(\gamma)) = h(\gamma)$ for every j and every $\gamma \in S$. Define $\varphi(\gamma) = \gamma + h(\gamma)$ for each $\gamma \in S$. It remains to show that φ is an injective map from S into B.

We first check that φ indeed maps S into B. Let $\gamma \in S$, then

$$\varphi(\gamma) = \gamma + h(\gamma) = \tau_j(\gamma) - \gamma_j + f(\tau_j(\gamma)) = \chi(\tau_j(\gamma)) - \gamma_j. \tag{4.10}$$

Since χ maps L_W into $L_{W'}$ then

$$p_2(\varphi(\gamma)) = p_2(\chi(\tau_j(\gamma))) - x_j \in W' - x_j, \tag{4.11}$$

and letting $j \to \infty$ we obtain that $p_2(\varphi(\gamma))$ lies in the closure of $W' - x$. In turn, using the assumption that $\partial W' - x$ does not intersect $p_2(\Gamma)$, we conclude that $p_2(\varphi(\gamma))$ must in fact lie in the interior of $W' - x$. As a consequence, $\varphi(\gamma) \in L_{W'-x} = B$.

Lastly, we show that φ is injective. Indeed, let $\gamma, \gamma' \in S$, then by (4.10) we have

$$\varphi(\gamma) = \chi(\tau_j(\gamma)) - \gamma_j, \quad \varphi(\gamma') = \chi(\tau_j(\gamma')) - \gamma_j. \tag{4.12}$$

Hence, if we assume that $\varphi(\gamma) = \varphi(\gamma')$ then $\chi(\tau_j(\gamma)) = \chi(\tau_j(\gamma'))$. Since χ is an injective map, it follows that $\tau_j(\gamma) = \tau_j(\gamma')$. But recalling that $\gamma = \tau_j(\gamma) - \gamma_j$ and $\gamma' = \tau_j(\gamma') - \gamma_j$ this implies that $\gamma = \gamma'$. Hence φ is an injective map, and the lemma is proved. □

4.5. Since W and W' are bounded sets in $\mathbb{R}^n$, and since the image $p_2(\Gamma)$ is dense in $\mathbb{R}^n$, we may choose a system of n linearly independent vectors $v_1, \dots, v_n \in p_2(\Gamma)$ which are large enough for W and W' to be contained in the parallelepiped

$$\Omega = \{t_1 v_1 + \cdots + t_n v_n : t_1, \dots, t_n \in [-\tfrac{1}{2}, \tfrac{1}{2})\}. \tag{4.13}$$

Let H be the subgroup of $\mathbb{R}^n$ generated by the vectors $v_1, \dots, v_n$. Then H is a lattice in $\mathbb{R}^n$ and a subgroup of $p_2(\Gamma)$, and Ω is a fundamental domain of H in $\mathbb{R}^n$.

We now consider the quotient space $X = \mathbb{R}^n/H$, and let μ be the Lebesgue measure on X normalized such that $\mu(X) = 1$. Then $G = p_2(\Gamma)/H$ is a finitely generated abelian group which induces a free pmp action on (X, μ) by translations. Since W, W' are contained in the fundamental domain Ω of H, we may also view W, W' as measurable subsets of X, and we observe that W, W' are G-equidecomposable (up to measure zero) considered as subsets of X, if and only if W, W' are $p_2(\Gamma)$-equidecomposable (up to measure zero) as subsets of $\mathbb{R}^n$.

We now wish to prove that W, W' (as subsets of X) satisfy Hall's condition a.e. with respect to G. It suffices to show that there is a finite set $F \subset \Gamma$ and a full measure subset $X' \subset X$, such that for every point $x \in X'$ there exists a bijection from $W \cap (G + x)$ onto $W' \cap (G + x)$ that moves elements using only actions of the set $p_2(F)$.

We choose $F := \{\gamma \in L : |\gamma| \leqslant K\}$ where K is the bounded distance equivalence constant of L_W and $L_{W'}$, and we let X' be the set of points $x \in X$ satisfying the condition (4.8) (note that this condition is invariant under translations by vectors in H, so it may be viewed as a condition on elements of X). Since W and W' are

Riemann measurable sets, their boundaries ∂W and $\partial W'$ are both sets of measure zero, which implies that X' is a full measure subset of X.

Fix $x \in X'$, and denote $A = W \cap (G + x)$ and $B = W' \cap (G + x)$. We observe that the mapping $\gamma \mapsto p_2(\gamma) + x \pmod{H}$ defines a bijection $\varphi : L_{W-x} \to A$, as well as a bijection $\psi : L_{W'-x} \to B$. We also recall that by Lemma 4.3 there is a bijection $\chi : L_{W-x} \to L_{W'-x}$ such that $\chi(\gamma) - \gamma \in F$ for all $\gamma \in L_{W-x}$. Hence $\psi \circ \chi \circ \varphi^{-1}$ defines a bijection from A onto B that moves points using only actions of the finite set $p_2(F)$. We conclude that W, W' satisfy Hall's condition a.e. with respect to G.

4.6. We now wish to invoke Theorem 2.4 in order to conclude that the two sets W, W' are G-equidecomposable up to measure zero with measurable pieces (as subsets of X). To this end, we need to verify that W, W' are G-uniform sets.

By the structure theorem for finitely generated abelian groups, there exists a direct sum decomposition $G = M \oplus \Delta$ where M is a free abelian group of rank d, and Δ is a finite abelian group. We observe that since $p_2(\Gamma)$ is dense in $\mathbb{R}^n$, then G is dense in X. In turn, this implies that also M must be dense in X (see [Rud62, Section 2.1]).

Let $e_1, \dots, e_d$ be some basis for M, and denote

$$F_k = P_k \oplus \Delta, \quad P_k = \Big\{ \sum_{j=1}^{d} m_j e_j : m_1, \dots, m_d \in \{0, 1, \dots, k-1\} \Big\}. \tag{4.14}$$

To prove that W is a G-uniform set, we must show that there exist positive constants c and k_0 such that for almost all $x \in X$ and every $k > k_0$ we have

$$|W \cap (F_k + x)| \geqslant c k^d. \tag{4.15}$$

Since W is a Riemann measurable set of positive measure, there is $\varepsilon > 0$ such that W contains some open ball U of radius 2ε. Since M is dense in X, there is a positive integer k_0 such that the set P_{k_0} forms an ε-net in X. This implies that also any translate of P_{k_0} is an ε-net in X. Now observe that for every $x \in X$ and every $k > k_0$, the set $P_k + x$ contains at least $\lfloor k/k_0 \rfloor^d$ disjoint translated copies of P_{k_0}, and each one of these translated copies must intersect the ball U. It follows that

$$|W \cap (F_k + x)| \geqslant |U \cap (P_k + x)| \geqslant \lfloor k/k_0 \rfloor^d \geqslant c k^d, \tag{4.16}$$

which verifies condition (4.15) and shows that W is a G-uniform set. In a similar way, one can show that also the set W' is G-uniform.

Finally, by an application of Theorem 2.4 we conclude that the two sets W, W' are G-equidecomposable up to measure zero with measurable pieces as subsets of X. This implies that W, W' are $p_2(\Gamma)$-equidecomposable up to measure zero with measurable pieces as subsets of $\mathbb{R}^n$, and completes the proof of Theorem 4.1.

5. One-dimensional cut-and-project sets

5.1. Notice that in the statement of Theorem 4.1, the sets W, W' are assumed to be Riemann measurable, yet the result only guarantees their $p_2(\Gamma)$-equidecomposability with measurable pieces. One may therefore ask whether the pieces in the equidecomposition may be chosen to be also Riemann measurable. One may also consider a variant of this question, which appears to be of practical importance: if the sets W, W' in Theorem 4.1 are assumed to be polytopes, can the pieces in the equidecomposition be chosen to be also polytopes?

Note that by a “polytope" in $\mathbb{R}^d$ we mean any finite union of d-dimensional simplices with disjoint interiors. Thus a polytope may be non-convex, or even disconnected.

In this section we establish a result which gives an affirmative answer to both questions above for *one-dimensional* cut-and-project sets.

Let Γ be a lattice in $\mathbb{R} \times \mathbb{R}^d$, such that if p_1 and p_2 denote the projections from $\mathbb{R} \times \mathbb{R}^d$ onto $\mathbb{R}$ and $\mathbb{R}^d$ respectively, then $p_1|_\Gamma$ is injective, while $p_2(\Gamma)$ is dense in $\mathbb{R}^d$. If $W \subset \mathbb{R}^d$ is a bounded set, then again we consider the model set in $\mathbb{R}$ defined by

$$\Lambda(\Gamma, W) = \{p_1(\gamma) : \gamma \in \Gamma,\ p_2(\gamma) \in W\}. \tag{5.1}$$

Theorem 5.1. *Let $W, W' \subset \mathbb{R}^d$ be two bounded Riemann measurable sets (resp. two polytopes). If the one-dimensional model sets $\Lambda(\Gamma, W)$ and $\Lambda(\Gamma, W')$ are bounded distance equivalent, then W, W' are equidecomposable up to measure zero with Riemann measurable pieces (resp. with polytope pieces) using translations from $p_2(\Gamma)$.*

The proof below does not rely on the measurable Hall's theorem which only gives equidecomposability with measurable pieces. It is rather based on the connection of the problem to bounded remainder sets and the results obtained in [GL15], [GL18].

5.2. **Lattices in general position.** We say that a lattice Γ in $\mathbb{R} \times \mathbb{R}^d$ is in *general position* if the restriction of p_1 to Γ is injective, and the image $p_2(\Gamma)$ is dense in $\mathbb{R}^d$.

In [GL18] the term “general position” was used to indicate that the restrictions of both p_1 and p_2 to Γ are injective, and both their images $p_1(\Gamma)$ and $p_2(\Gamma)$ are dense in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. These two definitions are in fact equivalent:

Lemma 5.2. *If $\Gamma \subset \mathbb{R} \times \mathbb{R}^d$ is a lattice in general position, then also the restriction of p_2 to Γ is injective, and the image $p_1(\Gamma)$ is dense in $\mathbb{R}$.*

Proof. Let $v_1, \dots, v_{d+1}$ be a basis for the lattice Γ. The assumption that $p_2(\Gamma)$ is dense in $\mathbb{R}^d$ implies that $p_2(v_1), \dots, p_2(v_d)$ must be linearly independent vectors in $\mathbb{R}^d$. Hence the vector $p_2(v_{d+1})$ admits a unique expansion $p_2(v_{d+1}) = \sum_{j=1}^d \alpha_j p_2(v_j)$. Using again the assumption that $p_2(\Gamma)$ is dense in $\mathbb{R}^d$ implies that the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent. As a consequence, the restriction of p_2 to Γ is injective.

Since the restriction of p_1 to Γ is injective, the numbers $p_1(v_1), \dots, p_1(v_{d+1})$ must be rationally independent. Hence these numbers generate a dense subgroup of $\mathbb{R}$. But this subgroup coincides with the image $p_1(\Gamma)$, so this image is dense in $\mathbb{R}$. $\square$

5.3. **Lattices in special form.** Following [GL18, Section 4] we define the notion of a lattice of special form.

Definition 5.3. We say that a lattice Γ in $\mathbb{R} \times \mathbb{R}^d$ is of *special form* if

$$\Gamma = \{(n + \beta^\top(n\alpha + m), n\alpha + m) : n \in \mathbb{Z}, m \in \mathbb{Z}^d\} \tag{5.2}$$

where α, β are column vectors in $\mathbb{R}^d$ satisfying the following conditions:

(i) The vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)^\top$ is such that the numbers $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are linearly independent over the rationals;

(ii) The vector $\beta = (\beta_1, \beta_2, \dots, \beta_d)^\top$ is such that the numbers $\beta_1, \beta_2, \dots, \beta_d, 1 + \beta^\top\alpha$ are linearly independent over the rationals.

It is easy to check that the conditions imposed on the vectors α and β are precisely those necessary and sufficient for Γ to be in general position.

Let a be a nonzero real scalar and B be a $d \times d$ invertible real matrix. We consider a linear and invertible transformation T from $\mathbb{R} \times \mathbb{R}^d$ onto itself given by

$$T(x, y) = (ax, By), \quad (x, y) \in \mathbb{R} \times \mathbb{R}^d. \tag{5.3}$$

Lemma 5.4 (see [GL18, Lemma 4.3])**.** *Assume that $L \subset \mathbb{R} \times \mathbb{R}^d$ is a lattice in general position. Then there exist a lattice Γ of special form* (5.2) *and an invertible linear transformation T of the form* (5.3) *such that $T(L) = \Gamma$.*

We argue that by Lemma 5.4 it suffices to consider lattices of special form. For suppose Theorem 5.1 holds in this case, and suppose $\Lambda(L, W)$ and $\Lambda(L, W')$ are bounded distance equivalent. Then so are the "lifted" sets

$$L_W = \{\ell \in L : p_2(\ell) \in W\}, \quad L_{W'} = \{\ell \in L : p_2(\ell) \in W'\},$$

and thus also the sets

$$T(L_W) = \{(ap_1(\ell), Bp_2(\ell)) : p_2(\ell) \in W\} = \Gamma_{BW}$$

and

$$T(L_{W'}) = \{(ap_1(\ell), Bp_2(\ell)) : p_2(\ell) \in W'\} = \Gamma_{BW'}.$$

It follows that the projected sets $p_1(\Gamma_{BW}) = \Lambda(\Gamma, BW)$ and $p_1(\Gamma_{BW'}) = \Lambda(\Gamma, BW')$ are bounded distance equivalent in $\mathbb{R}$. Since we assume that Theorem 5.1 holds for the lattice Γ of special form, this implies that the sets BW and BW' are equidecomposable up to measure zero using translations from $p_2(\Gamma) = Bp_2(L)$. It follows that W and W' are equidecomposable up to measure zero using translations from $p_2(L)$. Finally, since B is a linear and invertible map, properties of the pieces in the partition (such as Riemann measurability or them being polytopes) are preserved.

In what follows we will thus assume that Γ is a lattice of the special form (5.2).

5.4. **Point counting function.** If Λ is a uniformly discrete set in $\mathbb{R}$, then we define its point counting function $\nu(\Lambda, x)$ as

$$\nu(\Lambda, x) = \begin{cases} \#(\Lambda \cap [0, x)), & x \geqslant 0, \quad (5.4) \\ -\#(\Lambda \cap [x, 0)), & x < 0. \quad (5.5) \end{cases}$$

Lemma 5.5. *If two uniformly discrete sets $\Lambda, \Lambda' \subset \mathbb{R}$ are bounded distance equivalent, then there is a constant C such that $|\nu(\Lambda, x) - \nu(\Lambda', x)| \leqslant C$ for all $x \in \mathbb{R}$.*

This is obvious and so the proof is omitted.

5.5. **Cut-and-project sets.** Let W be a bounded set in $\mathbb{R}^d$, and

$$\Lambda(\Gamma, W) := \{p_1(\gamma) : \gamma \in \Gamma,\, p_2(\gamma) \in W\} \tag{5.6}$$

be the model set in $\mathbb{R}$ generated by the lattice Γ of special form (5.2) and the window W. It is well-known that $\Lambda(\Gamma, W)$ is a uniformly discrete set. We recall that

$$\chi_W(x) := \sum_{m \in \mathbb{Z}^d} \mathbb{1}_W(x+m), \quad x \in \mathbb{R}^d, \tag{5.7}$$

denotes the multiplicity function of the projection of W onto $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Lemma 5.6. *The counting function of* $\Lambda(\Gamma, W)$ *satisfies*

$$\nu(\Lambda(\Gamma, W), N) = \sum_{n=0}^{N-1} \chi_W(n\alpha) + O(1), \quad N \to +\infty. \tag{5.8}$$

Proof. Indeed, due to the special form (5.2), the elements $\gamma \in \Gamma$ may be parametrized by the vectors $(n, m) \in \mathbb{Z} \times \mathbb{Z}^d$ in such a way that

$$p_1(\gamma) = n + \beta^\top(n\alpha + m), \quad p_2(\gamma) = n\alpha + m. \tag{5.9}$$

Now the point $p_1(\gamma)$ belongs to $\Lambda(\Gamma, W)$ if and only if $n\alpha + m \in W$. In this case

$$p_1(\gamma) - n = \beta^\top(n\alpha + m) \in \beta^\top W, \tag{5.10}$$

and $\beta^\top W$ is a bounded subset of $\mathbb{R}$. Hence there is $C = C(\Gamma, W)$ such that

$$|p_1(\gamma) - n| \leqslant C \tag{5.11}$$

whenever $p_1(\gamma)$ is a point in $\Lambda(\Gamma, W)$. It follows that $\nu(\Lambda(\Gamma, W), N)$ differs from

$$\#\{(n, m) \in \mathbb{Z} \times \mathbb{Z}^d : 0 \leqslant n \leqslant N-1, n\alpha + m \in W\} \tag{5.12}$$

by a bounded magnitude, which is equivalent to (5.8). ☐

Lemma 5.7. *Let* W, W' *be two bounded, Riemann measurable sets in* $\mathbb{R}^d$. *If* $\Lambda(\Gamma, W)$ *and* $\Lambda(\Gamma, W')$ *are bounded distance equivalent, then there is a constant* C *such that*

$$\Big| \sum_{n=0}^{N-1} \chi_W(x + n\alpha) - \sum_{n=0}^{N-1} \chi_{W'}(x + n\alpha) \Big| \leqslant C \quad a.e. \tag{5.13}$$

holds for every N.

Proof. Define $f(x) = \chi_W(x) - \chi_{W'}(x)$ and $S_N(x) = \sum_{n=0}^{N-1} f(x + n\alpha)$. Lemmas 5.5 and 5.6 imply the existence of a constant C such that $|S_N(0)| \leqslant C$ for every N. We now use an argument from [GL15, Proposition 2.2]. The function S_N is $\mathbb{Z}^d$-periodic and we have $S_N(x + j\alpha) = S_{N+j}(x) - S_j(x)$, hence $|S_N| \leqslant 2C$ on the set $\{j\alpha\}_{j=1}^\infty$ which is dense in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Since W and W' are Riemann measurable sets, the function S_N is continuous at almost every point, so it follows that $|S_N| \leqslant 2C$ a.e. ☐

5.6. **Bounded distance equivalence and equidecomposability.** We can now use the observations made above in order to prove Theorem 5.1. Indeed, due to Lemma 5.4 we may assume that Γ is a lattice of the special form (5.2). By Lemma 5.7 there exists a constant C such that the estimate (5.13) holds for every N. We now invoke [GL15, Theorem 7.1] which asserts that the condition (5.13) is satisfied if and only if W, W' are equidecomposable up to measure zero with Riemann measurable pieces, using only translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^d = p_2(\Gamma)$; and moreover, if W, W' are two polytopes in $\mathbb{R}^d$ then the pieces in the equidecomposition be chosen to be also polytopes. This completes the proof of Theorem 5.1.

References

[Cas97] J. W. S. Cassels, An introduction to the geometry of numbers. Springer-Verlag, 1997.

[CS22] T. Cieśla, M. Sabok, Measurable Hall's theorem for actions of abelian groups. J. Eur. Math. Soc. (JEMS) 24 (2022), no. 8, 2751–2773.

[DO90] M. Duneau, C. Oguey, Displacive transformations and quasicrystalline symmetries. J. Physique **51** (1990), no. 1, 5–19.

[FG18] D. Frettlöh, A. Garber, Pisot substitution sequences, one dimensional cut-and-project sets and bounded remainder sets with fractal boundary. Indag. Math. **29** (2018), no. 4, 1114–1130.

[GMP17] Ł. Grabowski, A. Máthé, O. Pikhurko, Measurable circle squaring. Ann. of Math. (2) **185** (2017), no. 2, 671–710.

[Gre25a] S. Grepstad, Bounded distance equivalence of cut-and-project sets and equidecomposability. Int. Math. Res. Not. IMRN 2025, no. 4, 1–15.

[Gre25b] S. Grepstad, Corrigendum: Bounded distance equivalence of cut-and-project sets and equidecomposability.

[GK25] S. Grepstad, M. Kolountzakis, Bounded common fundamental domains for two lattices. Preprint, `arXiv:2507.00604`.

[GL15] S. Grepstad, N. Lev, Sets of bounded discrepancy for multi-dimensional irrational rotation. Geom. Funct. Anal. **25** (2015), no. 1, 87–133.

[GL18] S. Grepstad, N. Lev, Riesz bases, Meyer's quasicrystals, and bounded remainder sets. Trans. Amer. Math. Soc. **370** (2018), no. 6, 4273–4298.

[HV50] P. R. Halmos, H. E. Vaughan, The marriage problem. Amer. J. Math. **72** (1950), 214–215.

[HKK17] A. Haynes, M. Kelly, H. Koivusalo, Constructing bounded remainder sets and cut-and-project sets which are bounded distance to lattices, II. Indag. Math. **28** (2017), no. 1, 138–144.

[HK16] A. Haynes, H. Koivusalo, Constructing bounded remainder sets and cut-and-project sets which are bounded distance to lattices. Israel J. Math. **212** (2016), no. 1, 189–201.

[Lac90] M. Laczkovich, Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem. J. Reine Angew. Math. **404** (1990), 77–117.

[Rud62] W. Rudin, Fourier analysis on groups. Interscience, 1962.

[TW16] G. Tomkowicz, S. Wagon, The Banach-Tarski paradox, 2nd edition. Cambridge University Press, 2016.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel
Email address: mark.etkind@mail.huji.ac.il

Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway
Email address: sigrid.grepstad@ntnu.no

Department of Mathematics and Applied Mathematics, University of Crete, Voutes Campus, 70013 Heraklion, Greece and Institute of Computer Science, Foundation of Research and Technology Hellas, N. Plastira 100, Vassilika Vouton, 700 13, Heraklion, Greece
Email address: kolount@gmail.com

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel
Email address: levnir@math.biu.ac.il