BOUNDED COMMON FUNDAMENTAL DOMAINS FOR TWO LATTICES

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Abstract. We prove that for any two lattices $L, M \subseteq \mathbb{R}^d$ of the same volume there exists a measurable, bounded, common fundamental domain of them. In other words, there exists a bounded measurable set $E \subseteq \mathbb{R}^d$ such that E tiles $\mathbb{R}^d$ when translated by L or by M. In fact, the set E can be taken to be a finite union of polytopes. A consequence of this is that the indicator function of E forms a Weyl–Heisenberg (Gabor) orthogonal basis of $L^2(\mathbb{R}^d)$ when translated by L and modulated by M^*, the dual lattice of M.

Contents

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1. Introduction

1.1. **The Steinhaus tiling problem.** A question of Steinhaus from the 1950s [Mos81, Sie58] asks if there is a subset E of the plane $\mathbb{R}^2$ such that E tiles the plane when translated by $R_\theta \mathbb{Z}^2$, for any value of θ. Here R_θ denotes the 2×2 matrix which rotates the plane by the angle θ around the origin. Equivalently we are seeking a set E such that $R_\theta E$ tiles the plane when translated by $\mathbb{Z}^2$, for any θ.

For a set $E \subseteq \mathbb{R}^d$ to tile $\mathbb{R}^d$ when translated by the set $T \subseteq \mathbb{R}^d$ we mean that the T-translates of E partition $\mathbb{R}^d$. If the set T happens to be a subgroup of $\mathbb{R}^d$ this is the same as demanding that E contains exactly one element from each coset of T in $\mathbb{R}^d$. Clearly this definition of tiling makes sense in any abelian group.

The Steinhaus tiling problem bifurcated from the 1980s into two forms:

- the original, *set-theoretic* formulation where nothing else is expected from the set E but to contain one point from each coset of $R_\theta \mathbb{Z}^2$, and this for any θ, and
- the *measurable* formulation, where the set E is expected to be Lebesgue measurable but, in return, the tiling is demanded almost everywhere: for any θ we only ask that

$$\sum_{n \in R_\theta \mathbb{Z}^2} \mathbf{1}_E(x-n) = 1, \quad \text{for almost all } x \in \mathbb{R}^2. \tag{1}$$

We should add that the problem makes sense in $\mathbb{R}^d$, $d>2$, as well, where we are seeking a set E that tiles simultaneously with all linear transformations of $\mathbb{Z}^d$ by an orthogonal matrix (though we must admit that sensible forms of this problem may be stated even with smaller groups).

The set-theoretic question in the plane ($d=2$) was settled in a major result by Jackson and Mauldin [JM02a, JM02b, JM03] who proved the existence of such a set E in the plane.

The measurable question is still open in the plane. There have been many partial results, almost all of which are of the form "if a measurable Steinhaus set E exists it must be large near infinity". For example it is known [Bec89, Kol96] that such a set cannot be bounded. The best result so far in this direction is that in [KW99] where it is shown that

$$\int_E |x|^\alpha \, dx = +\infty \text{ for } \alpha > 46/27.$$

In an interesting lack of symmetry between the set-theoretic and measurable developments it is now known [KW99, KP02] that there are no measurable Steinhaus sets in dimensions $d>2$ but it is still unknown if there are "set-theoretic" Steinhaus sets for $d>2$.

The interested reader should consult the references in [KP17] as well as the most recent paper [KL24], for results on many variations of the Steinhaus question.

1.2. **Common fundamental domains for finitely many lattices.** A *fundamental domain* for an abelian group H within an abelian group G is a subset of G that contains exactly one element from every coset of H in G. So, the Steinhaus tiling problem for the plane asks for a common fundamental domain for all groups $R_\theta\mathbb{Z}^2$ inside $\mathbb{R}^2$, for $\theta \in [0, 2\pi)$.

From now on, we focus on the measurable version of the problem where we only ask E to satisfy the tiling equation (1) almost everywhere.

A sensible relaxation of the Steinhaus problem is to look for a common fundamental domain of only a finite family of lattices

$$L_1, \ldots, L_n \in \mathbb{R}^d. \tag{2}$$

Any measurable fundamental domain of a lattice has volume equal to the determinant (also called volume) of the lattice. Hence, we must require that all $L_1, \ldots, L_n$ have the same volume.

In [Kol97] it was proved that if the dual lattices of the collection (2) have a direct sum

$$L_1^* + \cdots + L_n^*$$

then we can find a measurable common fundamental domain for (2). And it was shown in [HW01] that for the case of two lattices only no condition is necessary: Any two lattices of the same volume in $\mathbb{R}^d$ have a measurable common fundamental domain. (See also [KP22] for several similar questions.)

In both [Kol97] and [HW01] the constructed fundamental domains are generally unbounded. Since then, it has been an open problem whether two lattices of the same volume in $\mathbb{R}^d$ have a measurable bounded common fundamental domain in $\mathbb{R}^d$. This question we answer in this paper:

Theorem 1. *Suppose L, M are lattices in $\mathbb{R}^d$ of the same volume. Then there is a bounded measurable $\Omega \subseteq \mathbb{R}^d$ which tiles with both L and M.*

The set Ω can be chosen as a finite union of polytopes.

The important technical breakthrough arises in the special case below when L and M have a direct sum. This is made possible using the main result of [Gre24].

Theorem 2. *If $L, M \subseteq \mathbb{R}^d$ are lattices of the same volume and $\overline{L+M} = \mathbb{R}^d$ then there is a bounded, measurable $E \subseteq \mathbb{R}^d$ such that $L \oplus E = M \oplus E = \mathbb{R}^d$ are both tilings. Moreover, the set E may be chosen to be a finite union of polytopes in $\mathbb{R}^d$.*

1.3. **An application to Weyl–Heisenberg orthogonal bases.** In [HW01] the existence of a measurable common fundamental domain for two lattices is used to show that whenever K, L are two lattices in $\mathbb{R}^d$ with $\det L \cdot \det K = 1$ then there exists a Gabor (or Weyl-Heisenberg) orthogonal basis of $\mathbb{R}^d$ with translation lattice L and modulation lattice K. In other words, there exists a function $g \in L^2(\mathbb{R}^d)$ such that the collection of time-frequency translates

$$e^{2\pi i \ell \cdot x} g(x-k), \quad \ell \in L, k \in K,$$

is an orthogonal basis of $L^2(\mathbb{R}^d)$. In their proof the function g is precisely the indicator function of a measurable common fundamental domain of the lattices K and L^*. Thus our Theorem 1 implies that this *window* function g may be chosen to be of compact support, a possibly significant property, since it offers the advantage of localization.

1.4. **Some notation.** A *lattice* is a discrete subgroup of $\mathbb{R}^n$ which linearly spans $\mathbb{R}^n$. The *rank* of a subgroup of $\mathbb{R}^n$ is the dimension of its linear span. Thus a lattice is a discrete subgroup of $\mathbb{R}^n$ of full rank, equal to n. We denote by $\operatorname{vol} L$ or $\det L$ the volume of any fundamental domain of the lattice L, and by $\operatorname{dens} L$ the lattice density $1/\operatorname{vol} L$. If L is a discrete subgroup of $\mathbb{R}^d$ of rank smaller than d we still write $\operatorname{vol} L$ or $\det L$ to denote the volume of the fundamental domain in the $\mathbb{R}$-linear space L spans.

Any lattice $L \subseteq \mathbb{R}^n$ is equal to $A\mathbb{Z}^n$ where A is a non-singular $n \times n$ matrix. This matrix A is not unique, but can be formed by taking as its columns any $\mathbb{Z}$-basis of L. The *dual lattice* of L is defined by

$$L^* = \left\{ x \in \mathbb{R}^d : x \cdot \ell \in \mathbb{Z} \text{ for all } \ell \in L \right\}$$

and it can be seen that $L^* = A^{-\top}\mathbb{Z}^d$.

When we write $A \oplus B$ for two sets A, B in an additive group we mean that all sums $a+b$, with $a \in A, b \in B$, are distinct. In this case we say the sum $A+B$ is *direct* or that $A+B$ is a *tiling*.

Plan. We prove Theorem 2 first in §2 and use it to prove then Theorem 1 in §3.

2. Bounded common fundamental domains when the sum is dense

The proof of Theorem 2 relies on certain results from the theory of so-called cut-and-project sets in $\mathbb{R}^d$. We therefore give a brief description of this point set construction, introducing necessary notation and terminology.

A discrete point set Λ in $\mathbb{R}^d$ is called a *Delone set* if it is both uniformly discrete and relatively dense, meaning there exist radii $r, R > 0$ such that any ball of radius r contains at most one point of Λ, and any ball of radius R contains at least one point of Λ. If Λ additionally satisfies

$$\Lambda - \Lambda \subseteq \Lambda + F,$$

for some finite set F in $\mathbb{R}^d$, then we say that Λ is a *Meyer set*.

A cut-and-project set, or *model set*, is constructed from a lattice $\Gamma \subset \mathbb{R}^m \times \mathbb{R}^n$ and a *window set* $W \subset \mathbb{R}^n$ by taking the projection into $\mathbb{R}^m$ of those lattice points whose projection into $\mathbb{R}^n$ is contained in W. Denoting the projections from $\mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$ and $\mathbb{R}^n$ by p_1 and p_2, respectively, we assume that $p_1|_\Gamma$ is injective, and that the image $p_2(\Gamma)$ is dense in $\mathbb{R}^n$, and denote by $\Lambda_W = \Lambda(\Gamma, W)$ the model set

$$\Lambda(\Gamma, W) = \{p_1(\gamma) \,:\, \gamma \in \Gamma,\, p_2(\gamma) \in W\}.$$

If the boundary ∂W of the window W has Lebesgue measure zero, then the model set Λ_W is called *regular*. In this case, the point set Λ_W in $\mathbb{R}^m$ has a number of desirable properties. One can show that Λ_W is a Meyer set with well-defined density

$$\operatorname{dens} \Lambda_W = \frac{|W|}{\det \Gamma} = |W| \cdot \operatorname{dens} \Gamma.$$

Moreover, if the model set is either generic (meaning that $p_2(\Gamma) \cap \partial W = \emptyset$) or if the window W is half-open as defined in [Ple00, Definition 2.2], then Λ_W is *repetitive*. Repetitivity is the crystal-like quality that every finite configuration appearing in Λ will reappear infinitely often, see e.g. [Ple00, Property 2] for a precise definition.

The cut-and-project construction is well-studied in the field of aperiodic order, and in the last 30 years there have been several results on when a model set (or more generally a Delone set) is at bounded distance from a lattice [DO91, FG18, Lac92]. We say that two point sets Λ and Λ' in $\mathbb{R}^n$ are *bounded distance equivalent* (or, at bounded distance from each other) if there exists a bijection $\varphi : \Lambda \to \Lambda'$ and a constant $C > 0$ such that

$$\|\varphi(\lambda) - \lambda\| < C$$

for all $\lambda \in \Lambda$.

Facts:

(1) Bounded distance equivalence is an equivalence relation.
(2) If a Delone set Λ in $\mathbb{R}^d$ has a well-defined density and is bounded distance equivalent to a lattice L in $\mathbb{R}^d$, then $\operatorname{dens} \Lambda = \operatorname{dens} L$.
(3) Any two lattices L and M in $\mathbb{R}^d$ of equal density are necessarily at bounded distance from each other ([DO90, Theorem 5.2], [DO91, Theorem 1], [Kol97, §3.2]).

The proof of Theorem 2 relies on the following result from [DO91] on model sets with parallelotope windows, as well as a more recent result from [Gre24] connecting bounded distance equivalence and equidecomposability (Theorem 5 below).

Theorem 3. *[DO91, Theorem 3.1] Let Γ be a lattice in $\mathbb{R}^m \times \mathbb{R}^n$. If $W \subset \mathbb{R}^n$ is a parallelotope*

$$W = \left\{ \sum_{k=1}^{n} t_k v_k \,:\, 0 \le t_k < 1 \right\}$$

spanned by n linearly independent vectors in $p_2(\Gamma)$, then the model set $\Lambda(\Gamma, W)$ is at bounded distance to a lattice in $\mathbb{R}^m$.

We say that two sets S and S' in $\mathbb{R}^m$ are *equidecomposable* if S can be partitioned into finitely many subsets which can be rearranged by translations to form a partition of S'. Given a subgroup $G \subset \mathbb{R}^m$ we will use the term *G-equidecomposable* to mean that we allow only translations in G for this rearrangement.

Theorem 3 above can be extended to hold for any reasonably well-behaved fundamental domain of a sublattice in $p_2(\Gamma)$ by the following result of Frettlöh and Garber.

Theorem 4. *[FG18, Theorem 6.1] Let Λ and Λ' be two model sets constructed from the same lattice Γ but with different windows W and W', respectively. If the windows W and W' are $p_2(\Gamma)$-equidecomposable, then Λ and Λ' are bounded distance equivalent.*

It turns out that for regular model sets, a converse of Theorem 4 can be established if we relax the equidecomposability condition to ignore sets of measure zero.

Definition 1. Let G be a group of translations in $\mathbb{R}^n$. We say that two measurable sets S and S' in $\mathbb{R}^n$ of equal Lebesgue measure are *G-equidecomposable up to measure zero* if there exists a partition of S into finitely many measurable subsets $S_1, \dots, S_N$, and a set of vectors $g_1, \dots, g_N \in G$, such that S' and $\bigcup_{j=1}^{N}(S_j + g_j)$ differ at most on a set of measure zero.

Theorem 5. *[Gre24, Theorem 1.1] Let $\Gamma \subset \mathbb{R}^m \times \mathbb{R}^n$ be a lattice and let W and W' be bounded, measurable sets in $\mathbb{R}^n$ where both ∂W and $\partial W'$ have measure zero and $|W| = |W'|$. If the model sets $\Lambda_W = \Lambda(\Gamma, W)$ and $\Lambda_{W'} = \Lambda(\Gamma, W')$ are bounded distance equivalent, then the windows W and W' are $p_2(\Gamma)$-equidecomposable up to measure zero.*

Remark 1. Note that in the proof of Theorem 5 in [Gre24], the partition of W is constructed by shifting W by certain elements $p_2(\gamma)$ of $p_2(\Gamma)$, and successively removing the intersection of (what remains of) $W + p_2(\gamma)$ and W'. Accordingly, if W and W' are both polytopes in $\mathbb{R}^m$, then the subsets in the partition of W may be chosen to be polytopes as well.

We are now equipped to prove Theorem 2.

Proof of Theorem 2. By abuse of notation let $L = L\mathbb{Z}^d$ and $M = M\mathbb{Z}^d$, where L and M are $d \times d$ non-singular matrices. Let Ω_L be the half-open parallelotope spanned by the columns of L and Ω_M be the half-open parallelotope spanned by the columns of M. Then Ω_L and Ω_M are fundamental domains of the lattices L and M, respectively. Since L and M are assumed to have equal volumes, we have $|\Omega_M| = |\Omega_L|$.

We now construct a lattice $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ (where Γ again denotes both the lattice itself and its matrix representation) by letting

$$\Gamma = \left[\begin{array}{cc} \multicolumn{2}{c}{K} \\ \hline L & M \end{array}\right],$$

where K may be chosen to be any $d \times 2d$ matrix which acts as an injective map on $\mathbb{Z}^{2d}$. With the cut-and-projection construction in mind, we note that this ensures that the projection p_1 is injective when restricted to the lattice Γ. Moreover, since $\overline{L+M} = \mathbb{R}^d$ by assumption, we know that $p_2(\Gamma)$ is dense in $\mathbb{R}^d$.

We now consider the two model sets

$$\Lambda_L = \Lambda(\Gamma, \Omega_L) = \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega_L\}$$

and

$$\Lambda_M = \Lambda(\Gamma, \Omega_M) = \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega_M\}.$$

Since $p_2(\Gamma) = L + M$, we see that both Ω_L and Ω_M are windows spanned by d linearly independent vectors in $p_2(\Gamma)$. Thus by Theorem 3, both Λ_L and Λ_M are bounded distance equivalent to a lattice. Moreover, by assumption we have $|\Omega_L| = |\Omega_M|$, so $\operatorname{dens}\Lambda_L = \operatorname{dens}\Lambda_M$. This implies that the model sets Λ_L and Λ_M are bounded distance equivalent to lattices of equal density, and thus also at bounded distance from each other. We thus conclude from Theorem 5 that we can find a partition of Ω_L into polytopal subsets $S_1, \dots, S_N$ and elements $\gamma_1, \dots, \gamma_N \in \Gamma$ such that

$$\Omega_M = \bigcup_{i=1}^{N} \underbrace{(S_i + p_2(\gamma_i))}_{S_i'} = \bigcup_{i=1}^{N} S_i', \tag{3}$$

where we understand this equality to hold up to measure zero.

Finally, we observe that

$$p_2(\gamma_i) = \ell_i + m_i,$$

for every $i = 1, \dots, N$, where $\ell_i \in L$ and $m_i \in M$. It follows that

$$E = \bigcup_{i=1}^{N} (S_i' - m_i) = \bigcup_{i=1}^{N} (S_i + \ell_i)$$

is a fundamental domain for both M and L by (3) and the fact that $(S_i)_{i=1}^N$ is a partition of Ω_L. We thus have

$$L \oplus E = M \oplus E = \mathbb{R}^d,$$

for a bounded measurable set $E \subset \mathbb{R}^d$. □

3. Bounded common fundamental domains in the general case

In this section we prove Theorem 1.

Lemma 1. *Suppose $L \subseteq \mathbb{Z}^m \times \mathbb{R}^n$ is a lattice in $\mathbb{R}^d$, where $d = m + n$. Then*

$$L_2 = L \cap \{0\}^m \times \mathbb{R}^n$$

has rank n.

Proof. Suppose $\operatorname{rank} L_2 = k < n$ and let $u_1, \ldots, u_k \in \{0\}^m \times \mathbb{R}^n$ be a $\mathbb{Z}$-basis of L_2. Let also $u_{k+1}, \ldots, u_d$ be an extension of this $\mathbb{Z}$-basis to a $\mathbb{Z}$-basis of L. This extension always exists [Cas96, Corollary 3, p. 14].

It follows that there are $g_j \in \mathbb{Z}^m$ and $r_j \in \{0\}^m \times \mathbb{R}^n$, for $j = 1, \ldots, d-k$, such that

$$u_{k+j} = g_j + r_j, \quad j = 1, \ldots d-k.$$

Since $m < d-k$ there are $n_j \in \mathbb{Z}$, not all 0, such that $\sum_{j=1}^{d-k} n_j g_j = 0$. This implies that $0 \neq \sum_{j=1}^{d-k} n_j u_{k+j} \in \{0\}^m \times \mathbb{R}^n$, hence this sum belongs to L_2, a contradiction, since $u_1, \ldots, u_d$ are linearly independent and L_2 is generated by $u_1, \ldots, u_k$.

□

Lemma 2. *Suppose G_1, G_2 are subgroups of the abelian group G of the same, finite index k. Then there are $g_1, \ldots, g_k \in G$ which are simultaneously a complete set of coset representatives of G_1 and G_2 in G. In other words*

$$G_1 + \{g_1, \ldots, g_k\} = G_2 + \{g_1, \ldots, g_k\} = G$$

are both tilings.

Proof. Define $s = [G : G_1 + G_2]$, so that $s \le k$, and let $x_1, \ldots, x_s$ be a complete set of coset representatives of $G_1 + G_2$ in G. It suffices to find a common fundamental domain E of G_1 and G_2 in $G_1 + G_2$ as, then, $E + \{x_1, \ldots, x_s\}$ is a common fundamental domain of G_1 and G_2 in G. Notice that $[G_1 + G_2 : G_1] = [G_1 + G_2 : G_2] = k/s$. Write $r = k/s$.

Case 1: $G_1 \cap G_2 = \{0\}$.

We enumerate $G_i = \left\{g_j^i : \ j = 1, \ldots, r\right\}$ for $i = 1, 2$, and let $F = \left\{g_j^1 + g_j^2 : \ j = 1, \ldots, r\right\}$. The elements of F are pairwise inequivalent mod G_1 and mod G_2 and $G_i + F = G_1 + G_2$, for $i = 1, 2$, so F is a complete set of coset representatives of G_1 and G_2 in $G_1 + G_2$.

Case 2: $G_1 \cap G_2 \neq \{0\}$.

Define then $\Gamma = (G_1 + G_2)/(G_1 \cap G_2)$ and $\Gamma_i = G_i/(G_1 \cap G_2)$, for $i = 1, 2$. By the previous case (we have $\Gamma_1 \cap \Gamma_2 = \{0\}$) we can find a complete set of coset representatives F for Γ_1, Γ_2 in Γ. Then F is also a complete set of coset representatives for G_1, G_2 in $G_1 + G_2$.

□

The proof of Theorem 1 follows.

The closed subgroups of $\mathbb{R}^d$ are, up to a non-singular linear transformation, of the form

$$\mathbb{Z}^m \times \mathbb{R}^n \tag{4}$$

where $m + n = d$, where $m = 0, 1, \ldots, d$ [HR12, Theorem 9.11]. Thus we may assume that $\overline{L + M} = \mathbb{Z}^m \times \mathbb{R}^n$ for some such decomposition $d = m + n$. Next we observe that it is enough to find a bounded common fundamental domain Ω' of L, M in $\mathbb{Z}^m \times \mathbb{R}^n$ which is measurable in $\mathbb{Z}^m \times \mathbb{R}^n$. Then we can take $\Omega = \Omega' + [0,1]^m \times \{0\}^n$. From the boundedness of Ω' we get that Ω will be a finite union of polytopes if Ω' is such a set on each slice $\{k\} \times \mathbb{R}^n$, $k \in \mathbb{Z}^m$.

3.1. **Case** $m = 0$**.** This is Theorem 2: $L + M$ is dense in $\mathbb{R}^d$ and they have the same volume, so there is a bounded common tile for them which is a finite union of polytopes.

3.2. **Case** $m = d$**.** We have $L + M = \mathbb{Z}^d$. The lattices have the same volume, hence the same index in $\mathbb{Z}^d$. By Lemma 2 there exists a finite set $F \subseteq \mathbb{Z}^d$ such that $L + F = M + F = \mathbb{Z}^d$ are tilings. Again, a finite set is considered as a finite union of polytopes.

3.3. **General case:** $0 < m < d$**.** Define the discrete subgroups of $\{0\}^m \times \mathbb{R}^n$

$$L_2 = (\{0\}^m \times \mathbb{R}^n) \cap L \quad \text{and} \quad M_2 = (\{0\}^m \times \mathbb{R}^n) \cap M.$$

By Lemma 1 the groups L_2, M_2 have rank n. It is clear that

$$\overline{L_2 + M_2} = \{0\}^m \times \mathbb{R}^n. \tag{5}$$

Write

$$L = L_1 \oplus L_2, \quad M = M_1 \oplus M_2,$$

where L_1, M_1 are discrete subgroups of $\mathbb{Z}^m \times \mathbb{R}^n$ of rank m. Since the sums are direct it follows that the points of L_1 are all different mod $\{0\}^m \times \mathbb{R}^n$ and so are all points of M_1. Therefore we have the group indices

$$[\mathbb{Z}^m \times \mathbb{R}^n : L_1 \oplus \{0\}^m \times \mathbb{R}^n] = \det L_1 \quad \text{and} \quad [\mathbb{Z}^m \times \mathbb{R}^n : M_1 \oplus \{0\}^m \times \mathbb{R}^n] = \det M_1. \tag{6}$$

We also have that

$$L_1 + M_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n, \tag{7}$$

since the left hand side is a subgroup of the right hand side. If it were a proper subgroup then we could not have $\overline{L + M} = \mathbb{Z}^m \times \mathbb{R}^n$.

Abusing notation we can write $L = L\mathbb{Z}^d$, $M = M\mathbb{Z}^d$, where L, M are $d \times d$ non-singular matrices. The columns of these matrices can be any basis of the lattices so we choose the first m to be a basis of L_1 (resp. M_1) and the last n to be a basis of L_2 (resp. M_2). The matrices L, M are now lower block triangular

$$L = \begin{pmatrix} L_1 & 0 \\ * & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_1 & 0 \\ * & M_2 \end{pmatrix},$$

where the $m \times m$ matrices L_1, M_1 have integer entries since these entries represent the first m coordinates of a basis of $L_1, M_1 \subseteq \mathbb{Z}^m \times \mathbb{R}^n$. It follows that

$$\det L = \det L_1 \cdot \det L_2 \quad \text{and} \quad \det M = \det M_1 \cdot \det M_2.$$

Since $\det L = \det M$ and $\det L_1, \det M_1 \in \mathbb{Z}$ we have that

$$\frac{\det L_2}{\det M_2} = \frac{\det M_1}{\det L_1} \in \mathbb{Q}. \tag{8}$$

All determinants in (8) are non-zero and can be assumed positive.

3.3.1. *A simple case. Not strictly necessary for the rest, but easier.* If, besides $\det L = \det M$, we also have that the ratios in (8) are equal to 1, so that $\det L_i = \det M_i$, $i = 1, 2$, then, using the case $m = 0$ above, we can find a bounded common tile E' of L_2 and M_2 in $\{0\}^m \times \mathbb{R}^n$:

$$L_2 \oplus E' = M_2 \oplus E' = \{0\}^m \times \mathbb{R}^n. \tag{9}$$

From (6) the groups $L_1 \oplus \{0\}^m \times \mathbb{R}^n$ and $M_1 \oplus \{0\}^m \times \mathbb{R}^n$ have the same finite index $\det L_1 = \det M_1$ in the group $\mathbb{Z}^m \times \mathbb{R}^n$, hence, from Lemma 2 we can find a common, finite tile F of them in $\mathbb{Z}^m \times \mathbb{R}^n$:

$$L_1 \oplus \{0\}^m \times \mathbb{R}^n \oplus F = M_1 \oplus \{0\}^m \times \mathbb{R}^n \oplus F = \mathbb{Z}^m \times \mathbb{R}^n. \tag{10}$$

From (9) and (10) we obtain

$$L_1 \oplus (L_2 \oplus E') \oplus F = M_1 \oplus (M_2 \oplus E') \oplus F = \mathbb{Z}^m \times \mathbb{R}^n,$$

so with $E = F \oplus E'$ we obtain the tilings

$$L \oplus E = M \oplus E = \mathbb{Z}^m \times \mathbb{R}^n.$$

This concludes the proof of this simple case.

In general the ratios in (8) are not necessarily 1. Take now L_2' and M_2' to be superlattices of L_2 and M_2 in $\{0\}^m \times \mathbb{R}^n$ such that

$$[L_2' : L_2] = \det M_1 \quad \text{and} \quad [M_2' : M_2] = \det L_1. \tag{11}$$

It follows from (8) that

$$\det L_2' = \frac{\det L_2}{\det M_1} = \frac{\det M_2}{\det L_1} = \det M_2'. \tag{12}$$

Since, because of (5), L_2' and M_2' also have a dense sum in $\{0\}^m \times \mathbb{R}^n$ it follows from the case $m = 0$ in this proof that there is a bounded common tile E' of L_2' and M_2' in $\{0\}^m \times \mathbb{R}^n$. E' is a finite union of polytopes. We have

$$|E'| = \det L_2' = \det M_2'.$$

Let the finite sets $J_2 \subseteq L_2'$ and $K_2 \subseteq M_2'$ be such that

$$L_2' = L_2 \oplus J_2 \ \text{ and } \ M_2' = M_2 \oplus K_2$$

are both tilings, so that it follows from (11) that $|J_2| = \det M_1$ and $|K_2| = \det L_1$.

Since $L_1 + M_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n$ from (7) we can find finite sets $J_1 \subseteq L_1$ of size $|J_1| = \det M_1$ and $K_1 \subseteq M_1$ of size $|K_1| = \det L_1$ (these sizes follow from (6)) such that

$$K_1 \oplus L_1 \oplus \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n \tag{13}$$

and

$$J_1 \oplus M_1 \oplus \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n. \tag{14}$$

Since $|J_1| = |J_2|$ and $|K_1| = |K_2|$ we can find bijections

$$\phi : K_1 \to K_2, \quad \psi : J_1 \to J_2.$$

Define the sum

$$E = \left\{x + y + \phi(x) + \psi(y) : x \in K_1,\ y \in J_1\right\} \ \oplus\ E' \ \subseteq\ \mathbb{Z}^m \times \mathbb{R}^n. \tag{15}$$

E is clearly a finite union of polytopes on each slice $\{k\} \times \mathbb{R}^n$, $k \in \mathbb{Z}^m$, since E' is a finite union of polytopes in $\mathbb{R}^n$. The fact that the sum in (15) is direct is a byproduct of the proof that follows in which we show that the set E is a common tile for the lattices L and M.

For reasons of symmetry we need only verify that

$$L \oplus E = L_1 \oplus L_2 \oplus E = \mathbb{Z}^m \times \mathbb{R}^n$$

is a tiling.

We first show that this is a packing. Let $\ell = \ell_1 + \ell_2$ and $\widetilde{\ell} = \widetilde{\ell_1} + \widetilde{\ell_2}$ be elements of $L = L_1 \oplus L_2$ and assume that the two translates $\ell + E$ and $\widetilde{\ell} + E$ overlap on positive measure. This means that there are

$$x, \widetilde{x} \in K_1, \quad y, \widetilde{y} \in J_1$$

such that

$$\ell_1 + \ell_2 + x + y + \phi(x) + \psi(y) + E' \ \text{ and } \ \widetilde{\ell_1} + \widetilde{\ell_2} + \widetilde{x} + \widetilde{y} + \phi(\widetilde{x}) + \psi(\widetilde{y}) + E'$$

overlap on positive measure. These can be rewritten as

$$\underbrace{\ell_1 + y}_{\in L_1} + x + \underbrace{\ell_2 + \phi(x) + \psi(y) + E'}_{\subseteq \{0\}^m \times \mathbb{R}^n}$$

and

$$\underbrace{\widetilde{\ell_1} + \widetilde{y}}_{\in L_1} + \widetilde{x} + \underbrace{\widetilde{\ell_2} + \phi(\widetilde{x}) + \psi(\widetilde{y}) + E'}_{\subseteq \{0\}^m \times \mathbb{R}^n}.$$

Since $x, \widetilde{x} \in K_1$ we get, because of tiling condition (13), that

$$\ell_1 + y = \widetilde{\ell_1} + \widetilde{y} \quad \text{and} \quad x = \widetilde{x}, \tag{16}$$

which of course implies that $\phi(x) = \phi(\widetilde{x})$. Thus the translates

$$\ell_2 + \psi(y) + E' \quad \text{and} \quad \widetilde{\ell_2} + \widetilde{\psi}(y) + E'$$

overlap on positive measure. But $\ell_2 + \psi(y), \widetilde{\ell_2} + \widetilde{\psi}(\widetilde{y}) \in L_2'$ and $L_2' \oplus E' = L_2 \oplus J_2 \oplus E'$ are tilings, so we get $\ell_2 = \widetilde{\ell_2}$ and $\psi(y) = \psi(\widetilde{y})$. The last equation implies $y = \widetilde{y}$ since ψ is a bijection. Finally from (16) we obtain $\ell_1 = \widetilde{\ell_1}$.

We have shown that the translates of E' that participate in the definition (15) of the tile E are all non-overlapping and, therefore,

$$\begin{aligned} |E| &= |E'| \cdot |K_1| \cdot |J_1| \\ &= \det L_2' \cdot \det L_1 \cdot \det M_1 \\ &= \det L_1 \cdot \det L_2' \cdot |J_2| \\ &= \det L_1 \cdot \det L_2 \\ &= \det L. \end{aligned} \tag{17}$$

We also showed that $L + E$ is a packing. Since the arrangement $L + E$ is periodic it follows that $L \oplus E = \mathbb{Z}^m \times \mathbb{R}^n$ is a tiling. By symmetry so is $M \oplus E = \mathbb{Z}^m \times \mathbb{R}^n$.

The final bounded common tile Ω of L and M in $\mathbb{R}^d$ is then given by

$$\Omega = E + [0, 1)^m \times \{0\}^n.$$

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