$f \in L^1(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$ $\qquad f*g(x) = \int \underbrace{g(y)\, f(x-y)}\, dy$

$$|g(y)| \leq \|g\|_\infty \qquad \underline{\|f*g\|_\infty \leq \|f\|_1 \|g\|_\infty}$$

<u>$f*g$ ομοιόμορφα συνεχής στο $\mathbb{R}$</u>

$$|f*g(x+h) - f*g(x)| \leq \left| \int g(y) \left(f(\overline{x-y}+h) - f(\overline{x-y}) \right) dy \right|$$

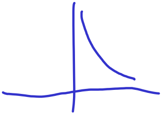
$$= \left| g * \left(f(\cdot+h) - f(\cdot) \right)(x) \right| \leq \|g\|_\infty \|f(\cdot+h) - f(\cdot)\|_1$$

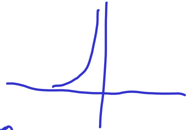
δεν εξ. από x

<u>Μεταφορά συνεχής στο L^1</u>: $\quad \|f(\cdot+h) - f(\cdot)\|_1 \xrightarrow{h\to 0} 0$

$$f, g \in L^1(\mathbb{R}) \not\Longrightarrow f * g \text{ συνεχής}$$

$$f(x) = \frac{1}{\sqrt{x}} \mathbb{1}(x > 0) \qquad g(x) = \frac{1}{\sqrt{-x}} \mathbb{1}(x < 0)$$





$$f * g(0) = \int f(y)\, f(y)\, dy = \int_0^\infty \frac{1}{x}\, dx = +\infty$$

$$f \in L^p(\mathbb{R}), \quad g \in L^q(\mathbb{R}), \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$\Longrightarrow \quad f * g \in L^\infty(\mathbb{R})$$

$$|f*g(x)| = \Big| \int \overbrace{f(y)}^{L^p} \, \overbrace{g(x-y)}^{L^q} \, dy \Big|$$

Hölder

$$\leq \|f\|_p \cdot \|g\|_q$$

Άρα $\quad \|f*g\|_\infty \leq \|f\|_p \|g\|_q$

$$\boxed{\|f * g\|_p \leq \|f\|_1 \|g\|_p} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$|f * g(x)| \leq \int |g(t)| \cdot |f(x-t)| \, dt = \int \underbrace{|g(t)| \cdot |f(x-t)|^{\frac{1}{p}}} \, \underbrace{|f(x-t)|^{\frac{1}{q}}} \, dt$$

$$\overset{\text{Hölder}}{\leq} \left(\int |g(t)|^p \, |f(x-t)| \, dt \right)^{1/p} \left(\int |f(x-t)| \, dt \right)^{1/q}$$

$$|f * g(x)|^p \leq \int |g(t)|^p \cdot |f(x-t)| \, dt \;\; \|f\|_1^{p/q}$$

$$\int |f * g(x)|^p dx \leq \|f\|_1^{p/q} \int |g(t)|^p \underbrace{\int |f(x-t)| \, dx} \, dt = \|f\|_1^{1+\frac{p}{q}} \, \|g\|_p^p$$

$$\|f * g\|_p = \left(\int |f * g(x)|^p dx \right)^{1/p} \leq \|f\|_1 \, \|g\|_p$$

$g \in L^p(\mathbb{R})$, $f \in L^q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$ $\Rightarrow$ $f * g$ ομοιόμ. συνεχώς

$q < \infty$

$$\|f * g\|_\infty \leq \|g\|_p \|f\|_q$$

$$|g * f(x+h) - g * f(x)| = \left| \int \overbrace{g(y)}^{p} \overbrace{\big(f(x-y+h) - f(x-y) \big)}^{q} \, dy \right|$$

$$= \left| g * \big(f(\cdot + h) - f(\cdot) \big) \right| \leq \|g\|_p \, \|f(\cdot + h) - f(\cdot)\|_q$$

Λόγω του $\lim_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_q = 0$

$f \in C^1(\mathbb{R})$, f' φραγμένη, $g \in L^1(\mathbb{R}) \Longrightarrow (f*g)' = f'*g$

$|f'| \leq M$

$$\left| \frac{1}{h} \left(f*g(x+h) - f*g(x) \right) - f'*g(x) \right| =$$

$$= \Big| \underbrace{\int g(y) \left[\frac{f(x-y+h) - f(x-y)}{h} - f'(x-y) \right]}_{\to 0 \text{ για } h \to 0} dy \Big|$$

$$= \Big| \int \underbrace{g(y) \left[f'(\xi_{x,y,h}) - f'(x-y) \right]}_{\leq 2M |g(y)| \in L^1(\mathbb{R})} dy \Big|$$

από ΘΚΣ το $\int \xrightarrow{h \to 0} 0$

Ανισότητα Young : $\|f*g\|_r \leq \|f\|_p \|g\|_q$

αν $$1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$$

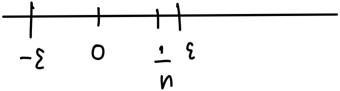
Καλός πυρήνας:

$$k_n \in L^1(\mathbb{R})$$

- $\displaystyle\int_{\mathbb{R}} k_n = 1$
- $\|k_n\|_1 \leq M$
- $\forall \epsilon > 0$: $\displaystyle\int_{|x|>\epsilon} |k_n(x)|\,dx \xrightarrow{n\to\infty} 0$

Παράδειγμα

$$k_n(x) = n\,\mathbb{1}_{[0,1/n]}$$



$$\frac{n}{2}\,\mathbb{1}_{[-1/n,\,1/n]}$$

$\underline{\Theta}$ Αν k_n καλός πυρήνας, $f \in C_0(\mathbb{R})$ [συνεχής που μηδενί- ζεται έξω από διάστημα]

τότε $f * k_n \longrightarrow f$ ομοιόμορφα στο $\mathbb{R}$

(δηλ. $\| f * k_n - f \|_\infty \xrightarrow{n\to\infty} 0$)

$$|f * k_n(x) - f(x)| = \left| \int f(x-y) k_n(y)\, dy - \int f(x) k_n(y)\, dy \right|$$

$$= \left| \int \big(f(x-y) - f(x)\big) k_n(y)\, dy \right| = \left| \int_{|y|<\varepsilon} \cdots + \int_{|y|\geq\varepsilon} \cdots \right|$$

Έστω $\delta > 0$:

$$\int_{|y| \leq \epsilon} |f(x-y) - f(x)| \, |k_n(y)| \, dy \leq \delta \int_{|y|\leq \epsilon} |k_n| \leq \delta \int_{\mathbb{R}} |k_n|$$

$$\leq \delta \cdot M \quad \underline{\underline{\forall n}}$$

αν ϵ αρκ. μικρό τότε

$$|f(x-y) - f(x)| \leq \delta$$

$|f| \leq M_1$

$$\int_{|y|>\epsilon} |f(x-y) - f(x)| \cdot |k_n(y)| \, dy \leq 2M_1 \int_{|y|>\epsilon} |k_n(y)| \, dy$$

$$\Big\downarrow \; n \to \infty$$

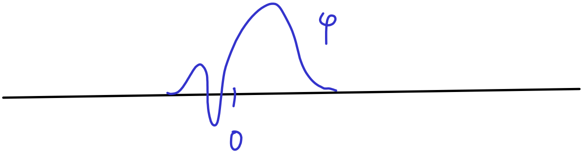
$$0$$

$$f * k_n \longrightarrow f$$

Υπάρχει καλός πυρήνας $k_n \in C^\infty$.

$\varphi = 0$ εκτός του $[-1,1]$

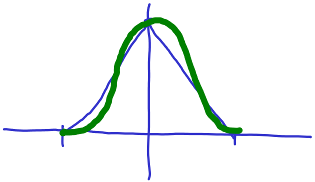
$\int \varphi = 1$

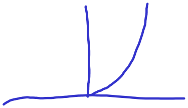


$\int k_n = 1$ $\qquad \int |k_n| = \int |\varphi|$

$$k_n(x) = n\,\varphi(nx)$$

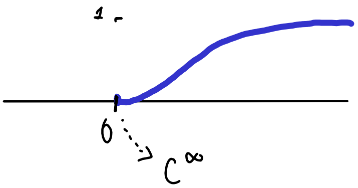
Αν $\varphi \in C^\infty(\mathbb{R}) \Longrightarrow k_n \in C^\infty$





$$y = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$



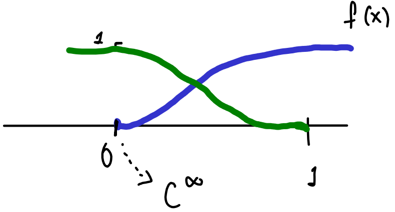
$$\left[f^{(n)}(x) = \frac{P_n(x)}{x^{2n}} \cdot f(x) \right.$$

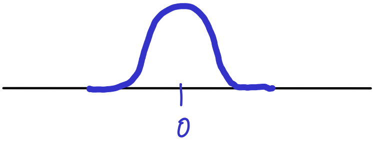
, P_n πολυώνυμο βαθμού $n-1$

$$x \to 0^+ \qquad \frac{1}{x^{2n}} \nearrow +\infty \qquad f(x) \to 0$$

$$f^{(n+1)}(x) = \frac{\left(P_n'(x) \cdot f(x) + P_n(x) \cdot \frac{p}{x^2} f(x)\right) x^2 - 2n\,x\,P_n(x)\,f(x)}{x^{2n+2}} = \frac{f(x)}{x^{2(n+1)}} \left[x^2 \cdot P_n'(x) + p \cdot P_n(x) - 2nx\,P_n(x) \right]$$

$$f'(x) = \frac{p}{x^2} f(x)$$





$f(1-x)$

$$g(x) = f(x)\,f(1-x)$$

$$g\left(x-\frac{1}{2}\right)$$

$$h(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x)\,dx}$$

$f \in C_0(\mathbb{R})$

$$\underbrace{f * k_n}_{C^\infty} \longrightarrow f \quad \text{on} \ \mu.$$

C^∞

Θ. Weierstrass $f : [a, b] \to \mathbb{R}$ συνεχής , $\varepsilon > 0$ τότε

υπάρχει πολυώνυμο $P(x)$ τ.ώ. $\|f - P\|_{L^\infty([a,b])} \leq \varepsilon$.

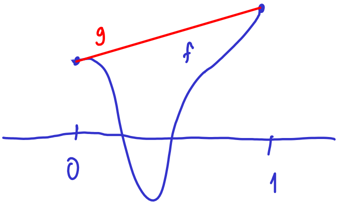
Υπάρχει ακολουθία πολυωνύμων $P_n(x) \longrightarrow f(x)$

ομοιόμορφα για $x \in [a, b]$.

- Αρκεί να το δείξουμε για $[a,b] = [0,1]$.

$$f\left(\frac{x-a}{b-a}\right)$$

Υποθέτουμε $f(0) = f(1) = 0$



$p \sim f - g$

$p + g \sim f$

$$L_n(x) = \int_{-\infty}^{\infty} f(x+t)\, K_n(t)\, dt = \int_{-x}^{1-x} f(x+t)\, K_n(t)\, dt$$

όπου $$K_n(t) = \begin{cases} c_n(1-t^2)^n & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

c_n τη διαλέγουμε ώστε $\int_{-\infty}^{\infty} K_n = 1$

L_n πολυώνυμα – Αλλαγή μεταβλητής $u = x+t$

$$L_n(x) = \int_0^1 f(u)\, K_n(u-x)\, du = c_n \int_0^1 f(u)\, \underbrace{\left(1-(u-x)^2\right)^n}_{\sum_{j=0}^{2n} q_j(u) x^j} du$$

$0 \leq x \leq 1$

$$= \sum_{j=0}^{2n} c_n \left(\int_0^1 f(u)\, q_j(u)\, du \right) x^j$$

$$K_n(t) = \begin{cases} c_n(1-t^2)^n & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

$$\int_{|t|>\varepsilon} K_n(t)\,dt \xrightarrow{n} 0$$

$$= 2\int_{\varepsilon}^{1} \underline{c_n}\,(1-t^2)^n\,dt$$