$$L^p(A) = \{ f : A \to \mathbb{C} \text{, μετρήσιμη, } \int_A |f|^p < \infty \} \qquad 1 \leq p \leq \infty$$

$A \subseteq \mathbb{R}^d$

$$L^\infty(A) = \{ f : A \to \mathbb{C} \text{, μετρ., } \operatorname*{ess\,sup}_A |f| < \infty \}$$

Ταυτίζουμε f, g αν $m\{f \neq g\} = 0$.

$$f \sim g \overset{\text{ορ.}}{\iff} m\{f \neq g\} = 0$$

$$\|f\|_p = \left(\int_A |f|^p \right)^{1/p} \quad 1 \leq p < \infty, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in A} |f(x)|$$

$d(f,g) = \|f-g\|_p$ μετρική

Θεωρήματα πυκνότητας

$C_0(\mathbb{R})$ = συνεχείς $f: \mathbb{R} \to \mathbb{C}$, που μηδενίζονται έξω από κάποιο διάστημα

$C_0(\mathbb{R}) \subseteq L^p(\mathbb{R})$, $1 \leq p \leq \infty$

Θ $C_0(\mathbb{R})$ πυκνός στον $L^p(\mathbb{R})$, $\underline{1 \leq p < \infty}$.

Αν $f \in L^p(\mathbb{R})$ τότε $\exists\, f_n \in C_0(\mathbb{R})$ τ.ώ. $f_n \to f$ δηλ.

$$\|f_n - f\|_p \xrightarrow{n} 0$$

(και στο $\mathbb{R}^d$)

Λάθος στο $L^\infty(\mathbb{R}^d)$, ακόμη και για το $C(\mathbb{R})$.

$$f_n \xrightarrow{L^\infty} f : \quad \|f_n - f\|_\infty \xrightarrow{n} 0$$

$$\mu_n = \underset{\mathbb{R}}{\text{esssup}} \, |f_n - f| \xrightarrow{n} 0$$

$$\text{σ}\forall x \quad |f_n(x) - f(x)| \to 0$$

$x \notin E^c$

$m(E) = 0$

$$\wedge\!|$$

μ_n <--- δεν εξαρτάται από το x

ομοιόμορφη σύγκλιση

$f = \mathbb{1}_{[0,1]}$ δεν προσεγγίζεται από συνεχείς συναρτήσεις.

Μετασχ. Fourier για $f \in L^1(\mathbb{R})$:

$$\hat{f} : \mathbb{R} \to \mathbb{C}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} f(x)\, dx$$

<u>Θ</u> $\hat{f}$ ομοιόμ. συνεχής στο $\mathbb{R}$.

$$\left|\hat{f}(\xi+h) - \hat{f}(\xi)\right| = \left| \int f(x)\, e^{-2\pi i (\xi+h) x}\, dx - \int f(x)\, e^{-2\pi i \xi x}\, dx \right|$$

$$= \left| \int f(x)\, e^{-2\pi i \xi x} \left(e^{-2\pi i h x} - 1 \right) dx \right| \leq \int \underbrace{|f(x)| \cdot \overbrace{\left| e^{-2\pi i h x} - 1 \right|}^{\leq 2}}_{\xrightarrow{h \to 0}\, 0} dx$$

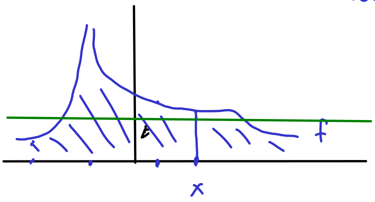
ΘΚΣ $\to$ τείνει στο 0.

$\underline{\Theta}$ (Αρχή Cavalieri) $f \geq 0$: $\displaystyle\int f(x)\,dx = \int \underbrace{m\{f \geq t\}}\,dt$

$\underline{\text{Πόρ}}$: $f, g \geq 0$, με $F_f = F_g \Rightarrow \int f = \int g$

$F_f(t)$ συνάρτηση κατανομής της f

$$\int f(x)\,dx = \int_{-\infty}^{\infty} \int_0^{f(x)} 1\,dt\,dx$$

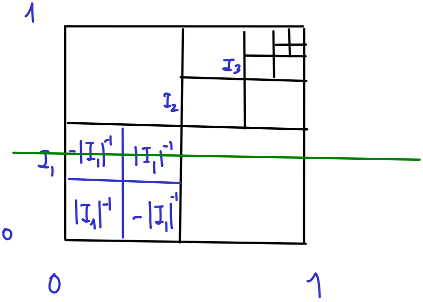


$$= \int_0^{\infty} \int_{f(x) \geq t} 1\,dx\,dt$$

$$= \int_0^{\infty} m\{x : f(x) \geq t\}\,dt$$

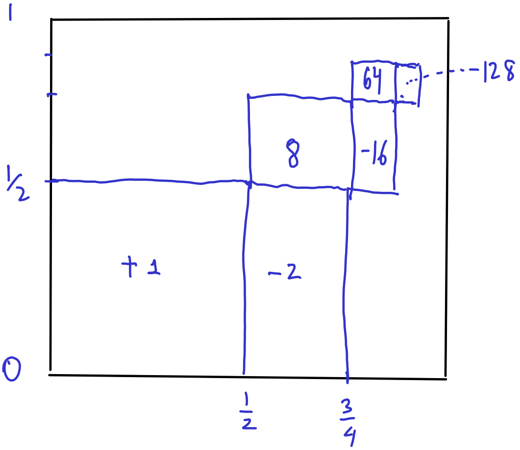
Fubini

$\int_{\mathbb{R}^2} |f| < \infty$: $\int_{\mathbb{R}^2} f = \int \int f(x,y)\, dy\, dx = \int \int f(x,y)\, dx\, dy$



$\int |f| = +\infty$

Επαναλαμβανόμενα ολοκληρώματα διαφορετικά



$$\iint f(x,y)\,dx\,dy = 0$$

$$\iint f(x,y)\,dy\,dx = +\infty$$

Ανισ. Hölder : $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$: $\int |fg| \le \|f\|_p \cdot \|g\|_q$

Cauchy-Schwarz : $p = q = 2$ $\quad \int |fg| \le \|f\|_2 \|g\|_2 \quad \left(|\langle f, g \rangle| \le \|f\|_2 \|g\|_2 \right)$

$$\langle f, g \rangle = \int f \bar{g}$$

Hölder $\to$ Minkowski: $\|f + g\|_p \le \|f\|_p + \|g\|_p \quad (1 \le p \le \infty)$

$$\Downarrow$$

$$d(f, g) = \|f - g\|_p$$

$$d(f, g) \le d(f, h) + d(h, g) \Leftarrow \|f - g\|_p \le \|f - h\|_p + \|h - g\|_p$$

Δεν ισχύει για $p < 1$.

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$ (Minkowski) $\quad \frac{1}{p}+\frac{1}{q}=1 \Rightarrow \frac{1-p}{p} = \frac{-1}{q}$

$\Rightarrow q = \frac{p}{p-1}$

$p=\infty$: $\quad \operatorname{esssup}|f+g| \leq \operatorname{esssup}|f| + \operatorname{esssup}|g|$

$p<\infty$:

$$\|f+g\|_p^p = \int |f+g|^{p-1}|f+g| \overset{\text{Tp.av.}}{\leq} \int |f+g|^{p-1}\cdot|f| + \int |f+g|^{p-1}\cdot|g|$$

$$\overset{\text{Hölder}}{\leq} \||f+g|^{p-1}\|_{\frac{p}{p-1}} \|f\|_p + \||f+g|^{p-1}\|_{\frac{p}{p-1}}\cdot\|g\|_p$$

$$= \left(\int |f+g|^p\right)^{\frac{p-1}{p}} \left(\|f\|_p + \|g\|_p\right) = \|f+g\|_p^{p-1}\left(\|f\|_p + \|g\|_p\right)$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \qquad \|x\|_\infty = \max_{j=1..n} |x_j|$$

Hölder: $\displaystyle \left| \sum_{j=1}^{n} x_j y_j \right| \leq \|x\|_p \, \|y\|_q \,, \quad \frac{1}{p} + \frac{1}{q} = 1$

<u>Minkowski</u> $\qquad \|x+y\|_p \leq \|x\|_p + \|y\|_p \qquad 1 \leq p \leq \infty$

Minkowski δεν ισχύει για $\underline{p < 1}$

$$f = \mathbb{1}_{[0,1/2)} \qquad g = \mathbb{1}_{[1/2,1]}$$

$$\|f+g\|_p = \|\mathbb{1}_{[0,1]}\|_p = \left(\int_{[0,1]} 1^p\right)^{1/p} = 1$$

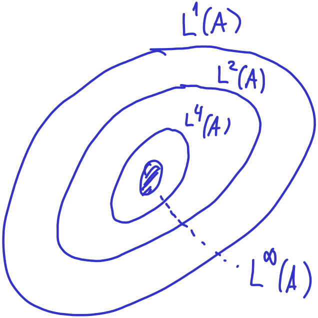
$$\|f\|_p = \left(\int_0^{1/2} 1^p\right)^{1/p} = \frac{1}{2^{1/p}} = \|g\|_p$$

$$1 \leq \frac{2}{2^{1/p}} = 2^{1-\frac{1}{p}} < 1$$ άτοπο, άρα δεν ισχύει η Minkowski

$\underline{\Theta}$ $A \subseteq \mathbb{R}^d$, $m(A) < \infty$: $1 \leq p_1 \leq p_2 \leq \infty \Rightarrow \underline{L^{p_2}(A) \subseteq L^{p_1}(A)}$

$$\int_A |f|^{p_2} < \infty \Longrightarrow \int_A |f|^{p_1} < \infty$$

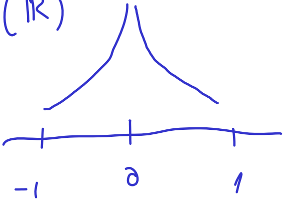
$$\|f\|_{p_2} \geq \|f\|_{p_1}$$



$$\forall \quad p_1 \neq p_2 \qquad L^{p_1}(\mathbb{R}) \not\subseteq L^{p_2}(\mathbb{R})$$

- $p_1 < p_2$ θέλω $f \in L^{p_1}(\mathbb{R}) \setminus L^{p_2}(\mathbb{R})$

$$f(x) = \frac{1}{|x|^{\alpha}} \mathbb{1}\left(|x| \leq 1 \right)$$



$$\alpha p_1 < 1 \implies \alpha p_1 = \frac{p_1}{p_2} < 1$$

$$\alpha p_2 \geq 1 \implies \alpha = \frac{1}{p_2}$$

$\underline{p_1 > p_2}$ δείξω $f \in L^{p_1}(\mathbb{R}) \setminus L^{p_2}(\mathbb{R})$

$$f(x) = \mathbb{1}(x \geq 1) \frac{1}{x^\alpha}$$

$$\alpha p_1 > 1 \quad \Rightarrow \quad \alpha p_1 = \frac{p_1}{p_2} > 1$$

$$\alpha p_2 \leq 1 \quad \Rightarrow \quad \alpha = \frac{1}{p_2}$$

$$x = (x_1, x_2, \dots) \qquad x_n \in \mathbb{C}$$

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \qquad \|x\|_\infty = \sup_n |x_n|$$

$$\ell^p(\mathbb{N}) = \{ x = (x_1, x_2, \dots) : \|x\|_p < \infty \}$$

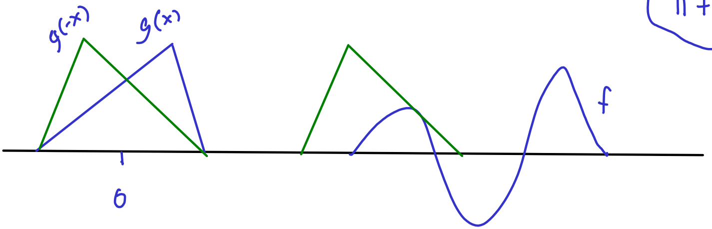
$$p_1 < p_2 \Rightarrow \ell^{p_1}(\mathbb{N}) \subseteq \ell^{p_2}(\mathbb{N})$$

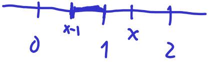
$$\|x\|_{p_1} < \infty \Rightarrow \|x\|_{p_2} < \infty$$

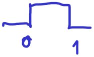
$$\sum_n |x_n|^{p_1} < \infty \Rightarrow \sum_n |x_n|^{p_2} < \infty$$

Συνέλιξη: $\underbrace{f*g}(x) = \int_{\mathbb{R}} \underbrace{f(y)}\, g(\underbrace{x-y})\,dy$

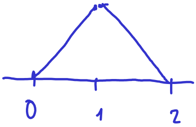
$\|f*g\|_1 \le \|f\|_1 \|g\|_1$







$f(x) = g(x) = \mathbb{1}_{[0,1]}$



$f*g(x) = \int_0^1 \cancel{\mathbb{1}_{[0,1]}(y)}\, \mathbb{1}_{[0,1]}(x-y)\,dy = \begin{cases} x & 0\le x\le 1 \\ 2-x & 1\le x\le 2 \end{cases}$

$0 \le x-y \le 1 \iff y \le x \le 1+y \iff \overset{0}{x-1} \le y \le \overset{1}{x}$