Θεώρημα κυριαρχημένης σύγκλισης

$f_n \to f$ σ.π. $\qquad |f_n| \leq g, \quad \int g < \infty$

Τότε $\qquad \int f_n \to \int f$.

$\int |f_n| \leq \int g < \infty$

$$f_n \to f \implies \left.\begin{array}{l} |f_n| \to |f| \\ |f_n| \leq g \end{array}\right\} \; |f| \leq g \quad \text{άρα} \quad \int |f| < \infty$$

$$|f_n| \le g \implies g \pm f_n \ge 0$$

$$f_n \to f$$

$$\int \liminf (g + f_n) \le \liminf \int g + f_n$$

$$\cancel{\int g} + \liminf f_n \le \cancel{\int g} + \liminf \int f_n$$

$$\underbrace{\int \liminf f_n}_{\int f} \le \liminf \int f_n \le \limsup \int f_n \le \underbrace{\int \limsup f_n}_{\int f}$$

$$\int \liminf g - f_n \le \liminf \int g - f_n \implies \cancel{\int g} - \limsup f_n \le \cancel{\int g} - \limsup \int f_n$$

$$\implies \underline{\limsup \int f_n \le \int \limsup f_n}$$

$$f_n(x) = \mathbb{1}_{[n,\, n+1]}(x) \longrightarrow 0 = f$$

$$\int f_n = 1 \qquad\qquad \int f = 0$$

$$\text{Δεν υπάρχει } \underbrace{g \geq \mathbb{1}_{[n,\, n+1]}}_{g(x) \geq 1 \quad (x \geq 0)} \ \forall n \text{ και } \int g < \infty$$

$\underline{\Theta}$ f_n τ.ώ. $\boxed{\sum_{n=1}^{\infty} \int |f_n| < \infty}$. Τότε

η $\sum_{n=1}^{\infty} f_n(x)$ συγκλίνει σ.π. σε μια συνάρτηση f και ← ολοκληρώσιμη

$$\int f = \sum_{n=1}^{\infty} \int f_n \quad \left(\text{ή} \quad \int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n \right)$$

$g = \sum_{n=1}^{\infty} |f_n|$ $\qquad g_N = \sum_{n=1}^{N} |f_n|$ $\qquad g_N \nearrow g$ $\qquad$ ΘΜΤ

$$\int g = \lim_N \int g_N = \lim_N \sum_{n=1}^{N} \int |f_n| = \sum_{n=1}^{\infty} \int |f_n| < \infty \quad \left(\text{άρα } g \text{ ολοκλ.} \right)$$

Άρα $g < \infty$ σ.π. Άρα σ.π. η $\sum f_n$ συγκλίνει, έστω $f = \sum f_n$

$$h_N = \sum_{n=1}^{N} f_n \qquad h_N \to f = \sum_{1}^{\infty} f_n$$

$$|h_N| \leqslant \sum_{1}^{N} |f_n| \leqslant \sum_{1}^{\infty} |f_n| = g$$

$$\Theta K \Sigma \implies$$

$$\begin{tikzcd}
\int h_N \arrow[r, "N"] \arrow[d, equal] & \int f \\
\int \sum_{1}^{N} f_n = \sum_{1}^{N} \int f_n \arrow[ur, dotted, "N"'] &
\end{tikzcd}$$

$$\sum_{1}^{\infty} \int f_n = \int f$$

1. (σελ. 24)

f ολοκλ. $\Rightarrow$ $f \neq \pm\infty$ σ.π.

$\int |f| < \infty$

Αν $|f| = +\infty$ στο E, $m(E) > 0$, τότε

επειδή $|f| \geq \mathbb{1}_E |f|$ έχουμε

$$\infty > \int |f| \geq \int \mathbb{1}_E |f| = m(E)(+\infty) = +\infty$$

2. f ολοκλ. $\quad \left|\int f\right| \le \int |f| \;=$ ανν $f \ge 0$ σ.π.

$f \le 0$ σ.π.

$$f = f^+ - f^- \;:\; \int f = \int f^+ - \int f^-$$

$$|f| = f^+ + f^- \;:\; \int |f| = \int f^+ + \int f^-$$

$$-\int |f| \le \int f \le \int |f|$$

$$-\int f^+ - \int f^- \le \int f^+ - \int f^- \le \int f^+ + \int f^-$$

$$\int f^- = 0$$

$f \ge 0$, $\int f = 0$

$\Longrightarrow f = 0$ σ.π.

$\exists\, n \in \mathbb{N} : m\{f > \frac{1}{n}\} > 0$

$m(f > 0) = 0 \leftarrow$ θέλω

$\{f > 0\} \subseteq \bigcup \{f > \frac{1}{n}\}$

Αν $m\{f > \frac{1}{n}\} > 0 \overset{n}{\Longrightarrow} \int f > \frac{1}{n}\, m(f >$

3. f ολοκλ. $E_1, E_2, \dots$ $E = \bigcup_{n=1}^{\infty} E_n$.

(a) Αν E_n ανά δύο ξένα $\Longrightarrow$ $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$

$\sum \int |g_n| < \infty \Longrightarrow$

$\int \sum g_n = \sum \int g_n$

$$\int_E f = \int f \mathbb{1}_E \overset{\text{ανά 2 ξένα}}{=} \int f \sum_{n=1}^{\infty} \mathbb{1}_{E_n} = \int \sum_{n=1}^{\infty} f \mathbb{1}_{E_n} = \sum_{n=1}^{\infty} \int f \mathbb{1}_{E_n} = \sum_{n=1}^{\infty} \int_{E_n} f$$

$$g_n = f \mathbb{1}_{E_n} \Longrightarrow |g_n| = |f| \mathbb{1}_{E_n} \Longrightarrow \int |g_n| = \int_{E_n} |f| \Longrightarrow$$

$$\Longrightarrow \sum \int |g_n| = \underbrace{\sum_{n=1}^{\infty} \int_{E_n} |f|}_{= \int_{\cup E_n} |f|} \leq \int |f|$$

(b) $E_n \subseteq E_{n+1}$: $\int_E f = \lim_n \int_{E_n} f$

$E = \bigcup_{n=1}^{\infty} E_n$

$$\int_{E_n} f = \int f \mathbb{1}_{E_n} \xrightarrow{n} f \mathbb{1}_E$$

$$\longrightarrow \int f \mathbb{1}_E = \int_E f$$

$$|f \mathbb{1}_{E_n}| = |f| \mathbb{1}_{E_n} \leq |f| \mathbb{1}_E \text{ ολοκλ.}$$

$\forall x$: $\mathbb{1}_{E_n}(x) \longrightarrow \mathbb{1}_E(x)$

$x \notin E \Rightarrow \forall n \; x \notin E_n \Rightarrow \mathbb{1}_{E_n}(x) = 0 \; \forall n$

$x \in E \Rightarrow \exists n : x \in E_n, E_{n+1}, \dots \Rightarrow \mathbb{1}_{E_n}(x)$ τελικά 1 $\to 1$

4. f ολοκλ.

$$\int_{-\infty}^{r} f \xrightarrow[r\to+\infty]{} \int f$$

$$\int_{(-\infty,r]} f = \int \underbrace{f\,\mathbb{1}_{(-\infty,r]}}_{f_r}$$

$$|f_r| \leq |f|$$

$$|f_r| = |f|\,\mathbb{1}_{(-\infty,r]} \leq |f|$$

$$f_r \xrightarrow{r\to+\infty} f$$

ΘΚΣ $\int f_r \to \int f$

ΘΚΣ $\underline{f_n} \to f \quad (n\to\infty)$

$$\Rightarrow \int \underline{f_n} \to \int f$$

$$\Downarrow$$

$$f_{\underline{t}} \to f \quad (t\to t_0)$$

$$\Rightarrow \int f_t \to \int f \quad (t\to t_0)$$

$$\varphi(x) \xrightarrow[x\to\xi]{} a$$

$$\Leftrightarrow \forall\, x_n \to \xi$$

$$\varphi(x_n) \to a$$

4'

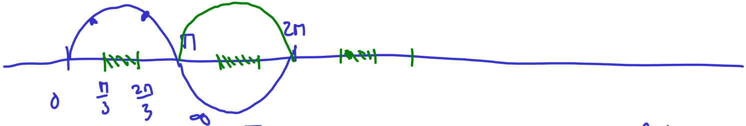
$$E_n = (-\infty, n] \subseteq E_{n+1} \qquad E = \bigcup E_n = \mathbb{R}$$

$$\int_{-\infty}^{n} f = \int_{E_n} f \longrightarrow \int_E f = \int f$$

5) $f(x) = \dfrac{\sin x}{x}$ δεν είναι ολοκληρώσιμη

αλλά $\lim\limits_{r\to+\infty} \int_0^r \dfrac{\sin x}{x}\,dx$ συγκλίνει.

$$\int_1^\infty \frac{dx}{x} = +\infty$$



$$\int_0^\infty \frac{|\sin x|}{x}\,dx = +\infty$$

$$T = \bigcup_{n=0}^{\infty} \left[n\pi + \frac{\pi}{3},\ n\pi + \frac{2\pi}{3}\right]$$

$|\sin x| \geq \dfrac{1}{10}$, $x \in T$

$$\int \frac{|\sin x|}{x} \geq \int_T \frac{|\sin x|}{x} \geq$$

$$\geq \frac{1}{10} \int_T \frac{dx}{x}$$

$$\int_T \frac{dx}{x} = \sum_{n=0}^{\infty} \int_{n\pi+\frac{\pi}{3}}^{n\pi+\frac{2\pi}{3}} \frac{dx}{x} \geq \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi}\,\frac{\pi}{3} = \frac{1}{3}\sum_{0}^{\infty} \frac{1}{n+1}$$

$$= +\infty$$

$$\int_1^r \frac{\sin x}{x}\,dx = \int_1^r \frac{(-\cos x)'}{x}\,dx = \frac{-\cos x}{x}\Big|_1^r - \int_1^r \cos x \,\frac{dx}{x^2}$$

$$= \underbrace{\frac{\cos 1}{1}}_{\text{Stał.}} + \underbrace{\frac{\cos r}{r}}_{\to\,\infty} - \int_1^r \frac{\cos x\,dx}{x^2} \xrightarrow{r\to+\infty} \text{Stał} - \int_1^{\infty} \frac{\cos x\,dx}{x^2}$$

$\underline{6|}$ f_n, g ολοκλ. $\underbrace{f_n \geq g}$ σ.π. $\forall n$. Τότε

$$\int \liminf f_n \leq \liminf \int f_n$$

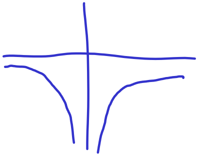
$$f_n - g \geq 0$$

$$\int \liminf f_n - g \leq \liminf \int f_n - g$$

$$\int (\liminf f_n) - g \leq \liminf \left(\int f_n - \int g \right)$$

$$\int \underline{\lim} f_n - \cancel{\int g} \leq \underline{\lim} \int f_n - \cancel{\int g}$$

$$g(x) = \frac{-1}{\sqrt{|x|}}$$



$$\int_0^1 \frac{dx}{|x|^\alpha} < \infty \iff \alpha < 1$$

$$\int_1^\infty \frac{dx}{x^\alpha} < \infty \iff \alpha > 1$$

8] f ολοκλ. $\forall \epsilon > 0 \ \exists \delta > 0$ τ.ώ.

$$m(E) < \delta \implies \left| \int_E f \right| < \varepsilon .$$

Υπόδ! Εύκολο αν f φραγμένη. Αν όχι $f \mathbb{1}_{\{|f|<n\}} \xrightarrow{n} f$.

$|f| \leq M$:

$$\left| \int_E f \right| \leq \int_E |f| \leq \int_E M = M\, m(E) < M\delta < \epsilon$$

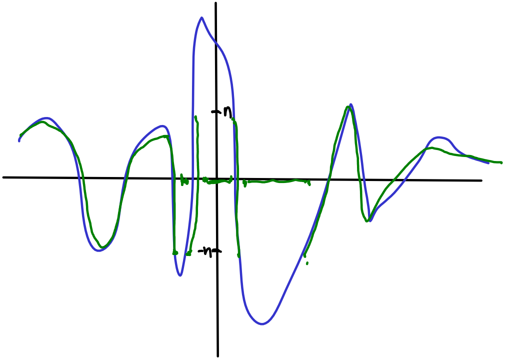
πάρε $\delta < \frac{\epsilon}{M}$.

$$\underbrace{f\,\mathbb{1}_{\{|f|<n\}}}_{} \xrightarrow{n} f$$

↘ φραγμένη από το n

$f(x) \neq \pm\infty$

$x \in \bigcup_{n=1}^{\infty} \{|f|<n\}$



$$\int f = \int \overbrace{f \cdot \mathbb{1}_{\{|f|<n\}}}^{\leq |f|} + \int R_n$$

$\int f \cdot \mathbb{1}_{\{|f|<n\}} \xrightarrow{\text{ΘΚΣ}} \int f$, $\quad \int R_n \to 0$

$\exists n: \quad \int R_n < \frac{\varepsilon}{2}$

$\left| \int_{|f|>n} f \right| < \frac{\varepsilon}{2}$

$\delta < \frac{\varepsilon/2}{n}$

9] f ολοκλ. $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt \quad$ (αόριστο ολοκλ. της f)

$\Rightarrow$ F συνεχής

$F(x+h) \longrightarrow F(x)$ όταν $h \to 0^{+}$

$F(x+h) - F(x) = \int_{x}^{x+h} f(t)\,dt$ $\longleftarrow$ μπορώ να το κάνω $< \varepsilon$ αν $h < \delta$

ομοιόμορφη συνέχεια

$:= \int_{[x,\,x+h]} f$ ..... m ..... $h < \delta$ $\longleftarrow$ ..... από άσκ. 8

$\boxed{10}$

$$\lim_{n\to\infty}\int_0^n \overbrace{\left(1+\frac{x}{n}\right)^n}^{\uparrow\, e^x} e^{-2x}\,dx = \int \underbrace{\mathbb{1}_{[0,n]}\underbrace{\left(1+\frac{x}{n}\right)^n}_{e^x} e^{-2x}}_{f_n(x)}$$

$$f_n(x)=\left(1+\frac{x}{n}\right)^n e^{-2x} \le \underbrace{M e^{-x}} \qquad (x>0)$$

$$\Uparrow$$

$$\left(1+\frac{x}{n}\right)^n \le M e^{x}$$

$$f_n(x)\ \xrightarrow{n\to+\infty}\ \mathbb{1}_{[0,+\infty)}\, e^{-x} = f(x)$$

$e^{-x}\left(1+\frac{x}{n}\right)^n$ είναι φραγ. ανεξ. του n

$$\left(e^{-x} \left(1+\frac{x}{n}\right)^n \right)' = -e^{-x}\left(1+\frac{x}{n}\right)^n + e^{-x} \left(1+\frac{x}{n}\right)^{n-1}$$

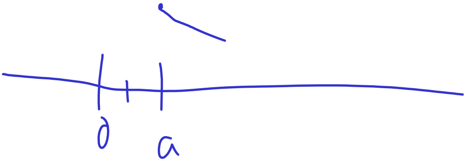
$$= e^{-x}\left(1+\frac{x}{n}\right)^{n-1}\left(-\left(1+\frac{x}{n}\right) + 1 \right) =$$

$$= -e^{-x}\left(1+\frac{x}{n}\right)^{n-1}\frac{x}{n} \qquad x = -n$$

$$\leq 1$$

$a \geq 0$

$$\lim_n \int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx$$



$e^{n^2x^2} \geq n^2x^2$

$n^2 x e^{-n^2x^2} \leq \quad n^2 x e^{-n^2x^2} \leq M$

$\leq \frac{n^2x}{n^2x^2} = \frac{1}{x} \leq \frac{1}{a}$

$$\varphi(y) = n \cdot \overbrace{nx}^{y} e^{-(nx)^2} \qquad \varphi\left(\tfrac{1}{\sqrt{2}}\right) = \underline{\underline{n}} \frac{1}{\sqrt{2}} e^{-1/2}$$

$$\left(n y e^{-y^2}\right)' = n e^{-y^2} + ny(-2y)e^{-y^2} = e^{-y^2} n\left(1-2y^2\right)$$

$y = 1/\sqrt{2} = nx \Rightarrow x = \frac{1}{\sqrt{2}n}$