Λήμμα Fatou

$0 \le f_n$

$$\int \varliminf f_n \le \varliminf \int f_n$$

Παραδείγματα

(a) $f_n(x) = \mathbb{1}_{[n,n+1]}(x)$

$$\varliminf f_n(x) = 0 \Rightarrow \int \varliminf f_n = 0$$

$$\int f_n = 1 \Rightarrow \varliminf \int f_n = 1$$

(b) $g_n(x) = \mathbb{1}_{[n,+\infty)}(x)$ $\quad \varliminf g_n(x) = 0$

$$\int g_n = +\infty \Rightarrow \lim \int g_n = +\infty$$

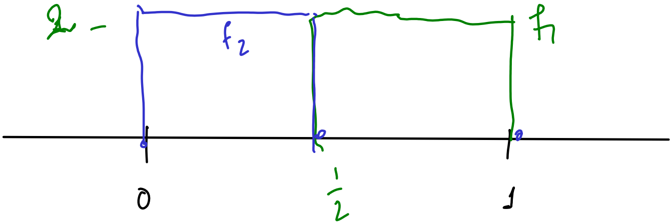
$h_n(x) = \mathbb{1}_{[n,2n]}(x)$

$$\int h_n = n$$

$$\varliminf \int h_n = \varliminf n = +\infty$$

(d) $\mathcal{L}_{TO}$ $[0,1]$. $\quad f_n(x) = 0$ av $x \notin [0,1]$

$\forall x \quad \underline{\lim}\, f_n(x) = 0 \qquad \int f_n = 1 \qquad f_n \geq 0$



$f_1, f_2, f_1, f_2, \ldots\ldots$

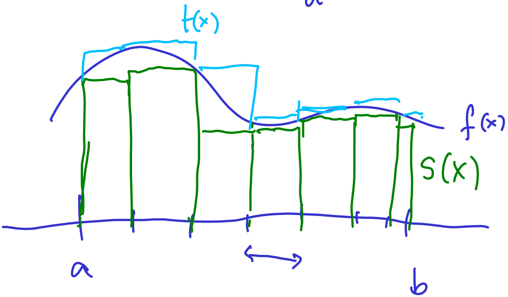
Ολοκλ. Riemann

$f : [a,b] \to \mathbb{R}^{\geq 0}$, φραγμ.

Riemann ολοκλ.

Τμε

$$\int_a^b f(x)\,dx = \int_{[a,b]} f = \int \mathbb{1}_{[a,b]} f$$



$$\underbrace{\int_{[a,b]} s \leq \int_{[a,b]} f \leq \int_{[a,b]} t}_{\leq \varepsilon}$$

$$\int_a^b f(x)\,dx = \int_{[a,b]} f$$

$\int f \qquad f \geqslant 0 \qquad\qquad \swarrow \int f+g = \int f + \int g$

$\Downarrow$

$$\int \underbrace{f-g} = \int f - \int g \iff \int \underbrace{f-g} + \int \underline{g} = \int \underline{f}$$

$f-g \geqslant 0$

$0 \leq f \leq g$

$\int f \leq \int g$

$f \geqslant 0$ είναι σ.π. $0 \Longrightarrow \int f = 0$

$E \subseteq \mathbb{R},\ m(E)=0$; $\quad x \notin E \Longrightarrow f(x)=0$

$$f = f \cdot \mathbb{1}_E + \underbrace{f \cdot \mathbb{1}_{E^c}}_{\to \sigma\pi.\ 0}$$

$$\int f \cdot \mathbb{1}_E \leq \int (+\infty) \mathbb{1}_E = 0$$

Αν $0 \le f$, $\int f = 0 \Rightarrow f = 0$ σ.π.

$$\Uparrow$$

$E = \{ f > 0 \}$

$m\{ f \geq \frac{1}{n} \} = 0, \ \forall n$

$E \subseteq \bigcup_{n=1}^{\infty} \{ f \geq \frac{1}{n} \}$

$m\{ f \geq \frac{1}{n} \} \geq c > 0$

$m(E) \le \sum_{n=1}^{\infty} m\{ f > \frac{1}{n} \} = \sum 0 = 0$

$g(x) = \mathbb{1}_{\{ f \geq \frac{1}{n} \}} \cdot \frac{1}{n}$

$\frac{1}{n} \overbrace{m\{ f \geq \frac{1}{n} \}}^{=0} = 0 = \int g \le \int f = 0 \Leftarrow g(x) \le f(x)$

$f \geq 0 \qquad a < b \qquad A \subseteq B \Rightarrow \int_A f \leq \int_B f$

$$\int_{[a,b]} f = \int_{[a,b)} f = \int_{(a,b]} f = \int_{(a,b)} f$$

$$(a,b) \subseteq [a,b) \subseteq [a,b] \Rightarrow \int_{(a,b)} f \leq \int_{[a,b)} f \leq \int_{[a,b]} f$$

$$\int_{(a,b)} f = \int_{[a,b]} f = \int f \cdot \mathbb{1}_{[a,b]}$$

$$\int_{\{a\}} f + \int_{(a,b)} f + \int_{\{b\}} f$$

$$\int_{[a,b]} f = \int f \, \mathbb{1}_{[a,b]} = \int f \left(\mathbb{1}_{\{a\}} + \mathbb{1}_{(a,b)} + \mathbb{1}_{\{b\}} \right) = \int f \, \mathbb{1}_{\{a\}} = f(a)\, \underbrace{m\{a\}}_{} = 0$$

$\int_{\{a\}} f = \int f \, \mathbb{1}_{\{a\}} = f(a)\, m\{a\} = 0$, $\int_{\{b\}} f = 0$

(4) $\displaystyle\int_{[1,+\infty)} \frac{1}{x} = \int_{(0,1]} \frac{1}{x} = +\infty$

$\forall M>0$ $\forall\!/$

$$\int_{[1,M]} \frac{1}{x} = \int_1^M \frac{dx}{x} = \ln x \Big|_1^M = \ln M$$

$$\int_{[1,+\infty)} \frac{1}{x} \geq \ln M, \quad \forall M>0 \implies \int_{[1,+\infty)} \frac{1}{x} = +\infty$$

$$\int_{(0,1]} \frac{1}{x} \geq \int_{[\varepsilon,1]} \frac{1}{x} = \int_{\varepsilon}^{1} \frac{dx}{x} = \ln x \Big|_{\varepsilon}^{1} =$$

$$= 0 - \ln \varepsilon = \ln \frac{1}{\varepsilon} \qquad \forall \varepsilon > 0$$

$$\int_1^{\infty} \frac{1}{x^\alpha} < \infty \Leftrightarrow \alpha > 1$$

$$\int_{(0,1]} \frac{1}{x} = +\infty$$

$$\int_{(0,+\infty)} \frac{1}{x^\alpha} = +\infty \qquad \int_0^1 \frac{1}{x^\alpha} < \infty \Leftrightarrow \alpha < 1$$

$$\alpha > 0$$

(5) $f_n \geq 0$ $\quad f_n \to f$, $\underbrace{f_n \leq f} \Longrightarrow \int f_n \to \int f$

$$\underbrace{\int f_n} \leq \underbrace{\int f} = \int \lim f_n = \int \underline{\lim}\, f_n \overset{F}{\leq} \underline{\lim} \int f_n$$

$$\Downarrow$$

$$\overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$$

$$\lim \int f_n = \int f$$

(6) $\quad 0 \leq f_n \downarrow \qquad f_n \to f, \quad \underbrace{\int f_1 < \infty}.$

Τότε $\quad \lim \int f_n = \int f$

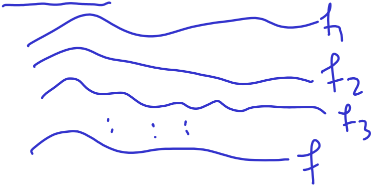
$A = \bigcup\limits_n^{\uparrow} A_n$

$m(A) = \lim m(A_n)$

$A = \bigcap^{\downarrow} A_n$

$m(A_1) < \infty$

$m(A) = \lim m(A_n)$



$$\underbrace{0 \leq f_1 - f_n} \quad \uparrow$$

$$\downarrow$$

$$0 \leq f_1 - f$$

Θ. Μον. Σ.

$$\int f_1 - f_n \to \int f_1 - f \implies \cancel{\int f_1} + \int f_n \to \cancel{\int f_1} + \int f$$

$$\int f_n \to \int f$$

(7) $0 \le f_n \to f$, $\int f_n \to \int f < \infty$

$\underline{\lim}\, a_n + \underline{\lim}\, b_n \le \underline{\lim}\,(a_n + b_n)$

Για κάθε μετρ. E τότε $\int_E f_n \to \int_E f = \int f \mathbb{1}_E$

$\int_E f_n = \int f_n \mathbb{1}_E$

Fatou $\int \underline{\lim}\, f_n \le \underline{\lim} \int f_n$

$\int \underline{\lim}\, f_n = \int f$, $\quad \underline{\lim} \int f_n = \int f$

$\forall E: \quad \int_E f \le \underline{\lim} \int_E f_n$ $\quad <$

$$\int \underline{\lim}\, f_n \mathbb{1}_E \le \underline{\lim} \int f_n \mathbb{1}_E = \underline{\lim} \int_E f_n$$

$$\int \underline{\lim}\, f_n \mathbb{1}_E = \int f \mathbb{1}_E = \int_E f$$

$$\int_{E^c} f \le \underline{\lim} \int_{E^c} f_n$$

$$\int f \le \underline{\lim} \int_E f_n + \underline{\lim} \int_{E^c} f_n$$

$\le \liminf \int_E f_n$

$$\int f < \underline{\lim} \int f_n = \lim \int f_n = \int f$$

סתירה

Ολοκλήρωμα προσημασμένης f

$$f^+ = \max\{0, f\}$$

$$|f| = f^+ + f^-$$

$$f = \underbrace{f^+}_{\geq 0} - \underbrace{f^-}_{\geq 0}$$

$$f^- = -\min\{0, f\}$$

f <u>ολοκληρώσιμη</u> αν $\left[\int f^+ < \infty, \int f^- < \infty\right] \Leftrightarrow \boxed{\int |f| < \infty}$

<u>Τότε ορίζουμε</u> $\int f = \int f^+ - \int f^-$

$\int f = +\infty$ όταν $\int f^+ = +\infty$, $\int f^- < \infty$

f, g ολοκληρώσιμες ($\int |f| < \infty$, $\int |g| < \infty$)

(α) $\alpha \in \mathbb{R} \Rightarrow \alpha f$ ολοκλ. και $\int \alpha f = \alpha \int f$

<u>$\alpha > 0$</u> $\alpha f = \alpha f^+ - \alpha f^- \Rightarrow (\alpha f)^+ = \alpha f^+$, $(\alpha f)^- = \alpha f^-$

$$\int |\alpha f| = \int |\alpha| \cdot |f| = \int \alpha \cdot |f| = \alpha \int |f| < \infty$$

$$\int \alpha f = \int (\alpha f)^+ - \int (\alpha f)^- = \int \alpha f^+ - \int \alpha f^- = \alpha \int f^+ - \alpha \int f^-$$

$$= \alpha \left(\int f^+ - \int f^- \right) = \alpha \int f$$

$\alpha < 0$ $\quad (\alpha f)^+ = -\alpha f^- \quad (\alpha f)^- = -\alpha f^+$

(b) $f+g$ ολοκλ. , $\int f+g = \int f + \int g$

$$\int |f+g| \leq \int |f| + \int |g| < \infty$$

$$f+g = \boxed{(f+g)^+ - (f+g)^-}$$

$$f+g = \boxed{f^+ - f^- + g^+ - g^-}$$

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\int (f+g)^+ + \int f^- + \int g^- = \underline{\int (f+g)^-} + \int f^+ + \int g^+$$

$$\int (f+g)^+ - \int (f+g)^- = \int f^+ - \int f^- + \int g^+ - \int g^-$$

$$\int f+g = \int f + \int g$$

(γ) $f = 0$ σ.π. $\Longrightarrow$ $\int f = 0$

$$f = f^+ - f^-$$

$$0 = f(x) = f^+(x) - f^-(x) \Longrightarrow f^+(x) = 0 = f^-(x)$$

$$\{f \neq 0\} \supseteq \{f^+ \neq 0\}$$

$$\left.\begin{array}{l} 0 = m\{f \neq 0\} \geq m\{f^+ \neq 0\} = 0 \Rightarrow \int f^+ = 0 \\ \qquad\qquad\qquad\qquad m\{f^- \neq 0\} = 0 \Rightarrow \int f^- = 0 \end{array}\right\} \int f = 0$$

(δ) $\underbrace{f \leq g \text{ σ.π.} \Rightarrow \int f \leq \int g}_{g - f \geq 0 \text{ σ.π.}} \iff \int g - f \geq 0$

$$\int g - f = \int (g-f)^+ - \underbrace{\int (g-f)^-}_{= 0 \text{ σ.π.}} = \int (g-f)^+ \geq 0$$

(ε) $A \cap B = \emptyset \Rightarrow \mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B$

$$\int_{A \cup B} f = \int_A f + \int_B f$$

$$\int_{A\cup B} f = \int f \mathbb{1}_{A \cup B} = \int f(\mathbb{1}_A + \mathbb{1}_B) = \int f \mathbb{1}_A + f \mathbb{1}_B =$$

$$= \int_A f + \int_B f.$$

(1) $\int |f| < \infty$. Τότε $f \neq \pm\infty$ σ.π.

Έστω όχι, δηλ. $m\{f = +\infty\} > 0$.

$f \geq 0 \Rightarrow f = f^+$

$\{f = +\infty\} = \{f^+ = +\infty\}$

$\int |f| < \infty \iff \int f^+ < \infty, \int f^- < \infty$

$$\infty > \int f^+ \geq \int (+\infty) \cdot \mathbb{1}_{\{f^+ = +\infty\}} = (+\infty) \overbrace{m\{f^+ = +\infty\}}^{> 0}$$

$= +\infty$ άτοπο

(2) $\int |f| < \infty$. Τότε $\left|\int f\right| \leq \int |f| \iff -\int|f| \leq \int f \leq \int |f|$

"=" $\iff$ ($f \geq 0$ σ.π.) ή ($f \leq 0$ σ.π.)

$\downarrow$

"=" $\iff$ $f \leq 0$ σ.π.

$$f = f^+ - f^- \implies \int f = \int f^+ - \int f^-$$

$$|f| = f^+ + f^- \implies \int |f| = \int f^+ + \int f^-$$

$$-\int f^+ - \int f^- \leq \int f^+ - \int f^- \leq \int f^+ + \int f^-$$