απλές συνάρτηση : παίρνει πεπ. στο πλήθος τιμές

$\varphi \geq 0$

$\{v_1, \dots, v_n\} \subseteq \overline{\mathbb{R}}$ : $\varphi(x) = \sum_{j=1}^{n} \overset{\geq 0}{v_j} \, \mathbb{1}_{\{\varphi = v_j\}}(x)$

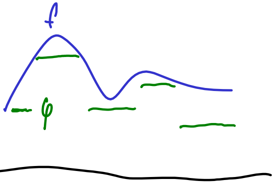
$\int \varphi = \sum_{j=1}^{n} v_j \, m\{\varphi = v_j\}$


$0 < v_1 \in \mathbb{R}$ $\quad m\{\varphi = v_j\} = +\infty \longrightarrow \int \varphi = +\infty$

$v_1 = +\infty$ $\quad m\{\varphi = +\infty\} > 0 \longrightarrow \int \varphi = +\infty$

Αν $\boxed{\int \varphi < \infty \Longrightarrow m\{\varphi = +\infty\} = 0}$

Αν $\int \varphi < \infty \Rightarrow \varphi \in \mathbb{R}$ σ.π.

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$f \geq 0$ : $\displaystyle \int f = \sup_{\substack{0 \le \varphi \le f \\ \varphi \text{ απλή}}} \int \varphi$

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$\varphi, \psi$ απλές : $\displaystyle \int_E \varphi + \psi = \int_E \varphi + \int_E \psi$

$\displaystyle \int_E \lambda \varphi = \lambda \int_E \varphi$

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$\displaystyle \int_E f \overset{\text{ορ.}}{=} \int f \cdot \mathbb{1}_E$

E ..... μετρ.

$E \subseteq \mathbb{R}$

E μετρ.

$\Updownarrow$

$\mathbb{1}_E$ μετρ. συνάρτηση

Λήμμα Fatou: $0 \leq f_n$ μετρήσιμες. Τότε

$$\int \liminf f_n \leq \liminf \int f_n$$

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$$\liminf \overset{\mathbb{R}}{\overset{\Downarrow}{a_n}} = \lim_{n \to +\infty} \underbrace{\left( \inf_{k \geq n} a_k \right)}_{\wedge\!\!\mid\!\mid} \quad \text{υπάρχει πάντα}$$

$b_n \uparrow$

$$\limsup a_n = \lim_{n \to +\infty} \left( \sup_{k \geq n} a_k \right) \quad \text{υπάρχει}$$

$\liminf a_n \leq \limsup a_n$ : Υπάρχει το $\lim a_n \Leftrightarrow$
$\liminf a_n = \limsup a_n$

$g_n(x) = \inf_{k \geq n} f_k(x)$ $\qquad$ $\liminf f_n = \underline{\lim} f_n = \lim g_n$

$g_n(x) \uparrow$ ακολουθία $\qquad$ $g_n(x) \uparrow \underline{\lim} f_n(x)$

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Αρκεί να δείξουμε ότι, αν $\varphi$ απλή με $0 \leq \varphi \leq \underline{\lim} f_n$,

τότε
$$\int \varphi \leq \underbrace{\underline{\lim} \int f_n}_{F}$$

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$$\sup_{\varphi} \int \varphi = \int \left( \underline{\lim} f_n \right) \leq F.$$

$0 \leq \varphi \leq \underline{\lim} f_n = \lim g_n \implies \int \varphi \leq \underline{\lim} \int f_n$

A. $\int \varphi = +\infty$

B. $\int \varphi < \infty$

• $\int \varphi = +\infty \overset{\text{θέλω}}{\implies} \underline{\lim} \int f_n = +\infty$

Ισχυρισμός: Για κάποιο $\alpha > 0$ το $m\{\varphi > \alpha\} = \infty$.

$$\varphi = \sum_{j=1}^{n} v_j \mathbb{1}_{\{\varphi = v_j\}} \qquad \int \varphi = \sum_{j=1}^{n} \cancel{v_j}^{\,0}\, m\{\varphi = v_j\}$$

π.χ. $v_1 m\{\varphi = v_1\} = +\infty$

$\underbrace{v_1 = +\infty}$ ή $\underbrace{m\{\varphi = v_1\} = +\infty}_{\alpha \text{ επιλέγω } = v_1/2}$

$\varphi = \infty \cdot \mathbb{1}_A + \dots$

$\varphi \geq \varphi' = M \, \mathbb{1}_A$

$\varphi \le \underline{\lim} f_n$ $\quad \int \varphi = +\infty \implies m\{\varphi > \alpha\} = +\infty$ (κάποιο $\alpha > 0$)

$\underline{\lim} \int f_n = +\infty$ $\quad g_n = \inf_{k \ge n} f_k \uparrow \underline{\lim} f_n$

$A = \{\varphi > \alpha\}$

$A_n = \{g_n > \alpha\}$ $\implies$ $\underbrace{A \subseteq \bigcup_{n=1}^{\infty} \uparrow A_n}$

$\underline{\lim} f_n(x)$

$\varphi(x)$

$\alpha$

$x \in A \implies \exists n : x \in A_n$

$\varphi(x) > \alpha \implies \exists n : g_n(x) > \alpha$ $\quad +\infty = m(A) \le m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_n m(A_n)$

$A_n \subseteq A_{n+1}$ : $\quad g_n(x) > \alpha \implies g_{n+1}(x) > \alpha$

$m(A_n) \to +\infty$

$$\int f_n \geqslant \int g_n \geqslant \int_{A_n} g_n \geqslant \int_{A_n} \alpha = \alpha\, m(A_n)$$

$$A_n = \{g_n > \alpha\}$$

$$f_n \geqslant g_n = \inf_{k \geqslant n} f_k$$

$$0 \leq f \leq g \Rightarrow \int f \leq \int g$$

$$\sup_{\varphi \leq f \leq g} \int \varphi$$

$$\Downarrow$$

$$\underline{\lim} \int f_n \geqslant \underline{\lim}\ \alpha\, m(A_n) = +\infty$$

$$0 \leq f\ ,\ E: \quad \int_E f \leq \int f$$

$$\int \underline{\lim}\, f_n \leq \underline{\lim} \int f_n$$

$$\int_E f = \int \mathbb{1}_E f\ ,\quad \mathbb{1}_E f \leq f$$

• $0 \leq \varphi \leq \underline{\lim} f_n$ θέλω $\int \varphi \leq \underline{\lim} \int f_n$ | $g_n = \inf_{k \geq n} f_k \uparrow \underline{\lim} f_n \geq \varphi$

$> \theta\varphi$

$\int \varphi < \infty$ $B = \{ \varphi > 0 \} = \operatorname{supp} \varphi \Rightarrow m(B) < \infty$

Αν $0 < v_1 < v_2 < \dots < v_n$ οι τιμές της $\varphi$ τότε

στο $B$ η $\varphi \geq v_1$, άρα $\infty > \int \varphi \geq \int_B \varphi \geq \int_B v_1 = v_1 m(B)$

Παίρνω $\theta \in (0,1)$ και ορίζω

$B_n = \{ g_n > \theta \varphi \}$ $B_{n+1} \supseteq B_n$ (αύξουσα)

(γιατί $g_{n+1} \geq g_n$)

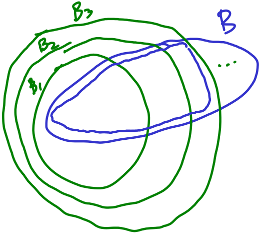
$\underline{\lim} f_n > \theta\varphi$

$B \subseteq \bigcup_{n=1}^{\infty} B_n$ : $x \in B \Rightarrow \exists n : x \in B_n$ : $\varphi(x) > 0 \Rightarrow \exists n : g_n(x) > \theta \varphi(x)$

$B \setminus B_n \downarrow$ καί $\bigcap_{n=1}^{\infty}(B\setminus B_n)=\emptyset$

$m(B)<\infty$

$m(B\setminus B_n)<\infty$




$$0 = m\left(\bigcap_{n=1}^{\infty}(B\setminus B_n)\right) = \lim m(B\setminus B_n)$$

$$m(B\setminus B) \xrightarrow{n} 0$$

$$\boxed{A\cap B=\emptyset \;\Big|\; \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B \qquad \int_{A\cup B}\varphi = \int_A\varphi + \int_B\varphi}$$

$$\int f_n \geq \int g_n \geq \int_{B_n} g_n \geq \int_{B_n}\theta\varphi = \theta\int_{B_n}\varphi = \theta\left(\int_B\varphi - \int_{B\setminus B_n}\varphi\right)$$

$$= \theta\int\varphi - \theta\int_{B\setminus B_n}\varphi \geq \theta\int\varphi - \theta\int_{B\setminus B_n} M = \theta\int\varphi - \theta\cdot M\cdot m(B\setminus B_n)$$

$M=\max\varphi(x)$

$$\int f_n \geq \theta \int \varphi - \underbrace{\theta \cdot M \cdot \overbrace{m(B \setminus B_n)}^{\to\infty}}$$

$$\underline{\lim} \int f_n \geq \theta \int \varphi - \underbrace{\overline{\lim}\ \theta \cdot M \cdot m(B \setminus B_n)}_{=0}$$

$$\underline{\lim} \int f_n \geq \theta \int \varphi \qquad \forall \theta < 1$$

$$\Downarrow$$

$$\underline{\lim} \int f_n \geq \int \varphi \qquad \#$$

**Θ** **Μονότονης σύγκλισης** $0 \leq f_n \uparrow f \quad [\, f_n(x) \nearrow f(x), \ \forall x \,]$

Τότε $$\lim \int f_n = \int f = \int \lim f_n$$

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$$f_n(x) \leq f(x) \Rightarrow \int f_n \leq \int f \Rightarrow \lim \int f_n \leq \int f$$

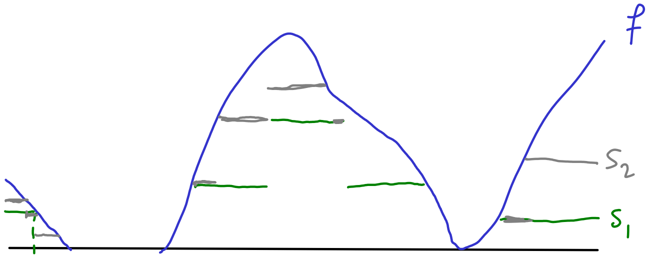
**Λ. Fatou** $$\int f = \int \overset{\underline{\lim}}{\overset{\|}{\lim}} f_n \leq \underline{\lim} \overline{\int f_n} = \lim \int f_n$$

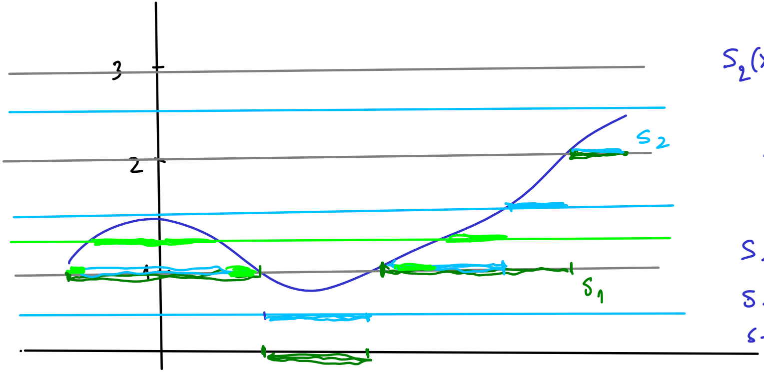
$$\int f \leq \lim \int f_n$$

$\underline{\Theta}$ $f \geq 0$ μετρ. Υπάρχουν $\underline{\text{απλές}}$ $s_n(x)$, $s_{n+1}(x) \geq s_n(x)$

και $s_n(x) \nearrow f(x)$, $\forall x \in \mathbb{R}$.

— Σε αυτή την περίπτωση $\int s_n \to \int f$ (Θ. Μονότονης σύγκλισης)






$$S_2(x) = \begin{cases} S_1(x) \\ S_1(x) + \frac{1}{2} \end{cases}$$

$$S_2(x) \geqslant S_1(x)$$

$S_1 \to 1$

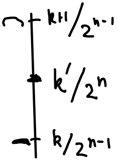
$S_2 \to 1/2$

$S_3 \to 1/4$

$S_n \to \frac{1}{2^{n-1}}$

$$S_n(x) = \frac{k(x)}{2^{n-1}} \quad \text{όπου} \quad f(x) \in \left[ \frac{k(x)}{2^{n-1}}, \frac{k(x)+1}{2^{n-1}} \right) \Longrightarrow S_n(x) \leqslant f(x)$$

$$S_{n+1}(x) \geqslant S_n(x)$$




$$0 \leqslant f(x) - S_n(x) < \frac{1}{2^{n-1}}$$

$0 \le f, g$ μετρ. Τότε $\displaystyle\int f+g = \int f + \int g$

$0 \le s_n \nearrow f$

απλές

$0 \le t_n \nearrow g$

$s_n + t_n$ απλή

$s_n + t_n \uparrow$

$s_n + t_n \nearrow f+g$

Από Θ. Μ. Σ.

$\int s_n \to \int f$

$+$

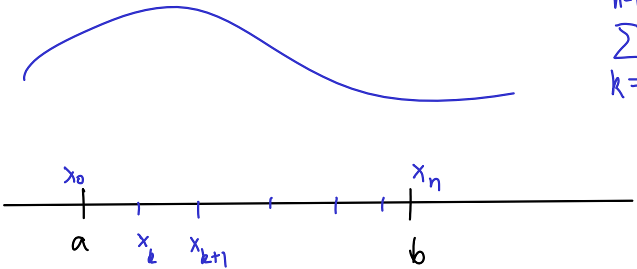
$\int t_n \to \int g$

$\| $

$\int t_n + s_n \to \int f+g$

Ολοκλ. Riemann  $f:[a,b]\to\mathbb{R}$, φραγμένη

$$\int_0^1 \frac{dx}{\sqrt{x}} \overset{\text{ορ.}}{=} \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{\sqrt{x}}$$




κάτω Riemann άθροισμα

$$\sum_{k=0}^{n-1}(x_{k+1}-x_k)\cdot \inf_{[x_k,x_{k+1}]} f$$

κάτω R α. $\leq$ άνω R.α.