GEOMETRIC IMPLICATIONS OF WEAK TILING

MIHAIL N. KOLOUNTZAKIS, NIR LEV, AND MÁTÉ MATOLCSI

Dedicated to the 100th anniversary of Bent Fuglede's birthday

Abstract. The notion of weak tiling played a key role in the proof of Fuglede's spectral set conjecture for convex domains, due to the fact that every spectral set must weakly tile its complement. In this paper, we revisit the notion of weak tiling and establish some geometric properties of sets that weakly tile their complement. If $A \subset \mathbb{R}^d$ is a convex polytope, we give a direct and self-contained proof that A must be symmetric and have symmetric facets. If $A \subset \mathbb{R}$ is a finite union of intervals, we give a necessary condition on the lengths of the gaps between the intervals.

1. Introduction

A bounded, measurable set $A \subset \mathbb{R}^d$ is called *spectral* if it admits an orthogonal basis consisting of exponential functions. Fuglede famously conjectured [Fug74] that A is a spectral set if and only if it can tile the space by translations. This conjecture inspired extensive research over the years, see [Kol24] for the history of the problem and an overview of the known (positive as well as negative) related results.

A major recent result states that the Fuglede conjecture holds for convex domains in all dimensions [LM22]. A key role in the proof is played by the concept of weak tiling, introduced in the same paper as a relaxation of proper tiling.

Definition 1.1. We say that a bounded, measurable set $A \subset \mathbb{R}^d$ *weakly tiles its complement* $A^{\complement} = \mathbb{R}^d \setminus A$, if there exists a positive, locally finite measure ν on $\mathbb{R}^d$ such that $\mathbb{1}_A * \nu = \mathbb{1}_{A^{\complement}}$ a.e. In this case, ν is called a *weak tiling measure* for the set A.

If the measure ν is a sum of unit masses, then the weak tiling becomes a proper tiling of the complement $A^{\complement}$ by translated copies of A.

The role of weak tiling in the theory of spectral sets is due to the fact that every spectral set must weakly tile its complement, see [LM22, Theorem 1.5]. This can be viewed as a weak form of the "spectral implies tile" part of Fuglede's conjecture.

In the present paper we revisit the notion of weak tiling, and establish some geometric properties of sets that weakly tile their complement. In particular, these

Date: June 30, 2025.

2020 *Mathematics Subject Classification.* 42B10, 42C05, 52C22.

Key words and phrases. Fuglede's conjecture, spectral set, tiling, convex polytopes.

N.L. is supported by ISF Grant No. 1044/21. M.M. is supported by the Hungarian National Foundation for Scientific Research, Grants No. K146387 and KKP 133819.

include all the spectral sets, as well as the sets which can tile properly by translations.

In Section 2 we consider the case where A is a convex polytope, and give a new self-contained proof of the fact that A must be centrally symmetric and have centrally symmetric facets. In Section 3 we consider the case where $A \subset \mathbb{R}$ is a finite union of intervals, and give a necessary condition on the lengths of the gaps appearing between the intervals. Finally, in Section 4 we pose some open problems.

2. Weak tiling by convex polytopes

2.1. In this section we focus on the case where $A \subset \mathbb{R}^d$ is a convex body, that is, a compact, convex set with nonempty interior. Note that while in general, a set that weakly tiles its complement need not tile properly, it was shown in [KLM23] that for the class of convex bodies weak tiling implies tiling.

Theorem 2.1 ([KLM23, Theorem 1.4])**.** *Let A be a convex body in $\mathbb{R}^d$, and assume that A weakly tiles its complement. Then A is a convex polytope which can also tile $\mathbb{R}^d$ properly by translations.*

The proof of this result consists of several ingredients. First, due to [Ven54], [McM80], in order to prove that a convex body $A \subset \mathbb{R}^d$ tiles by translations, it suffices to show that A satisfies the following four conditions:

(i) A is a convex polytope;

(ii) A is centrally symmetric;

(iii) all the facets of A are centrally symmetric;

(iv) each belt of A consists of either 4 or 6 facets.

It was proved in [LM22, Theorem 4.1] that if A is a convex body in $\mathbb{R}^d$, then the weak tiling assumption implies condition (i). It was also proved [LM22, Theorem 6.1] that the weak tiling assumption together with (i), (ii) and (iii) implies (iv).

Conditions (ii) and (iii) were shown in [Kol00], [KP02], [GL17] to follow from the assumption that the convex polytope A is spectral, but these proofs do not work if we only assume that A weakly tiles its complement.

The fact that the weak tiling assumption together with (i) implies both (ii) and (iii), was proved in [KLM23], based on an elaborate machinery developed in [LL21]. In fact, conditions (ii) and (iii) were obtained as a consequence of [KLM23, Theorem 1.3], which states that if a (convex or non-convex) polytope $A \subset \mathbb{R}^d$ weakly tiles its complement, then A is equidecomposable by translations to a cube of the same volume.

Our goal in the present section is to give a more direct and self-contained proof of the fact that the weak tiling condition together with (i) implies both (ii) and (iii).

Theorem 2.2. *Let A be a convex polytope in $\mathbb{R}^d$. If A weakly tiles its complement, then A must be centrally symmetric and have centrally symmetric facets.*

The proof requires some auxiliary results, which we will introduce first. Then, we will proceed with the proof of Theorem 2.2.

2.2. Zero-free regions of the Fourier transform. If $f \in L^1(\mathbb{R}^d)$ then its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle \xi, x\rangle} dx, \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

Given a vector $v \in \mathbb{R}^d$, a positive integer R, and $\varepsilon > 0$, we define the set

$$\begin{aligned} S &= S(v, R, \varepsilon) \\ &= \{nv + w : n \in \mathbb{Z},\ |n| \notin \{1, 2, \dots, R-1\},\ w \in \mathbb{R}^d,\ |w| < \varepsilon\}. \end{aligned}$$

The set S is thus a union of open balls of radius ε centered at integral multiples of the vector v, but only those multiples nv for which either $n = 0$ or $|n| \geqslant R$. We observe that $S(v, R, \varepsilon)$ is contained in a cylinder of width ε along the vector v.

Theorem 2.3. *Let A be a convex polytope in $\mathbb{R}^d$. Assume that A is not centrally symmetric, or that at least one of the facets of A is not centrally symmetric. Then there exist a nonzero vector $v \in \mathbb{R}^d$, a positive integer R and $\varepsilon > 0$, such that the Fourier transform $\widehat{\mathbb{1}}_A$ has no zeros in the set $S = S(v, R, \varepsilon)$.*

Proof. This result was essentially established in [GL17, Sections 3 and 4]. The proof is based on Minkowski's theorem, which states that a convex polytope A is centrally symmetric if and only if each facet F of A has a parallel facet F' such that $|F| = |F'|$ (see, for instance, [Gru07, Corollary 18.1]).

If A is not centrally symmetric, then the conclusion of Theorem 2.3 follows from standard estimates, see equation (3.5) in [GL17, Section 3], where an even larger zero-free region is given, which is obtained from a cylinder of width ε by the removal of a large ball around the origin.

The case where A is centrally symmetric but has a non-centrally symmetric facet, is more intricate. In this case, the conclusion of Theorem 2.3 is obtained as a combination of equations (4.6), (4.15) and (4.16) in [GL17, Section 4].

It remains to notice that $\widehat{\mathbb{1}}_A$ also has no zeros in some neighborhood of the origin, which is obvious as $\widehat{\mathbb{1}}_A$ is a continuous function with $\widehat{\mathbb{1}}_A(0) = m(A) > 0$. □

2.3. Translation-bounded measures. A (complex) measure μ on $\mathbb{R}^d$ is said to be *translation-bounded* if for every (or equivalently, for some) open ball B we have

$$\sup_{x \in \mathbb{R}^d} |\mu|(B + x) < +\infty. \tag{2.2}$$

If μ is a translation-bounded measure on $\mathbb{R}^d$, then μ is a tempered distribution.

If f is a function in $L^1(\mathbb{R}^d)$ and μ is a translation-bounded measure on $\mathbb{R}^d$, then the convolution $f * \mu$ is a translation-bounded measure which is also a locally integrable function on $\mathbb{R}^d$, which can be defined (uniquely up to equality a.e.) by the condition that $(f * \mu) * \varphi = f * (\mu * \varphi)$ for every continuous, compactly supported function φ.

If $f \in L^1(\mathbb{R}^d)$ then we denote by

$$Z(f) := \{\xi \in \mathbb{R}^d : \widehat{f}(\xi) = 0\} \tag{2.3}$$

the (closed) set of zeros of the Fourier transform $\widehat{f}$.

Theorem 2.4. *Let $f \in L^1(\mathbb{R}^d)$, $\int f \neq 0$, and let μ be a translation-bounded measure on $\mathbb{R}^d$. If we have $f * \mu = 1$ a.e., then $\widehat{\mu} = (\int f)^{-1} \cdot \delta_0$ in the open set $Z(f)^\complement$.*

Proof. This result can be proved in a similar way to [KL16, Theorem 4.1]. The details are as follows. First we observe that it suffices to prove the following claim: Let B be an open ball contained in $Z(f)^\complement$, and let ψ be a smooth function whose closed support is contained in B. Then $\widehat{\mu}(\psi) = (\int f)^{-1}\psi(0)$.

To prove this, let $K \subset B$ be a closed ball containing $\operatorname{supp}(\psi)$. Since $\widehat{f}$ does not vanish on K, then due to Wiener's theorem there exists $g \in L^1(\mathbb{R}^d)$ such that $\widehat{f} \cdot \widehat{g} = 1$ on K (see [Hel10, pp. 150–152] for a proof in the one-dimensional case; the multi-dimensional case is similar). There is a Schwartz function φ such that $\widehat{\varphi} = \psi$, so this function φ satisfies $\widehat{\varphi} \cdot \widehat{g} \cdot \widehat{f} = \widehat{\varphi}$. This implies that $\varphi * g * f = \varphi$, hence

$$\widehat{\mu}(\psi) = \mu(\widehat{\psi}) = \int \varphi(-t) d\mu(t) = \int (\varphi * g * f)(-t) d\mu(t) \tag{2.4}$$

$$= \int \Big(\int (\varphi * g)(-x) f(x-t) dx \Big) d\mu(t). \tag{2.5}$$

If we can exchange the order of integrals, we obtain

$$\widehat{\mu}(\psi) = \int (\varphi * g)(-x) \Big(\int f(x-t) d\mu(t) \Big) dx = \int (\varphi * g)(-x) dx, \tag{2.6}$$

because the inner integral is $(f * \mu)(x)$, which is 1 a.e. by assumption. Furthermore,

$$\int (\varphi * g)(-x) dx = \widehat{\varphi}(0)\widehat{g}(0) = \widehat{f}(0)^{-1}\widehat{\varphi}(0) = (\textstyle\int f)^{-1}\psi(0), \tag{2.7}$$

where the second equality holds since we have $\widehat{\varphi} \cdot \widehat{g} \cdot \widehat{f} = \widehat{\varphi}$, and $\widehat{f}(0) = \int f \neq 0$.

It thus remains to justify the exchange of integrals. Indeed, observe that

$$\int \Big(\int (|\varphi| * |g|)(-x) \cdot |f(x-t)| dx \Big) |d\mu(t)| \tag{2.8}$$

$$= \int (|f| * |g|)(-x) \Big(\int |\varphi(x-t)| \, |d\mu(t)| \Big) dx, \tag{2.9}$$

and the latter integral is finite, because $|f| * |g| \in L^1(\mathbb{R}^d)$, and the inner integral is a bounded function of x since φ is a Schwartz function and μ is a translation-bounded measure. This justifies the exchange of integrals and completes the proof. ◻

2.4. After these preliminary results we are ready to prove the main theorem.

Proof of Theorem 2.2. Let A be a convex polytope in $\mathbb{R}^d$, and assume that A weakly tiles its complement, so there is a positive measure ν such that $\mathbb{1}_A * \nu = \mathbb{1}_{A^\complement}$ a.e. According to [LM22, Lemma 2.4] the measure ν must be translation-bounded. Hence the measure $\mu := \delta_0 + \nu$ is also translation-bounded, and satisfies $\mathbb{1}_A * \mu = 1$ a.e. In turn, Theorem 2.4 implies that $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$ in the open set $Z(\mathbb{1}_A)^{\complement}$.

We must prove that A is centrally symmetric and has centrally symmetric facets. Suppose to the contrary that this is not the case. Then by Theorem 2.3 there exist a nonzero vector $v \in \mathbb{R}^d$, a positive integer R and $\varepsilon > 0$, such that $\widehat{\mathbb{1}}_A$ has no zeros in the set $S = S(v, R, \varepsilon)$. It follows that $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$ in the open set S.

Now suppose that we are given a real-valued Schwartz function g on $\mathbb{R}^d$ satisfying

$$\operatorname{supp}(g) \subset S, \quad \widehat{g} \geqslant 0. \tag{2.10}$$

Then we have

$$\int_{\mathbb{R}^d} g(x)dx = \widehat{g}(0) \leqslant \widehat{g}(0) + \int \widehat{g}(\xi)d\nu(\xi) = \int \widehat{g}(\xi)d\mu(\xi), \tag{2.11}$$

where the inequality in (2.11) is due to $\widehat{g}$ being a nonnegative function and ν being a positive measure. On the other hand, we have

$$\int \widehat{g}(\xi)d\mu(\xi) = \mu(\widehat{g}) = \widehat{\mu}(g) = m(A)^{-1}g(0), \tag{2.12}$$

where the last equality holds since we have $\operatorname{supp}(g) \subset S$ and $\widehat{\mu} = m(A)^{-1} \cdot \delta_0$ in the open set S. We conclude that

$$\int_{\mathbb{R}^d} g(x)dx \leqslant m(A)^{-1}g(0) \tag{2.13}$$

for every real-valued Schwartz function g satisfying (2.10). We will show that this leads to a contradiction, by constructing an example of a real-valued Schwartz function g satisfying (2.10), but such that (2.13) does not hold.

We choose a nonnegative Schwartz function φ such that $\int \varphi = 1$, φ is supported in the open ball of radius ε centered at the origin, and $\widehat{\varphi} \geqslant 0$. We define also the trigonometric polynomial $p_N(t) := K_N(R\,t)$, where K_N is the classical Fejér kernel,

$$K_N(t) = \sum_{|n|<N} \Big(1 - \frac{|n|}{N}\Big) e^{2\pi i n t}, \quad t \in \mathbb{R}. \tag{2.14}$$

Then p_N is nonnegative, $p_N(0) = N$, the Fourier coefficients $\widehat{p_N}(n)$, $n \in \mathbb{Z}$, are also nonnegative, $\widehat{p_N}(0) = 1$, and we have $\widehat{p_N}(n) = 0$ if $0 < |n| < R$.

Finally, we define the function

$$g_N(x) := \sum_{n \in \mathbb{Z}} \widehat{p_N}(n)\, \varphi(x - nv), \quad x \in \mathbb{R}^d. \tag{2.15}$$

Notice that there are only finitely many nonzero terms in the sum (2.15), and that the nonzero terms correspond to integers n such that either $n = 0$ or $|n| \geqslant R$. Hence g_N is a real-valued (in fact, nonnegative) Schwartz function such that $\operatorname{supp}(g_N)$ is contained in the set $S = S(v, R, \varepsilon)$. The Fourier transform of g_N is given by

$$\widehat{g_N}(\xi) = \widehat{\varphi}(\xi) \sum_{n \in \mathbb{Z}} \widehat{p_N}(n)\, e^{-2\pi i n \langle v, \xi \rangle} = \widehat{\varphi}(\xi)\, p_N(-\langle v, \xi \rangle), \tag{2.16}$$

hence $\widehat{g_N}$ is a nonnegative function. We conclude that g_N satisfies the conditions (2.10).

To complete the proof we will show that if N is sufficiently large, then g_N does not satisfy (2.13). Indeed, we may assume that $\varepsilon < \frac{1}{2}|v|$ which ensures that the terms in the sum (2.15) have pairwise disjoint supports. This implies that

$$g_N(0) = \varphi(0), \tag{2.17}$$

so that the value $g_N(0)$ does not depend on N. On the other hand, using (2.16) we have

$$\int_{\mathbb{R}^d} g_N(x)dx = \widehat{g_N}(0) = \widehat{\varphi}(0)p_N(0) = N, \tag{2.18}$$

which can be arbitrarily large, contradicting (2.13). We therefore arrive at the desired contradiction, and Theorem 2.2 is thus proved. □

3. Weak tiling by finite unions of intervals

In dimensions $d = 1$ and 2, Fuglede's spectral set conjecture is still open in both directions. The important special case where $\Omega \subset \mathbb{R}$ is a finite union of intervals was recently studied in [DDF25], where several necessary conditions were established for Ω to be a spectral set. One of these conditions states that the lengths of the gaps between the intervals must be representable as sums of lengths (with multiplicities) of some of the intervals composing Ω. In this section, in Theorem 3.6 below, we will prove this in more generality, assuming only that Ω weakly tiles its complement.

We remark here that Fuglede's conjecture was proved for a union of *two* intervals, see [Łab01], but it is still open for a union of a higher number of intervals.

3.1. Spectral unions of intervals. Let us begin by considering the more restrictive situation where Ω does not only weakly tile its complement, but moreover is assumed to be spectral. In this case, the following result was obtained in [DDF25].

Theorem 3.1 ([DDF25, Theorem 3.1(i)]). *Let $\Omega = \bigcup_{i=1}^{n}(a_i, b_i)$, $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$, and assume that Ω is a spectral set. Then each gap length $a_{k+1} - b_k$ is representable in the form $\sum_{i=1}^{n} p_i(b_i - a_i)$ where p_i are nonnegative integers.*

Below we give an alternative proof of this result, based on a simple argument combined with existing results in the literature.

Proof of Theorem 3.1. Since Ω is a spectral set, it admits a spectrum Λ, that is, there is a countable set of frequencies $\Lambda \subset \mathbb{R}$ such that the system of exponential functions $\{\exp 2\pi i\lambda x\}$, $\lambda \in \Lambda$, forms an orthogonal basis in $L^2(\Omega)$. It is known that for a finite unions of intervals, any spectrum Λ is periodic [BM11], [Kol12], [IK13]. In this case, the proof of [LM22, Theorem 1.5] yields that Ω admits a weak tiling measure ν with the additional property that $\widehat{\nu}$ is a periodic measure. This implies that ν is a pure point measure which is supported on some arithmetic progression. In particular, $\operatorname{supp}(\nu)$ is a locally finite set.

The weak tiling condition thus says that $\mathbb{1}_{\Omega^\complement} = \sum_t \nu(t)\mathbb{1}_{\Omega+t}$ a.e., where t goes through the atoms of ν. In particular, if t is an atom of ν, then $\Omega + t \subset \Omega^\complement$. If we now fix one of the gaps $I = (a, b)$, then $\mathbb{1}_I = \sum_t \nu(t)\mathbb{1}_{I\cap(\Omega+t)}$ a.e. We observe that in this sum it suffices that t only runs through a finite set of atoms, since $\operatorname{supp}(\nu)$ is a locally finite set and hence $\Omega + t$ does not intersect I for all but finitely many atoms t.

Note that for any atom t, the set $I \cap (\Omega + t)$ is a union of several of the intervals composing Ω. Hence the indicator function $\mathbb{1}_I$ is a finite linear combination with positive coefficients of indicator functions of intervals I_j, where the length of each I_j belongs to the finite set $L = \{b_1 - a_1, \dots, b_n - a_n\}$. Hence, if we denote $I_j = (x_j, y_j)$, then we have $\mathbb{1}_{(a,b)} = \sum_{j=1}^m w_j \mathbb{1}_{(x_j,y_j)}$ a.e., where w_j are strictly positive scalars.

This equality holds also in the sense of distributions, hence differentiating yields

$$\delta_a - \delta_b = \sum_{j=1}^{m} w_j(\delta_{x_j} - \delta_{y_j}). \tag{3.1}$$

The measure on the right hand side of (3.1) thus must assign zero mass to all the points x_j and y_j except for a and b, while it assigns the mass $+1$ to a and -1 to b.

It follows from (3.1) that at least one of the points x_j must be equal to a; let's say it is the point x_{j_1}. Then we have $a = x_{j_1} < y_{j_1}$. The j_1'th term on the right hand side of (3.1) gives a negative mass to y_{j_1}, hence if $y_{j_1} < b$ then there must exist j_2 such that $x_{j_2} = y_{j_1}$. So we have $a = x_{j_1} < y_{j_1} = x_{j_2} < y_{j_2}$.

Continuing this way, we obtain a sequence

$$a = x_{j_1} < y_{j_1} = x_{j_2} < y_{j_2} = x_{j_3} < y_{j_3} = \dots \tag{3.2}$$

which we can continue as long as $y_{j_s} < b$. Since there are finitely many points x_j, y_j, it follows that there must exist N such that $y_{j_N} = b$. This implies that

$$b - a = \sum_{s=1}^{N} (y_{j_s} - x_{j_s}), \tag{3.3}$$

so the gap length $b-a$ is a sum of lengths of intervals $I_{j_s} = (x_{j_s}, y_{j_s})$ and these lengths belongs to the finite set $L = \{b_1 - a_1, \dots, b_n - a_n\}$. This implies that the gap length $b-a$ is representable in the form $\sum_{i=1}^n p_i(b_i - a_i)$ where p_i are nonnegative integers. □

3.2. Weak tiling measures are pure point. We now turn to the more general case where $\Omega \subset \mathbb{R}$ is a finite union of intervals that weakly tiles its complement, but is not assumed to be spectral. This case is more intricate, since the weak tiling measure ν is not known a priori to be supported on a locally finite set, nor even to be purely atomic.

We begin by showing that any weak tiling measure must be purely atomic.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}$ be a finite union of intervals, and assume that ν is a weak tiling measure for Ω. Then ν must be a pure point measure.*

Proof. We may assume that Ω is a nonempty open set given by $\Omega = \bigcup_{j=1}^{n}(a_j, b_j)$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. By our assumption, the measure $\mu = \delta_0 + \nu$ satisfies $\mathbb{1}_\Omega * \mu = 1$ a.e. It is straightforward to verify that if φ is a smooth, compactly supported function on $\mathbb{R}$, then $\varphi * (\delta_{a_j} - \delta_{b_j}) = \varphi' * \mathbb{1}_{(a_j,b_j)}$ for each j. As a consequence,

$$\varphi * \sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu = \varphi' * \mathbb{1}_\Omega * \mu = \varphi' * 1 = 0. \tag{3.4}$$

As this holds for every smooth, compactly supported function φ, we conclude that

$$\sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu = 0. \tag{3.5}$$

The measure μ has a unique decomposition $\mu = \mu_d + \mu_c$ into a sum of a pure point measure μ_d and a continuous measure μ_c. We observe that $\sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu_d$ is a pure point measure, while $\sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu_c$ is a continuous measure. The sum of these two measures vanishes by (3.5), hence both measures must be zero. In particular,

$$\sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu_d = 0. \tag{3.6}$$

Now, if φ is again a smooth, compactly supported function on $\mathbb{R}$, then as before

$$\varphi' * \mathbb{1}_\Omega * \mu_d = \varphi * \sum_{j=1}^{n}(\delta_{a_j} - \delta_{b_j}) * \mu_d = 0. \tag{3.7}$$

If φ_1, φ_2 are two smooth, compactly supported functions, $\int \varphi_1 = \int \varphi_2$, then there is a smooth, compactly supported function φ such that $\varphi' = \varphi_2 - \varphi_1$. Hence (3.7) yields that $\varphi_1 * (\mathbb{1}_\Omega * \mu_d) = \varphi_2 * (\mathbb{1}_\Omega * \mu_d)$. This is possible only if the function $\mathbb{1}_\Omega * \mu_d$ is constant a.e. But now recall that $\mu = \delta_0 + \nu$, and therefore

$$\mathbb{1}_\Omega = \mathbb{1}_\Omega * \delta_0 \leqslant \mathbb{1}_\Omega * \mu_d \leqslant \mathbb{1}_\Omega * \mu = 1 \quad \text{a.e.} \tag{3.8}$$

As the function $\mathbb{1}_\Omega * \mu_d$ is constant a.e., it follows that its constant value must be 1. In turn, we conclude that $\mathbb{1}_\Omega * \mu_d = \mathbb{1}_\Omega * \mu$ a.e., which implies that $\mu = \mu_d$. This shows that μ, and hence also ν, must be a pure point measure. ◻

3.3. Weak tiling of an interval by intervals. We have thus shown that any weak tiling measure ν must be purely atomic. However, note that a pure point measure may have finite accumulation points of atoms. Hence, we are not guaranteed that $\operatorname{supp}(\nu)$ is a locally finite set, and so we cannot proceed as in the proof of Theorem 3.1.

To address this additional difficultly we now prove two results, which may also be of independent interest, about weak tiling of finite intervals and half-lines by infinitely many intervals.

Theorem 3.3. *Let $I \subset \mathbb{R}$ be a bounded open interval, and suppose that*

$$\mathbb{1}_I = \sum_j w_j \mathbb{1}_{I_j} \quad a.e. \tag{3.9}$$

(a finite or infinite sum) where $I_j \subset \mathbb{R}$ are nonempty open intervals, and $w_j > 0$. Assume that the lengths of the intervals I_j belong to some finite set of positive real numbers $L = \{l_1, \dots, l_n\}$. Then,

(i) *there exist nonnegative integers $p_1, \dots, p_n$ such that $|I| = \sum_{i=1}^n p_i l_i$;*

(ii) *if $I = (a, b)$, then each one of the endpoints of any of the intervals I_j is representable in the form $a + \sum_{i=1}^n p_i l_i$ where p_i are nonnegative integers;*

(iii) *there are only finitely many distinct intervals I_j (so that the sum* (3.9) *is essentially a finite sum).*

Proof. First, note that $I_j \subset I$ for all j. If we set $l := \min\{l_1, \dots, l_n\}$, then

$$|I| = \int \mathbb{1}_I = \sum_j w_j \int \mathbb{1}_{I_j} \geqslant l \sum_j w_j, \tag{3.10}$$

which implies (since $l > 0$) that $\sum_j w_j < +\infty$.

Second, if some of the intervals I_j coincide, then we may replace them with a single interval whose weight is the sum of the weights of the coinciding intervals. This allows us to assume, with no loss of generality, that the intervals I_j are distinct.

We may also suppose with no loss of generality that $I = (0, 1)$. Let Θ be the finite subset of $[0, 1]$ consisting of all the real numbers in $[0, 1]$ which are representable in the form $\sum_{i=1}^n p_i l_i$ where p_i are nonnegative integers.

We shall now construct by induction finite sets A_k and points $x_k \in \Theta$, such that the following conditions are satisfied:

(a) The intervals $\{I_j\}$, $j \in A_k$, have their endpoints in Θ;

(b) $\sum_{j \in A_k} w_j \mathbb{1}_{I_j} = 1$ a.e. in $[0, x_k]$;

(c) $\sum_{j \in A_k} w_j \mathbb{1}_{I_j} < 1$ a.e. in $[x_k, 1]$.

We begin by taking A_0 to be the empty set and $x_0 = 0$, so that (a), (b), (c) hold for $k = 0$. Now suppose that we have already constructed the finite sets $A_0, \dots, A_k$ and the points $x_0, \dots, x_k \in \Theta$, such that

$$0 = x_0 < x_1 < \dots < x_k < 1. \tag{3.11}$$

We will construct a finite set A_{k+1}, and a point $x_{k+1} \in \Theta$ with $x_k < x_{k+1} \leqslant 1$, such that the conditions (a), (b), (c) are satisfied with k replaced by $k + 1$.

We observe that $\sum_{j \in A_k} w_j \mathbb{1}_{I_j}$ is a piecewise constant function, with only a finite number of discontinuity points. It thus follows from (c) that there is $0 \leqslant \lambda_k < 1$ such that $\sum_{j \in A_k} w_j \mathbb{1}_{I_j} = \lambda_k$ a.e. in some interval $[x_k, y_k]$, where $x_k < y_k \leqslant 1$. This and (b) imply that the sum of the remainder terms $\sum_{j \notin A_k} w_j \mathbb{1}_{I_j}$ vanishes a.e. in $[0, x_k]$, and is equal to $1 - \lambda_k$ a.e. in $[x_k, y_k]$.

By taking y_k smaller if needed, we may assume that $x_k < y_k \leqslant \min\{1, x_k + l\}$.

Let C_k be the set of all j such that the left endpoint of I_j is x_k. Note that C_k is a finite set with at most n elements, since the intervals I_j are distinct and their lengths belong to $L = \{l_1, \dots, l_n\}$. We claim that if we set $B_k := C_k \setminus A_k$ then $\sum_{j \in B_k} w_j = 1 - \lambda_k$.

First, it is obvious that $\sum_{j \in B_k} w_j$ cannot exceed $1 - \lambda_k$. Indeed, otherwise the sum $\sum_{j \in B_k} w_j \mathbb{1}_{I_j}$ would exceed $1 - \lambda_k$ in the interval $(x_k, x_k + l)$, which in turn implies that the sum $\sum_{j \in A_k \cup B_k} w_j \mathbb{1}_{I_j}$ must exceed $\lambda_k + (1 - \lambda_k) = 1$ a.e. in the interval (x_k, y_k). But this is not possible since $\sum_j w_j \mathbb{1}_{I_j} = \mathbb{1}_I$ a.e.

Hence, to prove that $\sum_{j \in B_k} w_j$ is equal to $1 - \lambda_k$, it remains to show that this sum cannot be less than $1 - \lambda_k$. Suppose to the contrary that $\sum_{j \in B_k} w_j = 1 - \lambda_k - \eta$ for some $\eta > 0$. Let $D_k(\varepsilon)$ be the set of all j such that the left endpoint of I_j belongs to $(x_k, x_k + \varepsilon)$. We choose and fix $\varepsilon > 0$ small enough so that $\sum_{j \in D_k(\varepsilon)} w_j < \eta$.

Now recall that the sum $\sum_{j \notin A_k} w_j \mathbb{1}_{I_j}$ vanishes a.e. in $[0, x_k]$. Hence, if for some $j \notin A_k$ the interval I_j intersects $[x_k, x_k + \varepsilon)$, then the left endpoint of I_j must be in $[x_k, x_k + \varepsilon)$, and therefore either $j \in B_k$ or $j \in D_k(\varepsilon) \setminus A_k$. Assuming that ε was chosen small enough so that we also have $x_k + \varepsilon \leqslant y_k$, this implies that

$$1 - \lambda_k = \sum_{j \notin A_k} w_j \mathbb{1}_{I_j} = \sum_{j \in B_k} w_j \mathbb{1}_{I_j} + \sum_{j \in D_k(\varepsilon) \setminus A_k} w_j \mathbb{1}_{I_j} \quad \text{a.e. in } [x_k, x_k + \varepsilon). \tag{3.12}$$

The first sum on the right hand side does not exceed $\sum_{j \in B_k} w_j = 1 - \lambda_k - \eta$, while the second sum does not exceed $\sum_{j \in D_k(\varepsilon)} w_j < \eta$. Hence the right hand side cannot be equal to $1 - \lambda_k$ on any set of positive measure. We thus arrive at a contradiction, which shows that we must have $\sum_{j \in B_k} w_j = 1 - \lambda_k$.

We now set $A_{k+1} := A_k \cup B_k$, then A_{k+1} is a finite set. We first observe that the intervals $\{I_j\}$, $j \in A_{k+1}$, have their endpoints in Θ. Indeed, if $j \in A_k$ then this holds by the inductive hypothesis, while if $j \in B_k$ then the left endpoint of I_j is equal to x_k which is in Θ, while the right endpoint lies in the set $x_k + \{l_1, \dots, l_n\}$ which again consists of elements of Θ.

Next, we consider the sum $\sum_{j \in A_{k+1}} w_j \mathbb{1}_{I_j}$ and observe that it is again a piecewise constant function, all of whose discontinuity points lie in Θ. Moreover, we have

$$\sum_{j \in A_{k+1}} w_j \mathbb{1}_{I_j} = \sum_{j \in A_k} w_j \mathbb{1}_{I_j} = 1 \quad \text{a.e. in } [0, x_k], \tag{3.13}$$

and

$$\begin{aligned} \sum_{j \in A_{k+1}} w_j \mathbb{1}_{I_j} &= \sum_{j \in A_k} w_j \mathbb{1}_{I_j} + \sum_{j \in B_k} w_j \mathbb{1}_{I_j} \\ &= \lambda_k + (1 - \lambda_k) = 1 \quad \text{a.e. in } [x_k, y_k]. \end{aligned} \tag{3.14}$$

It follows that there exists a point $x_{k+1} \in \Theta$ satisfying $x_k < y_k \leqslant x_{k+1} \leqslant 1$ and such that $\sum_{j \in A_{k+1}} w_j \mathbb{1}_{I_j} = 1$ a.e. in $[0, x_{k+1}]$, while $\sum_{j \in A_{k+1}} w_j \mathbb{1}_{I_j} < 1$ a.e. in $[x_{k+1}, 1]$. We conclude that the conditions (a), (b), (c) are satisfied with k replaced by $k + 1$.

Our inductive process may continue as long as $x_k < 1$. Since Θ is a finite set, it follows that there must exist N such that $x_N = 1$. Hence $|I| = x_N$ is an element of the set Θ, which yields condition (i). Moreover, the condition (b) with $k = N$ implies that $\sum_{j\in A_N} w_j \mathbb{1}_{I_j} = \mathbb{1}_I$ a.e., hence the intervals $\{I_j\}$, $j \in A_N$, in fact constitute all the intervals in the system, and so condition (iii) of the theorem is established. Finally, by condition (a) with $k = N$, each one of the endpoints of any of the intervals I_j must lie in Θ, which shows that also condition (ii) holds, and completes the proof. □

The next result is a version of Theorem 3.3 for semi-infinite intervals.

Theorem 3.4. *Let $I = (a, +\infty) \subset \mathbb{R}$ be an open half-line, and suppose that*

$$\mathbb{1}_I = \sum_{j=1}^{\infty} w_j \mathbb{1}_{I_j} \quad a.e. \tag{3.15}$$

where $I_j \subset \mathbb{R}$ are bounded open intervals, and $w_j > 0$. Assume that the lengths of the intervals I_j belong to some finite set of positive real numbers $L = \{l_1, \dots, l_n\}$. Then,

(i) *each one of the endpoints of any of the intervals I_j is representable in the form $a + \sum_{i=1}^{n} p_i l_i$ where p_i are nonnegative integers;*

(ii) *any bounded subset of $\mathbb{R}$ contains only finitely many distinct intervals I_j.*

Proof. This can be proved in a similar way as Theorem 3.3. We now proceed with the details. Again we denote $l := \min\{l_1, \dots, l_n\}$. First, we note that if $x \in (a, +\infty)$ then

$$x - a = \int \mathbb{1}_{(a,x)} \geqslant \sum_{I_j \subset (a,x)} w_j \int \mathbb{1}_{I_j} \geqslant l \sum_{I_j \subset (a,x)} w_j, \tag{3.16}$$

hence the sum of the weights w_j over all the intervals I_j contained in any fixed interval (a, x), must be finite. Second, again by replacing coinciding intervals with a single interval, we may assume with no loss of generality that the intervals I_j are distinct.

We may also suppose that $a = 0$, and therefore $I = (0, +\infty)$.

Let Θ be the set of all real numbers which are representable in the form $\sum_{i=1}^{n} p_i l_i$ where p_i are nonnegative integers. In other words, Θ is the semigroup generated by the numbers $l_1, \dots, l_n$. We note that Θ is a locally finite subset of $[0, +\infty)$.

Let us now perform the inductive construction from the proof of Theorem 3.3. It yields finite sets A_k and points $x_k \in \Theta$ with

$$0 = x_0 < x_1 < x_2 < \dots, \tag{3.17}$$

such that the conditions (a), (b), (c) are satisfied for each k. The fact that Θ is a locally finite set, implies that we must have $x_k \to +\infty$.

We now observe that the intervals $\{I_j\}$, $j \in \bigcup_{k=0}^{\infty} A_k$, must in fact constitute all the intervals in the system. Indeed, given any j_0 we can find k such that $I_{j_0} \subset (0, x_k)$. Due to condition (b), we have

$$1 = \sum_{j \in A_k} w_j \mathbb{1}_{I_j} \leqslant \sum_j w_j \mathbb{1}_{I_j} = 1 \quad \text{a.e. in } [0, x_k], \tag{3.18}$$

which implies that the system $\{I_j\}$, $j \in A_k$, contains all the intervals I_j that intersect $[0, x_k]$. In particular, $j_0 \in A_k$ which establishes our claim. As a consequence, it follows from condition (a) that each one of the endpoints of any of the intervals I_j must lie in the locally finite set Θ. This implies both (i) and (ii) and concludes the proof. □

Remark 3.5. An analogous result is true also for an open half-line of the form $I = (-\infty, a)$, which can be proved in a similar way.

3.4. Weak tiling by a finite union of intervals. We now apply the previous results to the case where a finite union of intervals weakly tiles its complement.

Theorem 3.6. *Let $\Omega = \bigcup_{i=1}^{n}(a_i, b_i)$, $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, and assume that Ω weakly tiles its complement. Then each gap length $a_{k+1} - b_k$ is representable in the form $\sum_{i=1}^{n} p_i(b_i - a_i)$ where p_i are nonnegative integers.*

In particular, this result applies if Ω tiles the space (properly) by translations, or if Ω is a spectral set. In the latter case, the result recovers Theorem 3.1 above.

Proof. Let ν be a weak tiling measure for Ω, then $\mathbb{1}_\Omega * \nu = \mathbb{1}_{\Omega^\complement}$ a.e. By Theorem 3.2, ν is a pure point measure, hence $\mathbb{1}_{\Omega^\complement} = \sum_t \nu(t)\mathbb{1}_{\Omega+t}$ a.e., where t goes through the atoms of ν. In particular, this implies that if t is an atom of ν, then $\Omega + t \subset \Omega^\complement$.

If we now fix one of the gaps $I = (b_k, a_{k+1})$, then $\mathbb{1}_I = \sum_t \nu(t)\mathbb{1}_{I\cap(\Omega+t)}$ a.e. If t is an atom of ν, then the set $I \cap (\Omega + t)$ consists of finitely many disjoint open intervals whose lengths belong to the finite set $L = \{b_1 - a_1, \dots, b_n - a_n\}$. An application of Theorem 3.3 thus allows us to conclude that the gap length $a_{k+1} - b_k$ is representable in the form $\sum_{i=1}^{n} p_i(b_i - a_i)$, where p_i are nonnegative integers. □

Next, we derive some consequences on the support of the weak tiling measure.

Theorem 3.7. *Let $\Omega = \bigcup_{i=1}^{n}(a_i, b_i)$, $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, and assume that ν is a weak tiling measure for Ω. Then the support of ν is contained in the locally finite set consisting of all the nonzero real numbers which are representable in the form*

$$\pm \sum_{i=1}^{n} p_i(b_i - a_i) \tag{3.19}$$

where p_i are nonnegative integers.

Proof. Let Θ denote the set of all real numbers which are representable in the form $\sum_{i=1}^{n} p_i(b_i - a_i)$ where p_i are nonnegative integers, that is, Θ is the semigroup generated by the numbers $b_1 - a_1, \dots, b_n - a_n$. Then Θ is a locally finite subset of $[0, +\infty)$.

By Theorem 3.2, ν is a pure point measure, hence $\mathbb{1}_{\Omega^\complement} = \sum_t \nu(t)\mathbb{1}_{\Omega+t}$ a.e., where t goes through the atoms of ν. In particular, for any atom t we have $\Omega + t \subset \Omega^\complement$.

Now consider the half-line $I = (b_n, +\infty)$ which is contained in $\Omega^\complement$, then we have $\mathbb{1}_I = \sum_t \nu(t)\mathbb{1}_{I\cap(\Omega+t)}$ a.e. We note that in this sum it suffices that t only runs through the atoms of ν that are contained in $(0, +\infty)$, for otherwise $\Omega + t$ does not intersect I. For any such t, the set $I \cap (\Omega + t)$ is a union of several of the intervals

$$(a_1 + t, b_1 + t), \quad (a_2 + t, b_2 + t), \quad \dots, \quad (a_n + t, b_n + t), \tag{3.20}$$

whose lengths belong to the finite set $L = \{b_1 - a_1, \dots, b_n - a_n\}$. In particular, the last interval $(a_n + t, b_n + t)$ must be one of the components of $I \cap (\Omega + t)$. An application of Theorem 3.4 then yields that each one of the endpoints of any of the intervals that constitute the set $I \cap (\Omega + t)$, is an element of $b_n + \Theta$. In particular, this is the case for the right endpoint of the interval $(a_n + t, b_n + t)$, which means that $t \in \Theta$.

We have thus shown that $\operatorname{supp}(\nu) \cap (0, +\infty)$ must be a subset of Θ. In a similar way, it follows that $\operatorname{supp}(\nu) \cap (-\infty, 0)$ is contained in $-\Theta$. Since the origin obviously cannot be an atom of ν, due to the weak tiling assumption, we conclude that

$$\operatorname{supp}(\nu) \subset (\Theta \cup (-\Theta)) \setminus \{0\}. \tag{3.21}$$

This means that the assertion of the theorem holds, which completes the proof. □

As a special case of Theorem 3.7 we obtain:

Corollary 3.8. *Let Ω be a finite union of disjoint intervals with integer lengths. Then any weak tiling measure ν for Ω satisfies* $\operatorname{supp}(\nu) \subset \mathbb{Z} \setminus \{0\}$.

If the set Ω consists of a single interval, then we can say a bit more:

Corollary 3.9. *The unit interval $I = (0, 1)$ admits a unique weak tiling measure, namely, the measure $\nu = \sum_{n \in \mathbb{Z} \setminus \{0\}} \delta_n$.*

Proof. Due to Corollary 3.8 we know that $\operatorname{supp}(\nu) \subset \mathbb{Z} \setminus \{0\}$. This implies that we have $\mathbb{1}_{I^\complement} = \sum_{n \neq 0} \nu(n) \mathbb{1}_{I+n}$. But the translates $I + n$ are pairwise disjoint intervals, so this is possible only if $\nu(n) = 1$ for every $n \in \mathbb{Z} \setminus \{0\}$, which proves our claim. □

4. Open problems

We conclude the paper with a few open problems.

4.1. We say that a set $\Lambda \subset \mathbb{R}$ has *bounded density* if we have

$$\sup_{x \in \mathbb{R}} \#(\Lambda \cap (x, x + 1)) < +\infty. \tag{4.1}$$

Let $\Omega \subset \mathbb{R}$ be a finite union of intervals, and let ν be a weak tiling measure for Ω. Is it true that $\operatorname{supp}(\nu)$ must be a set of bounded density?

4.2. Let $\Omega \subset \mathbb{R}$ be a finite union of intervals. If Ω weakly tiles its complement, must it also tile properly? A positive answer would imply that the "spectral implies tile" direction of Fuglede's conjecture is true for finite unions of intervals.

4.3. Let $\Omega \subset \mathbb{R}$ be a finite union of intervals, and let ν be a weak tiling measure for Ω. Is it true that ν must be a convex linear combination of proper tilings? If it is the case, then this implies a positive answer also to the previous question.

A similar question for convex polytopes in $\mathbb{R}^d$ was posed in [KLM23, Section 7.3].

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Department of Mathematics and Applied Mathematics, University of Crete, Voutes Campus, 70013 Heraklion, Greece and Institute of Computer Science, Foundation of Research and Technology Hellas, N. Plastira 100, Vassilika Vouton, 700 13, Heraklion, Greece
Email address: kolount@gmail.com

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel
Email address: levnir@math.biu.ac.il

HUN-REN Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053, Budapest, Hungary and Department of Analysis and Operations Research, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
Email address: matomate@renyi.hu