MAXIMALITY AND COMPLETENESS OF ORTHOGONAL EXPONENTIALS ON THE CUBE

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Dedicated to the memory of Bent Fuglede

ABSTRACT. It is possible to have a packing by translates of a cube that is maximal (i.e. no other cube can be added without overlapping) but does not form a tiling. In the long running analogy of packing and tiling to orthogonality and completeness of exponentials on a domain, we pursue the question whether one can have maximal orthogonal sets of exponentials for a cube without them being complete. We prove that this is not possible in dimensions 1 and 2, but is possible in dimensions 3 and higher. We provide several examples of such maximal incomplete sets of exponentials, differing in size, and we raise relevant questions. We also show that even in dimension 1 there are sets which are spectral (i.e. have a complete set of orthogonal exponentials) and yet they also possess maximal incomplete sets of orthogonal exponentials.

1. INTRODUCTION

1.1. If $\Omega \subset \mathbb{R}^d$ is a measurable set of finite measure, we call it *spectral* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that the system of exponential functions

$$E(\Lambda) = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\} \tag{1.1}$$

forms an orthogonal basis in $L^2(\Omega)$. The set of frequencies Λ is then called a *spectrum* of Ω. The properties of spectral sets, especially in comparison with properties of sets that tile by translations, have been a subject of intense research for decades (see, for instance, the recent survey [Kol24]). The Fuglede conjecture [Fug74] stated that the

spectral sets are precisely those that can tile the space by translations (this means that one can find a collection of translates of the set such that almost every point of the space belongs to exactly one translate), but this conjecture is now known to be false in dimensions 3 and higher [Tao04], [Mat05], [KM06], [FMM06], [FR06]. Still a lot more is known and continues to be discovered about the connection of spectrality to tiling, a major recent result being the truth of the Fuglede conjecture for convex sets [LM22].

For the unit cube $Q = [-\frac{1}{2}, \frac{1}{2}]^d$, which is of course both a spectral set and a translational tile in $\mathbb{R}^d$, a lot more is known. In particular it is known that the spectra Λ of Q are precisely the tiling complements of Q, namely the translation sets $\Lambda \subset \mathbb{R}^d$ such that $\{Q + \lambda\}$, $\lambda \in \Lambda$, constitutes a tiling [LRW00], [IP98], [Kol00b].

For any set $\Omega \subset \mathbb{R}^d$ the two exponentials $e^{2\pi i \langle \lambda, x \rangle}$ and $e^{2\pi i \langle \mu, x \rangle}$ are easily seen to be orthogonal in $L^2(\Omega)$ if and only if $\widehat{\mathbb{1}}_\Omega(\lambda - \mu) = 0$, where $\widehat{\mathbb{1}}_\Omega(\xi) = \int_\Omega e^{-2\pi i \langle \xi, x \rangle} dx$ is the Fourier transform of the indicator function $\mathbb{1}_\Omega$. Since the unit cube Q is a product set, we can easily compute $\widehat{\mathbb{1}}_Q$ and find that its zero set is

$$G = \{(\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d : \text{there is } j \text{ such that } \xi_j \in \mathbb{Z} \setminus \{0\}\}.$$
$$(1.2)$$

At the same time a collection of translates $\{Q + t\}$, $t \in T$, forms a packing (i.e. the translates overlap at measure zero only) if and only if $(T - T) \cap (-1, 1)^d = \{0\}$. Hence if Λ is an orthogonal set for the cube Q then $\Lambda - \Lambda \subset G \cup \{0\}$, which in turn implies, via the characterization (1.2), that $\{Q + \lambda\}$, $\lambda \in \Lambda$, is a packing. In short, any orthogonal set for Q is also a packing set for Q (note, however, that the converse is not true).

1.2.   We now come to the question of maximality. A packing of Q is called *maximal* if it is not possible to add another translated copy of Q so that it remains a packing. Clearly every tiling is a maximal packing. It is easy to see though that there exist maximal packings of Q by a set of translates T which are not tilings. For instance, in dimension one, consider T to contain all numbers of the form $\pm(n - \frac{1}{4})$, $n = 1, 2, 3, \ldots$. The same is true in all dimensions.

An orthogonal set of frequencies Λ is similarly called maximal if it is not possible to add another point to it so

that it remains orthogonal. Any spectrum of Q (i.e. an orthogonal and complete set of frequencies) is of course maximal. But now it is not clear whether maximality fails to imply completeness, just as maximality of a packing fails to imply tiling.

In fact, it is easy to see that in dimension one, the maximality of an orthogonal set of frequencies does imply completeness. Indeed, if the exponential system $E(\Lambda)$ is orthogonal in $L^2([-\frac{1}{2}, \frac{1}{2}])$ then $\Lambda - \Lambda \subset \mathbb{Z}$. Fix $\lambda_0 \in \Lambda$. It follows that $\Lambda \subset \lambda_0 + \mathbb{Z}$ and, since all frequencies in $\lambda_0 + \mathbb{Z}$ are orthogonal to each other, maximality of Λ implies that $\Lambda = \lambda_0 + \mathbb{Z}$, that is, Λ is a spectrum of $[-\frac{1}{2}, \frac{1}{2}]$.

Is the same still true in higher dimensions? That is, if Λ is a maximal orthogonal set for the unit cube in $\mathbb{R}^d$, must it be also complete? In §2 we show that every maximal orthogonal set for the square in 2 dimensions is also complete, just as in dimension 1. Then in §3 we show that this is not the case in dimension 3. The maximal incomplete set we find there has the maximum possible density. In §4 we give an alternative construction, again in dimension 3, where our maximal set is now much thinner, almost a finite union of planes. We then show in §5 how we can extend the results from dimension 3 to higher dimensions. In §6 we examine some general properties that maximal orthogonal sets for the cube must have. Closing, in §7 we show that if we are willing to examine spectral sets other than the cube then even in dimension 1 we can find an example of a spectral set which has a maximal incomplete set of exponentials.

2. Maximal orthogonal sets in two dimensions

First we show that in two dimensions, the maximality of an orthogonal set for the unit square implies completeness, similar to the one-dimensional case.

Theorem 2.1. *Any maximal orthogonal set for the unit square in $\mathbb{R}^2$ is also complete, i.e. it is a spectrum. More generally, any orthogonal set for the unit square can be embedded as a subset of some spectrum.*

The proof will require a lemma which can be found in [Kol00a, Observation 1] as well as in [GL17, Lemma 11.4].

Lemma 2.2. *Let X be a subset of an abelian group H, and let H_1 and H_2 be two subgroups of H. Assume that $X - X \subset H_1 \cup H_2$. Then $X - X \subset H_1$ or $X - X \subset H_2$.*

Now let $\Lambda \subset \mathbb{R}^2$ be an orthogonal set for the unit square. Then $\Lambda - \Lambda \subset G \cup \{0\}$, where G is the zero set from (1.2) (with $d = 2$). It follows that if we define $H_1 = \mathbb{Z} \times \mathbb{R}$ and $H_2 = \mathbb{R} \times \mathbb{Z}$, which are both subgroups of $\mathbb{R}^2$, then $\Lambda - \Lambda \subset H_1 \cup H_2$. Lemma 2.2 then implies that $\Lambda - \Lambda \subset \mathbb{Z} \times \mathbb{R}$ or $\Lambda - \Lambda \subset \mathbb{R} \times \mathbb{Z}$.

Let us consider the case where $\Lambda - \Lambda \subset \mathbb{Z} \times \mathbb{R}$ (the other case is similar). By translating the set Λ we may assume that it contains the origin, so this implies that $\Lambda \subset \mathbb{Z} \times \mathbb{R}$. We now observe that if (n, t) and (n, s) are two distinct points in Λ, then orthogonality implies that $t - s$ must be an integer. Hence Λ is contained in a set of the form

$$\{(n, k + t(n)) : n, k \in \mathbb{Z}\} \tag{2.1}$$

where $t(n)$ are real numbers. But it is known, see [JP99, Theorem 5], that any set of the form (2.1) is a spectrum of the unit square. Hence Λ can be embedded as a subset of a spectrum. In particular, Λ is maximal if and only if it coincides with a spectrum.

3. A MAXIMAL INCOMPLETE ORTHOGONAL SET IN THREE DIMENSIONS

3.1. Next we show that Theorem 2.1 does not extend to three dimensions. Namely, we will prove the following result.

Theorem 3.1. *There exists an orthogonal set $\Lambda \subset \mathbb{R}^3$ for the unit cube, which is maximal but incomplete.*

Our strategy is to start from the integer lattice $\mathbb{Z}^3$ which is a spectrum for the unit cube. We then remove a small part of this spectrum, contained in the union of the three coordinate planes, which leaves us with a subset B of the integer lattice. The set B is still orthogonal, but of course it is not maximal as it embeds in a spectrum. However, we will show that one can add a small set A to B such that the orthogonality is still preserved, but at the same time it eliminates the possibility of further extending $A \cup B$ to a spectrum, or stronger, to any larger orthogonal set.

3.2.   We now turn to the details of the construction. Fix three real numbers α, β, γ such that none of them is an integer. Let A be the set in $\mathbb{R}^3$ consisting of all points that have one of the following forms:

$$(0, \beta - k, \gamma), \quad (\alpha, 0, \gamma - k), \quad (\alpha - k, \beta, 0) \qquad (3.1)$$

where k is a nonzero integer. Notice that A is contained in the union of three lines.

Let B be the set of all vectors in $\mathbb{R}^3$ whose coordinates are nonzero integers. This set contains all integer vectors except those lying in a union of three planes.

The sets A and B are disjoint. Let $\Lambda = A \cup B$ be their union. It is straightforward to check that $\Lambda - \Lambda \subset G \cup \{0\}$, where G is the zero set from (1.2) (with $d = 3$). Hence Λ is an orthogonal set for the unit cube Q in $\mathbb{R}^3$. In particular, $Q + \Lambda$ is a packing.

On the other hand we claim that Λ is not a complete set of frequencies. Indeed, if it was complete then $Q + \Lambda$ would be a tiling, see [LRW00], [IP98], [Kol00b]. But notice that each one of the three slabs

$$[-\tfrac{1}{2}, \tfrac{1}{2}] \times \mathbb{R} \times \mathbb{R}, \quad \mathbb{R} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \times \mathbb{R}, \quad \mathbb{R} \times \mathbb{R} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \ (3.2)$$

is scarcely covered by the cubes in the packing $Q + \Lambda$, so this packing is not a tiling.

3.3.   To continue we need the following lemma.

Lemma 3.2. *Let $s \in \mathbb{R}^3$ and suppose that $s - B \subset G$. Then at least one of the coordinates of s is zero.*

Proof. Let $s = (u, v, w)$ and suppose to the contrary that all the coordinates u, v, w are nonzero. Since we have $(u, v, w) - (1, 1, 1) \in G$ then at least one of the coordinates u, v, w must be an integer. Hence, say, $u = j$ for some nonzero integer j. Next, since we also have $(u, v, w) - (j, 1, 1) \in G$, then at least one of the other two coordinates v, w must be an integer. Hence, say, $v = k$ for some nonzero integer k. Finally, since $(u, v, w) - (j, k, 1) \in G$, then w must be an integer, so $w = l$ for some nonzero integer l. But this implies that the zero vector $(u, v, w) - (j, k, l)$ is in G, a contradiction. $\square$

3.4.   Finally we show that Λ is a maximal orthogonal set for the unit cube Q. Suppose to the contrary that this is not the case, and let $s = (u, v, w) \in \mathbb{R}^3$ be a point not

belonging to Λ, such that $\Lambda \cup \{s\}$ is still an orthogonal set. In other words, this means that we have $s - \Lambda \subset G$.

In particular, we have $s - B \subset G$, hence by Lemma 3.2 at least one of the coordinates u, v, w is zero. Let us consider the case where $u = 0$ (the other two cases are similar). We then have $(0, v, w) - (0, \beta - k, \gamma) \in G$ for every nonzero integer k. This implies that either $w - \gamma$ is a nonzero integer, or that $v = \beta$.

If $v = \beta$, then $s = (0, \beta, w)$. We now have $(0, \beta, w) - (\alpha, 0, \gamma - k) \in G$ for every nonzero integer k. This implies that $w = \gamma$. We thus obtain $s = (0, \beta, \gamma)$, but this contradicts the condition that $s - (\alpha - k, \beta, 0) \in G$ for every nonzero integer k. This case cannot therefore happen.

Hence we must have $w = \gamma - l$ where l is a nonzero integer. Then $s = (0, v, \gamma - l)$. The condition $s - (\alpha, 0, \gamma - l) \in G$ now implies that $v = m$ for some nonzero integer m. We therefore obtain that $s = (0, m, \gamma - l)$, but this again contradicts the condition that $s - (\alpha - k, \beta, 0) \in G$ for every nonzero integer k. Hence this case cannot happen either.

This shows that Λ is indeed a maximal orthogonal set for the unit cube.

4. A "THIN" MAXIMAL SET IN THREE DIMENSIONS

4.1.   The maximal incomplete orthogonal set constructed in Section 3 is, in a sense, almost as large as a spectrum. Indeed, this set contains all integer vectors except those lying in the union of the three coordinate planes, and in particular, the set has the maximum possible density.

In this section we give an alternative construction which, somewhat surprisingly, produces a much thinner set.

Theorem 4.1. *There is an orthogonal set $\Lambda \subset \mathbb{R}^3$ for the unit cube, which is maximal but contained in a finite union of planes (and, as a consequence, is incomplete).*

4.2.   Fix again three real numbers α, β, γ such that none of them is an integer. Let Λ be the set in $\mathbb{R}^3$ consisting of all points that have one of the following forms:

$$(0, 0, 0), \quad (n, \beta - k, \gamma), \quad (\alpha, n, \gamma - k), \quad (\alpha - k, \beta, n) \quad \textbf{(4.1)}$$

where n and k are nonzero integers.

It is straightforward to check that $\Lambda - \Lambda \subset G \cup \{0\}$, hence Λ is an orthogonal set for the unit cube Q in $\mathbb{R}^3$. On the

other hand, Λ is contained in the union of three planes plus the origin, hence $Q + \Lambda$ is not a tiling and Λ is not a spectrum for Q.

4.3.   On the other hand we now claim that Λ is a maximal orthogonal set for the unit cube. Indeed, if not then there is a point $s = (u, v, w) \in \mathbb{R}^3$ not belonging to Λ, such that $\Lambda \cup \{s\}$ is still an orthogonal set. In other words, this means that $s - \Lambda \subset G$.

The condition $(u, v, w) - (0, 0, 0) \in G$ implies that at least one of u, v, w is a nonzero integer. Let us consider the case where $u = j$ for some nonzero integer j (the other two cases are similar). We then have $(j, v, w) - (j, \beta - k, \gamma) \in G$ for every nonzero integer k. This implies that either $w - \gamma$ is a nonzero integer, or that $v = \beta$.

If $v = \beta$, then $s = (j, \beta, w)$. We have $(j, \beta, w) - (\alpha, n, \gamma - k) \in G$ for every two nonzero integers n and k. This implies that $w = \gamma$. We thus obtain $s = (j, \beta, \gamma)$, but this contradicts the condition that $s - (\alpha - k, \beta, n) \in G$ for every two nonzero integers n and k. So this case cannot happen.

Hence we must have $w = \gamma - l$ where l is a nonzero integer. Then $s = (j, v, \gamma - l)$. The condition $s - (\alpha, n, \gamma - l) \in G$ for every nonzero integer n now implies that $v = 0$. We therefore obtain that $s = (j, 0, \gamma - l)$, but this again contradicts the condition that $s - (\alpha - k, \beta, n) \in G$ for every two nonzero integers n and k. It follows that this case cannot happen either.

This shows that Λ is indeed a maximal orthogonal set for the unit cube.

4.4.   We leave open the following question, which arises naturally.

Problem 4.2. *Does there exist a maximal orthogonal set $\Lambda \subset \mathbb{R}^3$ for the unit cube, such that Λ is contained in a union of finitely many lines?*

5. EXTENSIONS TO HIGHER DIMENSIONS

We will now use the results of Sections 3 and 4 in order to construct examples of maximal incomplete orthogonal sets for the cube also in dimensions greater than 3. We will present two methods that allow us to use lower-dimensional examples in order to produce examples in higher dimensions.

5.1.   Suppose that $\Lambda \subset \mathbb{R}^n$ is a maximal and incomplete orthogonal set for the unit cube in $\mathbb{R}^n$. We will show how to use this set Λ in order to construct a set $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ which is a maximal and incomplete orthogonal set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$.

We take $\Gamma = A \cup B$ to be the union of two disjoint sets A and B, defined as follows. Let A be the set of all vectors in $\mathbb{R}^n \times \mathbb{R}^m$ of the form $(\lambda, 0)$ where $\lambda \in \Lambda$. We also let B be the set of vectors $(p, q) \in \mathbb{Z}^n \times \mathbb{Z}^m$ such that the vector q has at least one nonzero coordinate, i.e. q is not the zero vector.

It is clear that A and B are indeed two disjoint sets. It is also straightforward to check that their union $\Gamma = A \cup B$ is an orthogonal set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 5.1. *If $\Lambda \subset \mathbb{R}^n$ is a maximal and incomplete orthogonal set for the unit cube in $\mathbb{R}^n$, then the set Γ constructed above is a maximal and incomplete orthogonal set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$.*

Proof. Let us first show that Γ is maximal. Suppose to the contrary that this is not the case, so there is a point $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ which is orthogonal to both A and B. If v is the zero vector, then orthogonality to A implies that the vector u is orthogonal to Λ (with respect to the unit cube in $\mathbb{R}^n$) which contradicts the maximality of Λ. Hence this cannot happen, so v must be a nonzero vector. We now define a vector $q = (q_1, \ldots, q_m)$ by taking the coordinate q_j to coincide with the corresponding coordinate v_j of v if it is an integer, and otherwise we let q_j be an arbitrary nonzero integer. This implies that q is a nonzero vector in $\mathbb{Z}^m$ which is not orthogonal to v (with respect to the unit cube in $\mathbb{R}^m$). But now (u, v) is orthogonal to all vectors of the form (p, q), $p \in \mathbb{Z}^n$, since these vectors are in B. However this is possible only if u is orthogonal to all vectors $p \in \mathbb{Z}^n$, which contradicts the completeness of the spectrum $\mathbb{Z}^n$ of the unit cube in $\mathbb{R}^n$. This shows that Γ is indeed maximal.

It remains to show that Γ is not complete. Indeed, if it was complete then Γ would be a tiling set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$, again due to the result in [LRW00], [IP98], [Kol00b]. In turn this would imply that Λ is a tiling set for the unit cube in $\mathbb{R}^n$. As a consequence, Λ would be a spectrum, contrary to our assumption. $\qquad\square$

5.2.   Another possibility for using lower-dimensional examples in order to construct examples in higher dimensions, is to utilize cartesian products.

Theorem 5.2. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two maximal orthogonal sets for the unit cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then the cartesian product $A \times B$ is a maximal orthogonal set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$. Moreover, $A \times B$ is a complete set if and only if both A and B are complete sets.*

Proof. It is straightforward to check that $A \times B$ is an orthogonal set for the unit cube in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose to the contrary that $A \times B$ is not maximal, and let $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ be orthogonal to all the points of $A \times B$. The maximality of A implies the existence of a point $a \in A$ which is not orthogonal to u. Similarly, by the maximality of B there is $b \in B$ which is not orthogonal to v. It follows that (u, v) is not orthogonal to the point (a, b) which belongs to $A \times B$, a contradiction. Hence $A \times B$ must be a maximal set.

Finally, it is well known, see e.g. [JP99, Theorem 4], that if both A and B are complete sets, then also $A \times B$ is complete. The converse is also true, see [JP99, Lemma 2]. $\square$

5.3.   It is natural to pose the following question, which generalizes Problem 4.2. Let us define the *affine dimension* of a discrete set $\Lambda \subset \mathbb{R}^d$ to be the smallest integer k such that Λ can be covered by a finite number of translated k-dimensional subspaces.

Problem 5.3. *How small can be the affine dimension of a maximal orthogonal set for the unit cube in $\mathbb{R}^d$?*

If we write $d = 3q + r$ where $r \in \{0, 1, 2\}$, then the set $\Gamma = \Lambda \times \cdots \times \Lambda \times \mathbb{Z}^r$ is a maximal incomplete orthogonal set for the unit cube in $\mathbb{R}^d$, where $\Lambda \subset \mathbb{R}^3$ is the set from Theorem 4.1, which appears q times in the product. We observe that this set Γ is contained in a finite union of translated subspaces of dimension $k = 2q + r$.

6. Some general properties of maximal orthogonal sets

6.1.   In the previous sections, we have constructed examples of maximal incomplete orthogonal sets for the unit cube in $\mathbb{R}^d$, $d \geqslant 3$. Moreover, we have seen examples of

rather "thin" maximal orthogonal sets. A natural question arises as to how small can these sets be. The following proposition gives a necessary condition on a maximal set, which in particular shows that it cannot be "too small" in a sense.

Theorem 6.1. *Let $\Lambda \subset \mathbb{R}^d$ be a maximal orthogonal set for the unit cube. Then for every point $(\lambda_1, \lambda_2, \ldots, \lambda_d) \in \Lambda$ and for every integer n, the number $\lambda_1 + n$ must appear as the first coordinate of some point from Λ. Similarly, $\lambda_2 + n$ must appear as the second coordinate of some point from Λ, and so on.*

Proof. By translating the set Λ we may assume that the point $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ lies at the origin. By symmetry it also suffices to prove that every integer n must appear as the first coordinate of some point from Λ.

Suppose to the contrary that n is an integer which does not appear as the first coordinate of any point from Λ. We then claim that the point $\xi = (n, 0, 0, ..., 0)$ is orthogonal to all points of Λ, which contradicts the maximality of Λ. Indeed, n must be a nonzero integer and hence ξ is orthogonal to the origin. Now let μ be any point of Λ other than the origin. Then μ is orthogonal to the origin and so at least one of the coordinates of μ is a nonzero integer. If this coordinate is not the first one then ξ is orthogonal to μ and we are done. Otherwise the first coordinate of μ is a (nonzero) integer, which cannot be equal to n by assumption. This once again implies that ξ is orthogonal to μ, and so the assertion is established. $\square$

As an immediate consequence of Theorem 6.1 we obtain:

Corollary 6.2. *Any maximal orthogonal set for the unit cube must be an infinite set.*

Moreover, it follows that a maximal orthogonal set cannot be "too localized" in space. Specifically, consider an axis-aligned slab of width one, i.e. a set of the form

$$S_j(a) = \{(x_1, x_2, \ldots, x_d) : a \leqslant x_j \leqslant a+1\}, \quad 1 \leqslant j \leqslant d, \quad a \in \mathbb{R}. \tag{6.1}$$

Since any closed interval of length 1 contains an integer, Theorem 6.1 implies:

Corollary 6.3. *A maximal orthogonal set for the unit cube must intersect any axis-aligned slab of width one.*

6.2. Assume now that Λ is a *finite* orthogonal set for the unit cube in $\mathbb{R}^d$. Then Λ is not maximal (Corollary 6.2). A standard application of Zorn's lemma shows that Λ can always be embedded as a subset of a maximal orthogonal set. Is it always possible to embed Λ as a subset of an orthogonal and complete set (i.e. a spectrum) of the cube?

Theorem 2.1 tells us that the answer is 'yes' in two dimensions. Let us show that it is 'no' in dimensions 3 and higher:

Theorem 6.4. *There exists a finite orthogonal set for the unit cube in $\mathbb{R}^d$, $d \geqslant 3$, which cannot be embedded as a subset of any spectrum of the cube.*

Proof. By our previous results there exists a maximal but incomplete orthogonal set Λ for the unit cube. For each n define $\Lambda_n := \Lambda \cap [-n, n]^d$. The set Λ_n is finite for every n. We claim that there is n such that the set Λ_n cannot be embedded as a subset of any spectrum. Indeed, if this is not the case then each Λ_n is contained in some spectrum Γ_n of the cube. It is known, see [GL16, Section 3], [Kol16], that there exists a weakly convergent subsequence Γ_{n_j} whose weak limit Γ is also a spectrum for the cube. We have $\Lambda_m \subset \Lambda_{n_j} \subset \Gamma_{n_j}$ for $n_j > m$, and letting $j \to +\infty$ it follows that the weak limit Γ contains the set Λ_m. Since m is arbitrary we conclude that $\Lambda \subset \Gamma$, that is, Λ can be embedded in a spectrum, in contradiction to its maximality and incompleteness. $\qquad\square$

7. SPECTRAL SETS OTHER THAN THE CUBE

As we saw in the introduction, in the case of the cube in dimension 1 (i.e. the unit interval), any maximal orthogonal set of frequencies necessarily forms a spectrum. A natural question arises whether this is true for any spectral set $\Omega \subset \mathbb{R}$. Below, we utilize an idea of the previous sections to show that this is not the case.

Theorem 7.1. *There exists a spectral set $H \subset \mathbb{R}$ (a union of finitely many intervals with integer endpoints) which admits a maximal but incomplete orthogonal set.*

We motivate the concrete example given below by a short informal discussion. We are looking for a set $H \subset \mathbb{R}$ such that H is spectral, but there exists a maximal orthogonal set of exponentials in $L^2(H)$ which is not complete. As Fuglede's conjecture is still open in $\mathbb{R}$, we may

as well assume that H tiles $\mathbb{R}$ by translations. If so, it is known that any tiling of $\mathbb{R}$ by a bounded region H is periodic and is essentially equivalent to a tiling of $\mathbb{Z}$ (see [LW96]). Furthermore, any tiling of $\mathbb{Z}$ is known to arise as a periodic extension of a tiling of a finite cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ (see [New77]). It is therefore natural to look for the set H in the form $H = \cup_{j=1}^{k}[h_j, h_j + 1)$, where the set $H_0 = \{h_1, \ldots, h_k\} \subset \{0, 1, 2, \ldots, N - 1\}$ is a tile (and a spectral set) in the group $\mathbb{Z}_N$ (where we have identified $\mathbb{Z}_N$ with the set $\{0, 1, 2, \ldots, N - 1\}$ in the natural way).

However, we note that if the cardinality of the set H_0 is a prime number, then the structure theory of tiling of finite abelian groups indicates that any maximal orthogonal set for H_0 is necessarily a spectrum. Presumably the same holds if the number of elements of the set H_0 is a prime power, or even a product of two prime powers, see [CM99]. This suggests that we should be looking for a more complicated example.

In the concrete choice of N and the set $H_0 \subset \mathbb{Z}_N$ we shall therefore mimic the "thin" example of Section 4, given in dimension 3 for the cube. Let $p < q < r$ be distinct odd primes, and let $N = p^2 q^2 r^2$. Then $\mathbb{Z}_N$ is isomorphic to the product group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$, with a group isomorphism being the mapping $\phi : \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2} \to \mathbb{Z}_N$ given by

$$\phi(a, b, c) = q^2 r^2 a + p^2 r^2 b + p^2 q^2 c \tag{7.1}$$

(clearly this mapping is a homomorphism, and one can easily check that it has a trivial kernel, hence it is an isomorphism).

Let us now consider the "discrete cube" $H_0 \subset \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$ given by

$$H_0 = \{(a, b, c) : \ 0 \leqslant a \leqslant p-1, \ 0 \leqslant b \leqslant q-1, \ 0 \leqslant c \leqslant r-1\}. \tag{7.2}$$

The Fourier transform $\widehat{\mathbb{1}}_{H_0}$ is a function on the dual group, which we can also identify with $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$. The zeros of $\widehat{\mathbb{1}}_{H_0}$ are then the vectors whose first coordinate is a nonzero multiple of p, or the second coordinate is a nonzero multiple of q, or the third coordinate is a nonzero multiple of r. As a consequence, H_0 is a spectral set in the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$ and the set $\Gamma_0 = \{(u, v, w) : p|u, \ q|v, \ r|w\}$ serves as a spectrum for H_0. Indeed, Γ_0 is an orthogonal set of frequencies for H_0 and it has the same cardinality as H_0 (see also [AGK18]).

We will now exhibit a maximal but incomplete set of frequencies for H_0. We let $\Lambda_0 \subset \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$ be the set of all vectors that have one of the following forms:

$$(0,0,0), \quad (n, 1-k, 1), \quad (1, k, 1-m), \quad (1-n, 1, m) \quad \textbf{(7.3)}$$

where n is a nonzero multiple of p, k is a nonzero multiple of q and m is a nonzero multiple of r. (Notice that these elements of Λ_0 are analogous to those in (4.1).) It is then straightforward to check that Λ_0 forms an orthogonal set of frequencies for H_0, and one can repeat the proof of Section 4 verbatim to show that Λ_0 is maximal. It is also clear that Λ_0 is not a spectrum of H_0, since we have $|H_0| = pqr$ while

$$|\Lambda_0| < pq + qr + rp < 3qr \leqslant pqr. \quad \textbf{(7.4)}$$

Via the isomorphism (7.1) we deduce that $\phi(H_0) \subset \mathbb{Z}_N$ is a spectral set which has a maximal, incomplete set of frequencies Λ_0'. We view Λ_0' as a subset of $\mathbb{Z}_N$ via the identification of $\mathbb{Z}_N$ with its dual group.

Finally, let $H \subset \mathbb{R}$ be the set defined by $H = \cup_{h \in \phi(H_0)}[h, h+1)$, or equivalently, $\mathbb{1}_H = \mathbb{1}_{\phi(H_0)} * \mathbb{1}_{[0,1)}$. This implies that the zero set of the Fourier transform $\widehat{\mathbb{1}}_H$ satisfies

$$\left(\{\widehat{\mathbb{1}}_H = 0\} \cup \{0\}\right) \cap \frac{1}{N}\mathbb{Z} = \frac{1}{N}\left(\{\widehat{\mathbb{1}}_{\phi(H_0)} = 0\} \cup \{0\}\right) + \mathbb{Z}, \quad \textbf{(7.5)}$$

where we view $\{\widehat{\mathbb{1}}_{\phi(H_0)} = 0\}$ as a subset of $\{0, 1, 2, \ldots, N-1\}$. Define also

$$\Lambda = \frac{1}{N}\Lambda_0' + \mathbb{Z} \quad \textbf{(7.6)}$$

and observe that this is an orthogonal set for H because of (7.5). For the same reason we have that if Γ_0' is a spectrum of $\phi(H_0)$ in $\mathbb{Z}_N$ then $\Gamma = \frac{1}{N}\Gamma_0' + \mathbb{Z}$ is a spectrum of H in $\mathbb{R}$. The orthogonality of Γ follows from (7.5) and the completeness follows from (a) having the right density and (b) being periodic, as (a) and (b) together imply that the "packing" $|\widehat{\mathbb{1}}_H|^2 + \Gamma$ is actually a tiling (see [Kol24, Section 2]).

It remains to show the maximality of Λ (the incompleteness of Λ is guaranteed since its density is too small). Assume that $\Lambda \cup \{x\}$ is an orthogonal set of frequencies for H, where $x \in \mathbb{R} \setminus \Lambda$. By the $\mathbb{Z}$-periodicity of Λ we may assume that $x \in [0, 1)$. Now, if we knew that $x = k/N$ for some $k \in \{0, 1, 2, \ldots, N-1\} \setminus \Lambda_0'$, then again by (7.5) this would imply that $\Lambda_0' \cup \{k\}$ is an orthogonal set of frequencies for $\phi(H_0)$ in $\mathbb{Z}_N$, which contradicts the maximality of

Λ_0'. To conclude the proof we therefore need to rule out the possibility that x is a real number in $[0, 1)$ which is not of the form k/N.

To this end we will need the following lemma.

Lemma 7.2. *If $H_0 \subset \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_{r^2}$ is the set defined by (7.2) and ϕ is given by (7.1) then, viewing the set $\phi(H_0) \subset \{0, 1, 2, \ldots, N-1\}$ as a subset of the integer group $\mathbb{Z}$, the Fourier transform of its indicator function (viewed as a function on the dual group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) has no zeros outside the set*

$$\frac{1}{p^2 q^2 r^2} \mathbb{Z}. \tag{7.7}$$

Proof. We must show that if $\xi \in \mathbb{R}$ and $\widehat{\mathbb{1}}_{\phi(H_0)}(\xi/N) = 0$ then $\xi \in \mathbb{Z}$.

Let $A = \{0, 1, \ldots, p-1\}$, $B = \{0, 1, \ldots, q-1\}$ and $C = \{0, 1, \ldots, r-1\}$ viewed as subsets of the integers. We make the following observation: if we interpret the expression (7.1) as an integer (and not as an element of $\mathbb{Z}_N$), then all the values attained by this expression for $a \in A$, $b \in B$ and $c \in C$ lie in the subset $\{0, 1, 2, \ldots, N-1\}$ of the integers. Indeed, all these values are nonnegative and not greater than

$$
\begin{aligned}
q^2 r^2 p + p^2 r^2 q + p^2 q^2 r &= pqr(qr + pr + pq) \\
&< pqr(3qr) \\
&\leqslant p^2 q^2 r^2 \\
&= N.
\end{aligned}
$$

This means that as the triple (a, b, c) goes through the elements of $A \times B \times C$, then the expression (7.1) not only goes through the elements of $\phi(H_0)$ viewed as congruence classes modulo N, but it also goes through the representatives in $\{0, 1, 2, \ldots, N-1\}$ (as a subset of $\mathbb{Z}$) of the elements of the set $\phi(H_0)$.

This observation allows us to compute the Fourier transform $\widehat{\mathbb{1}}_{\phi(H_0)}(\xi/N)$. Assume that $\xi \in \mathbb{R}$ is not an integer, then we have

$$\widehat{\mathbb{1}}_{\phi(H_0)}(\xi/N) = \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \exp(-2\pi i \phi(a,b,c)\xi/N) \qquad (7.8)$$

$$= \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \exp(-2\pi i (q^2 r^2 a + p^2 r^2 b + p^2 q^2 c)\xi/N) \tag{7.9}$$

$$= \sum_{a=0}^{p-1} e^{-2\pi i a \xi/p^2} \sum_{b=0}^{q-1} e^{-2\pi i b \xi/q^2} \sum_{c=0}^{r-1} e^{-2\pi i c \xi/r^2} \tag{7.10}$$

$$= \frac{1 - e^{-2\pi i \xi/p}}{1 - e^{-2\pi i \xi/p^2}} \cdot \frac{1 - e^{-2\pi i \xi/q}}{1 - e^{-2\pi i \xi/q^2}} \cdot \frac{1 - e^{-2\pi i \xi/r}}{1 - e^{-2\pi i \xi/r^2}}, \tag{7.11}$$

where the observation made above was used in (7.8) and (7.9). It remains to notice that the quantity in (7.11) is nonzero, as otherwise ξ would have to be an integer divisible by p, q or r. Hence the lemma is established. $\qquad \square$

Finally, we use the lemma to finish the proof of the maximality of Λ. Recall that we have assumed that $\Lambda \cup \{x\}$ is an orthogonal set of frequencies for H, where x is a real number in $[0,1) \setminus \Lambda$. Since $0 \in \Lambda$, Lemma 7.2 allows us to conclude that $x = k/N$ for some $k \in \{1, 2, \ldots, N-1\} \setminus \Lambda_0'$. As we have seen, this implies that $\Lambda_0' \cup \{k\}$ is an orthogonal set of frequencies for $\phi(H_0) \subset \mathbb{Z}_N$, contradicting the maximality of Λ_0'.

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