CS476/676: Numerical Computation for Financial Modeling

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Backgrounds:

1. Programming (CS370/CS371)
2. Calculus and Linear Algebra
3. Statistics
4. No finance background is assumed.

Outline:

1. Basic financial derivatives: Definitions and uses
2. Financial derivative theory (Low arbitrarge pricing)
3. Stochastic Asset Pricing Model:
   - Discrete Time/State Binomial Model
   - Continuous Time Model
4. Mathematics:
   - Stochastic Differential Equation (SDE)
   - Itô's calculus
   - Partial Differential Equations
   - Optimisation Theory
5. Computation and Algorithm
   - Binomial Lattice Option Pricing
   - Monte Carlo Methods
   - Dynamic programming

Notation: Italic letters are non-random and upright letters are random variables (e.g. x vs. x). This is consistent with *Deep Learning Book*.

"*Everything we do is underpinned by math and software*" – CFO of Goldman Sachs

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Caput 1

Financial Derivative Theory

Two things in finance: Value and Risk.

Definition

Risk is uncertainty/randomness in market price.

Risk assessment is a major agenda of governments. Computing is key to assessing the performance of dynamic trading strategies and valuation of financial instruments.

Key component: Derivative market.

Note. Options have a long history. Thales (Mathematician and Astronomer) in 332 BC observed the stars on the sky and predicted olive harvest based on the stars. When the harvest is predicted to be good, the demand of olive press will be high.

The nature of Thales' scheme is a contract with olive-press owners. Thales gives the olive press owners a fixed amount of money and then the owner lends Thales the olive-presses a few months after the contract is signed.

The fixed amount of money here is the price of this financial derivative.

In the 17th century, Holland had a *Tulip Mania*. The price of Tulip has bubbled to extremely high prices but the bubble eventually bursted. Tulip growers during the mania anticipates uncertainties in the price of Tulip.

Another type of business during Tulip Mania is reselling Tulips.

Option trading is a risky business. From 1733 – 1860, it was illegal in Britain to trade derivatives.

The pricing of options is not solved until 1973 when the Black-Scholes formula won a Nobel Memorial Prize in Economics.

1.1 European Options

Definition

A **financial derivative/option** is a financial contract whose value at the future expiry time T is determined exactly by the market price of an underlying asset at T.

The unit of T is year.

An option consists of:

- A underlying price S (tulip, stock, index)
- Expiry time T
- $\mathsf{payoff}(S_T)$: Payoff based on the future price S_T of the asset.

The value of the option at the expiry time is $V_T = \mathsf{payoff}(S_T)$. The question of option pricing is to evaluate V_0, the value of the option at the present time.

The seller of the option is the **writer** and the buyer is the **holder**. In essence, the holder buys insurance, and the writer buys risk.

Definition

A **European call** option is the right (which can be or not be exercised) to *buy* an underlying asset at the present strike price K. The right can only be exercised at the expiry time T.

A **European put** option gives is the right to *sell* an underlying asset at a preset strike price K. The right can only be exercised at expiry.

- The holder enters a **long** position.
- The writer enters a **short** position.

There is an asymmetry in this configuration. The holder has the option to exercise the right of the option, but the writer does not. Let $\mathrm{S}(t)$ denote the underlying price at time t. The function S is a **Stochastic process**.

Let $\mathrm{V}(t)$ ro $\mathrm{V}(\mathrm{S}(t), t)$ denote the option value at time t.

Example: Tulip Pricing

Consider an European call option with price $K = 0.5$. At expiry T, the underlying price $S(T) = 1$. In this case, $S(T) > K$, so the holder exercises the option.

However, if $S(T) = 0.2$, then the holder would not exercise the option, rendering the option worthless.

The value of a European call option at expiry time is:

$$V(T) = \max(S(T) - K, 0) = \begin{cases} S(T) - K & \text{if } S(T) > K \\ 0 & \text{if } S(T) \leq K \end{cases}$$

The value of a European put option is

$$V(T) = \max(K - S(T), 0)$$

This value is the **payoff**. The option specifies the payoff function payoff.

We focus on stock option. We assume $T \leq 1$ year (this allows us to ignore the variation in interest rate)

Definition

A **stock** is a share in the ownership of a company.

A stock may pay **dividend** to stock holder from the profits.

Dividend is usually specified as a proportion of the stock price. Note: The holder of a stock option does not receive dividend.

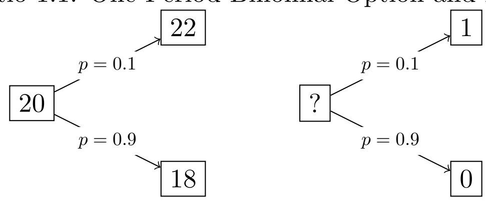
The question of option pricing: Given

- Expiry time T
- $S(0)$
- Payoff function payoff

Evaluate $V(0)$.

- Following Sage's realisation, focus on determine the relationship between S and V at any time t and any S, i.e. $V(\mathrm{S}, t)$.
- If this can be done, today's option value $V(0) = V(S(0), 0)$ is known.

Descriptio 1.1: One-Period Binomial Option and its value



Note. The hundred year discovery of option pricing:

1. In Louis Bachelier's PhD thesis, $\mathrm{S}(t)$ is modelled as a random walk.
2. In 1950, Samuelson modeled $\mathrm{S}(t)$ using log-normal random walk and solved the negative price problem.
3. In 1970, Merton, Black-Scholes formula.

1.2 1-Period Binomial Model

Assume underlying asset behaves in the following way:

- $S(0) = 20$
- Assume $T = 1$
- Assume that one of the following happens at $t = T$:

$$\begin{cases} P(\mathrm{S}(T) = 22) & = 0.1 \\ P(\mathrm{S}(T) = 18) & = 0.9 \end{cases}$$

Consider the call with $K = 21$ and expiry $T = 1$. The value of this option at $t = T$ is

$$\begin{cases} P(\mathrm{V}(T) = 1) & = 0.1 \\ P(\mathrm{V}(T) = 0) & = 0.9 \end{cases}$$

A naïve pricing at $t = 0$ is $\hat{V}(0) = 0.1 \cdot 1$, but this is not correct for a number of reasons:

1. There is no interest rate
2. The underlying asset can be traded. The option also can be traded.

1.2.1 Bonds

Definition

A **riskless bond** is a financial instrument whose value $\beta(t)$ at time t

$$\lim_{\delta \to 0} \frac{\beta(t+\delta) - \beta(t)}{\beta(t)\delta} = r$$

where **continuously compounding interest rate** is a rate $r \geq 0$.

The continuous compounding rate can also be expressed as

$$\frac{\mathrm{d}\beta(t)}{\beta(t)} = r \,\mathrm{d}\tau$$

This can also be expressed as an *ordinary differential equation* (ODE)

$$\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = r\beta$$

so $\beta(t) = \beta(0)\mathrm{e}^{rt}$.

When lending (depositing) money to a bank, this is equivalent to buying a bond from the bank. Borrowing money from a bank is equivalent to selling a bond to the bank.

1.2.2 Arbitrage

When pricing an option, we do not want any party to get "free lunch".

Definition

An **arbitrage** is a trading strategy to make a *riskless* rate of return which is greater than the risk-free rate $r \geq 0$.

The **risk-free rate** is the bank overnight lending rate (bank's rate of return).

Note. There is also statistical arbitrage which is not covered in this course.

Definition

The **fair value** of a financial instrument such as an option, is the price which does not lead to arbitrage.

Arbitrage can only occur momentarily. Under no arbitrage, two instruments have the same value at a future time, they must be priced at the same price today.

We represent a *trading strategy* as a **portfolio**, which is a collection of positions (long, short) in a collection of assets. Example: Buy 1 share of a stock and borrow \$100 (selling a bond) today. The profit of the trading strategy is

$$\pi(0) = \underbrace{1}_{\text{long}} \cdot s(0) \quad \underbrace{-100}_{\text{short (selling)}}$$

A long position benefits from increased prices and a short position benefits from decreased prices.

At time $t > 0$,

$$\pi(t) = 1 \cdot s(t) - 100\mathrm{e}^{rt}$$

An *arbitrage strategy* can be described as

- A portfolio with today's value $\pi(0) = 0$ but $\pi(t) > 0$ for $t > 0$.
- A portfolio with today's value $\pi(0) < 0$ but $\pi(t) \geq 0$ for some $t > 0$.

1.2.1 Put-Call Parity.
Assume stock $\mathrm{S}(t)$ *does not pay dividend, interest rate* $r \geq 0$ *constant, and no arbitrage. Then at any time* $0 \leq t \leq T$*, a European call* $C(t)$ *and put* $P(t)$ *with the same strike price and expiry* T *on the same underlying* S*, satisfy*

$$C(t) = P(t) + S(t) - K\mathrm{e}^{-r(T-t)}$$

Proof. At time T, we have

$$C(T) = \max(S(T) - K, 0)$$
$$P(T) = \max(K - S(T), 0)$$

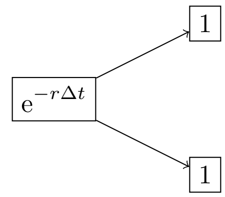
so

$$C(T) - P(T) = \max(S(T) - K, 0) - \max(K - S(T), 0) = S(T) - K$$

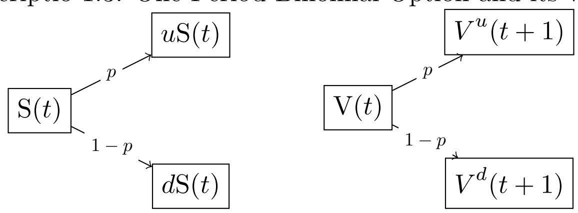
This is a trading strategy expressed in two different ways:

- Left: A long on $\mathrm{C}(T)$ and a short on $\mathrm{P}(T)$

Descriptio 1.2: One-Period Bond and its value



Descriptio 1.3: One-Period Binomial Option and its value



- Right: Holding the share S(T).

The value of S(t) $- K$ at time t is

$$\mathrm{S}(t) - K\mathrm{e}^{-r(T-t)}$$

Hence

$$C(t) - P(t) = S(t) - K\mathrm{e}^{-r(T-t)}$$

□

1.3 Pricing of 1-Period Binomial Model

Assume S(t) > 0 and $0 < d < u$ (market down and up), and $0 < p < 1$ such that

$$\begin{cases} P(\mathrm{S}(t+1) = u\mathrm{S}(t)) & = p \\ P(\mathrm{S}(t+1) = d\mathrm{S}(t)) & = 1-p \end{cases}$$

Assumption:

- Frictionless: No transaction fees during trading
- Short selling: Can sell something not owned.
- Asset is infinitely divisible. (In real markets assets have granularity)
- No arbitrage exists.
  Note that under this assumption, $d \leq \mathrm{e}^{r\Delta t} \leq u$.
- No dividend.
- Risk-free rate $r \geq 0$ is constant.

1.3.1 Proposition.
The value of the option today is

$$V(t) = \delta \cdot S(t) + \eta \cdot \beta(t)$$

where δ, η are the unique coëfficients such that

$$\begin{bmatrix} 1 & uS(t) \\ 1 & dS(t) \end{bmatrix} \begin{bmatrix} \eta \\ \delta \end{bmatrix} = \begin{bmatrix} V^u(t+1) \\ V^d(t+1) \end{bmatrix}$$

and $\beta(t)$ is a risk-free bond with $\beta(t+1) = 1$ (and so $\beta(t) = \mathrm{e}^{-r}$)

Proof. Replication: Consider a linear system

$$\begin{bmatrix}1 & uS(t)\\ 1 & dS(t)\end{bmatrix}\begin{bmatrix}\eta\\ \delta\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}\eta + \begin{bmatrix}uS(t)\\ dS(t)\end{bmatrix}\delta = \begin{bmatrix}V^u(t+1)\\ V^d(t+1)\end{bmatrix}$$

Since we assume $d < u$, this linear system has a unique solution.

- The LHS represents a portfolio of η bonds and δ shares of S.
  $\begin{bmatrix}1\\ 1\end{bmatrix}$ is the bond value at $t+1$ (not t) and $\begin{bmatrix}uS(t)\\ dS(t)\end{bmatrix}$ is the stock value at time $t+1$.
- The RHS is the option value at expiry.

This linear system indicates there exists a unique portfolio trading strategy $\pi(t) = \{\delta \cdot \mathrm{S}(t), \eta \cdot \beta(t)\}$ such that $\pi(t+1) \equiv V_{t+1}$.

No arbitrage indicates that the value of option today must be equivalent to the replicating portfolio,

$$V(t) = \delta \cdot S(t) + \eta \cdot \beta(t)$$

which is the no-arbitrage pricing of the option at present time. □

1.3.2 Corollary.
If the bond price at present time is not $V(t) = \delta \cdot S(t) + \eta \cdot \beta(t)$, an arbitrage trading strategy exists.

Note. A arbitrage strategy generated by Proposition 1.3.1 eliminates the randomness in an option. This is *hedging* (reduces risk).

Notice that the uncertainty p does not appear in the linear equation for η and δ.

In the example for $S(0) = 20$, we have

$$\begin{bmatrix}1 & 22\\ 1 & 18\end{bmatrix}\begin{bmatrix}\eta\\ \delta\end{bmatrix} = \begin{bmatrix}1 & uS(t)\\ 1 & dS(t)\end{bmatrix}\begin{bmatrix}\eta\\ \delta\end{bmatrix} = \begin{bmatrix}V^u(t+1)\\ V^d(t+1)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

which evaluates to $\delta = 0.25, \eta = -4.5$ (borrow), and the call option value is

$$C(0) = \delta \cdot S(0) + \eta \cdot \beta(0) = 0.25 \cdot 20 - 4.5 = 0.5$$

If the price is different, arbitrage exists:

- If $C(0) = 0.95$,

  Position at t is

  $$\pi(0) = -0.95 + 0.25 \cdot 20 - 4.5 = -0.45$$

  i.e. sell call for 0.95, buy 0.25 share, borrow -4.5.

  Then

  $$\pi(T) = -C(T) + 0.25S(T) - 4.5 = 0$$

1.3.1 Risk Neutral Valuation

Consider another linear system:

$$\begin{bmatrix}1\\ u\mathrm{S}(t)\end{bmatrix}\psi^u + \begin{bmatrix}1\\ d\mathrm{S}(t)\end{bmatrix}\psi^d = \begin{bmatrix}\mathrm{e}^{-r\Delta t}\\ \mathrm{S}(t)\end{bmatrix} = \begin{bmatrix}\text{Bond price today}\\ \text{Stock price today}\end{bmatrix} \tag{1.1}$$

This coëfficient is transposed from Proposition 1.3.1.

The first term ψ^u is the "up" state and ψ^d is the "down" state. The $\psi^{u,d}$ coëfficients are weights which relate the future value to today's value.

Solving the system gives

$$\psi^u = \mathrm{e}^{-r\Delta t}q^*, \qquad \psi^d = \mathrm{e}^{-r\Delta t}(1-q^*)$$

where

$$q^* = \frac{e^{r\Delta t} - d}{u - d}$$

No arbitrage assumption implies $0 \le q^* \le 1$.

Substituting into the second row of Equation 1.1,

$$\begin{aligned} S(t) &= e^{-r\Delta t}(q^* \cdot uS(t) + (1 - q^*) \cdot dS(t)) \\ &= e^{-r\Delta t}\mathbb{E}_Q[\mathrm{S}(t+1)] \end{aligned}$$

where $\mathbb{E}_Q$ uses q^* as a probability.

From the first row of Equation 1.1 (note that β is deterministic)

$$\beta(t) = e^{-r\Delta t}\mathbb{E}_Q[\beta(t+1)]$$

Recall that the pricing of 1-period binomial option is

$$V(t) = \delta \cdot S(t) + \eta \cdot \beta(t)$$

Substituting,

$$\begin{aligned} V(t) &= \delta \cdot S(t) + \eta \cdot \beta(t) \\ &= \delta \cdot e^{-r\Delta t}(q^* \cdot uS(t) + (1 - q^*)d(t)) + \eta \cdot e^{-r\Delta t}(q^* \cdot 1 + (1 - q^*) \cdot 1) \\ &= e^{-r\Delta t}q^*(\delta \cdot uS(t) + \eta \cdot 1) + e^{-r\Delta t}(1 - q^*)(\delta \cdot dS(t) + \eta \cdot 1) \\ &= e^{-r\Delta t}q^* V^u(t+1) + e^{-r\Delta t}(1 - q^*)V^d(t+1) \\ &= e^{-r\Delta t}\mathbb{E}_Q[\mathrm{V}(t+1)] \end{aligned}$$

Under no-arbitrage assumption, there exists a unique probability q^* such that today's option value $V(t)$ is the discounted expected value of the future option value using q^*.

Note. p is the probability of actual asset movement. q^* is a probability construct derived from no-arbitrage assumption.

1.3.3 Proposition.
Assume no arbitrage, then there exists a risk neutral probability $q^* = \frac{e^{r\Delta t} - d}{u - d} \in [0, 1]$ *such that*

$$\begin{aligned} \beta(t) &= e^{-r\Delta t} \cdot \mathbb{E}_Q[\beta(t+1)] \\ S(t) &= e^{-r\Delta t} \cdot \mathbb{E}_Q[\mathrm{S}(t+1)] \\ V(t) &= e^{-r\Delta t} \cdot \mathbb{E}_Q[\mathrm{V}(t+1)] \end{aligned}$$

(converse is also true)

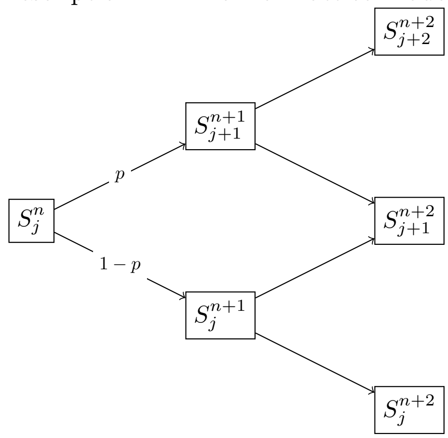
Risk neutral pricing indicates we can compute option value today by taking the expectation of option value in the future. In a world modeled by q^* and $\mathbb{E}_Q$, $\beta, \mathrm{S}, \mathrm{V}$ all grow at the risk-free rate.

Back to the example with $u = 1.1, d = 0.9$, we have

$$q^* = \frac{1 - 0.9}{1.1 - 0.9} = 0.5, \qquad C(0) = 0.5 \cdot 1 + 0.5 \cdot 0 = 0.5$$

This is a very simple model and is unrealistic given the long time interval Δt. However, if we split the time interval into many short periods and assume that the asset price fluctuates to one of two values in each period, the model becomes more realistic.

Descriptio 1.4: Binomial Lattice Model



1.4 Multi-Period Binomial Lattice Model

We divide $[0, T]$ into

$$T_0 = 0 < T_1 < \cdots < T_N = T$$

where $T_n = n\Delta t$ and $\Delta t = \frac{T}{N}$.

In the binomial model, let

$$S_0^0 = S_0, \qquad \mathrm{S}^n \coloneqq \mathrm{S}(T_n)$$

and let S_j^n be the price at T_n with $j = 0, 1, \ldots, n$.

Note. Convention: If only a subscript is used, the subscript denotes time. If both superscript and subscript are used, the superscript denotes time.

From S_j^n in Δt, the price at the next step S^{n+1} satisfy

$$\begin{cases} P(\mathrm{S}^{n+1} = S_{j+1}^{n+1} = u\mathrm{S}_j^n | \mathrm{S}^n = S_j^n) & = p \\ P(\mathrm{S}^{n+1} = S_j^{n+1} = d\mathrm{S}_j^n | \mathrm{S}^n = S_j^n) & = (1-p) \end{cases}$$

Note that the subscript for falling down is j, not $j+1$.

In one-period binomial model, we can replicate an option by holding a mixture of bond and stock. In multi-period binomial model, we must dynamically trade at each time step to ensure the portfolio is a replication of the option.

1.4.1 Logarithmic Return

Questions:

- How to determine u, d, p?
- What happens to binomial lattice if $\Delta t \to 0$?

In finance, instead of price at T_n, we consider the logarithmic price

$$\mathrm{X}(T_n) \coloneqq \log \mathrm{S}(T_n)$$

This can be expressed as

$$\mathrm{X}_{n+1} = \log \mathrm{S}_{n+1} = \log \mathrm{S}_n + \log \frac{\mathrm{S}_{n+1}}{\mathrm{S}_n} = \mathrm{X}_n + \Delta \mathrm{X}_n$$

X is **log return**, and

$$\begin{cases} P(\Delta \mathrm{X}_n = \log u) & = p \\ P(\Delta \mathrm{X}_n = \log d) & = 1-p \end{cases}$$

Logarithmic return is similar to simple return, using Taylor series expansion.

$$\frac{S_{n+1} - S_n}{S_n} \simeq \log \frac{S_{n+1}}{S_n} = \log \left(1 + \frac{S_{n+1} - S_n}{S_n} \right) \simeq \frac{S_{n+1} - S_n}{S_n}$$

In the case for which

$$\begin{cases} P(\Delta X_n = +\Delta h) & = p \\ P(\Delta X_n = -\Delta h) & = 1 - p \end{cases}$$

we have $u = e^{\Delta h}, d = e^{-\Delta h}, u = 1/d$. Therefore a "$u$" step is precisely negated by a "d" step. In this case the binomial lattice **recombine** and does not *drift* (can always return to the origin X_0).

Cumulative logarithmic return on $[0, T_n]$ is

$$X_0 + \sum_{k=0}^{n-1} \Delta X_k$$

Properties:

- ΔX_n has identical distribution for any n.
- $\Delta X_i, \Delta X_j$ are independent.

  At T_n, the next logarithmic price is independent of its history. Hence

  $$\mathbb{E}[\Delta X_i \Delta X_j] = \mathbb{E}[\Delta X_i]\mathbb{E}[\Delta X_j]$$

- $$\begin{aligned} \mu_0 &\coloneqq \mathbb{E}[\Delta X_i] = p\Delta h - (1-p)\Delta h = (2p-1)\Delta h \\ \mathbb{E}[(\Delta X_i)^2] &= p\Delta h^2 + (1-p)\Delta h^2 = \Delta h^2 \\ \sigma_0^2 &\coloneqq \operatorname{var}(\Delta X_i) = \Delta h^2(1 - (2p-1)^2) \end{aligned}$$

- On the entire period $[0, T]$,

  $$X_N = \sum_{n=0}^{N-1} \Delta X_n$$

Definition

A **random walk** or **Markov Process** is a process described by

$$X_0 + \sum_{k=0}^{n-1} \Delta X_k$$

where

- ΔX_n has identical distribution for any n.
- $\Delta X_i, \Delta X_j$ are independent.

Now if we let $N \to \infty$, Central Limit Theorem implies that the resulting lattice has a normal distribution:

$$\frac{\sum_{n=0}^{N-1} \Delta X_n - N\mu_0}{\sigma_0 \sqrt{N}} \to N(0, 1)$$

where the standard normal is given by

$$\Phi(x) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds, \qquad (x \in \mathbb{R})$$

1.4.2 Up-down ratio

How to obtain u, d? Assume the underlying stock has the following statistics:

- The average logarithmic return per year is μ
- The **volatility** (standard deviation of annual return) is σ

Then the expected log return in Δt should be $\mu\Delta t$ and the variance $\sigma\Delta t$.

Require the binomial model statistics to match the numbers:

$$\mathbb{E}[\log \frac{\mathrm{S}(t+\Delta t)}{\mathrm{S}(t)}] = p \log u + (1-p)\log d = \mu\Delta t$$
$$\mathrm{var}(\log \frac{\mathrm{S}(t+\Delta t)}{\mathrm{S}(t)}) = \sigma^2\Delta t$$

Solving for (u, d, p) does not yield unique solutions.

From Cox-Ross-Rubinstein ('79),

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}}, \qquad d = \mathrm{e}^{-\sigma\sqrt{\Delta t}}, \qquad p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t}$$

Note:

- Under this construction, no arbitrage optimal value depends only on volatility, not mean (log) return
- CRR is non-drift
- There are different solutions (u, d, p)

Rolling backwards, we have the algorithm

procedure ROLLBACK
  $V_j^N \leftarrow \mathsf{payoff}(S_j^N)$
  for $n = N-1, \dots, 0$
    for $j = 0, \dots, n$
      $V_j^n = \mathrm{e}^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$
    end for
  end for
  return V_0^0
end procedure

1.4.3 Dividend

Assume that at $t = 0$, it is known that the underlying stock pays a dividend $\mathrm{D} = \rho \cdot \mathrm{S}(t_d\nearrow)$, where ρ is a constant, $0 \le t_d < T$, and $t\nearrow$ is limit from the left. The dividend is paid at time t_d.

No arbitrage implies $\mathrm{S}(t_d\searrow) = \mathrm{S}(t_d\nearrow) - \mathrm{D}$, since the stock must lose value after paying dividend. Paying dividend does not affect option contract. No arbitrage implies $\mathrm{V}(\mathrm{S}(t), t)$ is a continuous function of time.

$$\mathrm{V}(\mathrm{S}(t_d\nearrow), t_d\nearrow) = \mathrm{V}(\mathrm{S}(t_d\searrow), t_d\searrow) = \mathrm{V}(\mathrm{S}(t_d\nearrow) - \mathrm{D}, t_d\searrow)$$

To extend the pricing algorithm for pricing options with dividends, we use

$$V(S_1^1, t_d\nearrow) = V(S_1^1 - \rho \cdot S_1^1, t_d\searrow)$$

We can use linear interpolation from $(S_0^1, V_0^1), (S_1^1, V_1^1)$ to determine the value at (S, V'). Note that the interpolation is valid since Δt is a very small value in practice.

Fix t_d, we know

$$V(S_1^1, t_d\searrow) = f(S_1^1), \qquad V(S_0^1, t_d\searrow) = f(S_0^1)$$

can evaluate $V(S, t_d\searrow) = f(S)$ by ensuring no arbitrage across dividend payments.

procedure PRICE-EUROPEANOPTION

```
    V_j^N ← payoff(S_j^N)
    for n = N − 1, ..., 0
        for j = 0, ..., n
            V_j^n = e^{−rΔt}(q^* V_{j+1}^{n+1} + (1 − q^*) V_j^{n+1})
        end for
        if t_n is dividend date
            Compute V_j^n(t_n↗) by continuity and interpolation
        end if
    end for
    return V_0^0
end procedure
```

1.5 American Options

Definition

An **American call** option is the right (which can be or not be exercised) to *buy* an underlying asset at the present strike price K. The right can be exercised at any discrete time $0 \leq t_n < T$.

An **American put** option gives is the right to *sell* an underlying asset at a preset strike price K. The right can be exercised at any discrete time $0 \leq t_n < T$.

American options are more powerful so their prices are higher than European options.

Pricing American options requires dynamic programming. At each point of the lattice, the option can either be exercised, or not exercised.

```
procedure PRICE-AMERICANOPTION
    V_j^N ← payoff(S_j^N)
    for n = N − 1, ..., 0
        for j = 0, ..., n
            (V_j^n)^* = e^{−rΔt}(q^* V_{j+1}^{n+1} + (1 − q^*) V_j^{n+1})     ▷ Continuation Value, not exercise option
            V_j^n = max((V_j^n)^*, payoff(S_j^n))
        end for
    end for
    return V_0^0
end procedure
```

1.6 Computational Complexity and Continuous Time Model

The computational complexity when pricing an option is $\Theta(N^2)$. The space complexity can be $\Theta(n)$.

It can be shown that the binomial lattice option value converge to the Black-Scholes formula. When $N \to +\infty$, the European call option price converges to

$$C(S, t) = S \cdot \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

where Φ is the CDF of $N(0, 1)$, and

$$d_1 := \frac{\log(S/k) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 := \frac{\log(S/k) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$(T - t)$ is **time to expiry**. There is a corresponding explicit formula for put.

There is no explicit formula for American options.

Caput 2

Stochastic Calculus

2.1 Brownian Motion

Consider a discrete random walk with $X_0 = 0$ and

$$\begin{cases} P(\Delta X_n = +\sqrt{\Delta t}) & = \frac{1}{2} \\ P(\Delta X_n = -\sqrt{\Delta t}) & = \frac{1}{2} \end{cases}$$

Now we let $\Delta t \to 0$. This gives the standard Brownian motion.

Definition

The **Standard Brownian Motion** is the limit $Z(t)$ of X_n with $\Delta t \to 0$ with the properties:

- $Z(0) = 0$
- For all $t \geq 0$ and $\Delta t > 0$,
$$Z(t + \Delta t) - Z(t) \sim N(0, \Delta t)$$
- For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, $Z(t_2) - Z(t_1), Z(t_4) - Z(t_3)$ are independent.

Note that $Z(t) \sim N(0, t)$. The Markovian property is constant with the market efficient assumption.

Definition

An **Itô Process** is a function

$$dX(t) = a \cdot dt + b \cdot dZ(t)$$

where a, b are stochastic processes.

The **Black-Scholes Model** is a *stochastic differential equation* (SDE)

$$\frac{dS(t)}{S(t)} = \mu \, dt + \sigma \, dZ(t)$$

where μ, σ are constants.

Notice that the Black-Scholes model is a Itô process with $a = \mu S(t), b = \sigma S(t)$.

Recall the bond ODE has a solution

$$\log \beta(t) - \log \beta(0) = \int_0^t d\beta(t) = rt$$

Assume ϕ_t is standard normal, the definition of Brownian motion implies

$$\Delta Z(t) = \phi(t) \cdot \sqrt{\Delta t}$$

When $\Delta t \to 0$,

$$\mathrm{dZ}(t) = \phi(t) \cdot \sqrt{\mathrm{d}t}$$

$\mathrm{Z}(t)$ is continuous but not differentiable.

Recall the moments of standard normal:

$$\mathbb{E}[\phi(t)] = 0, \qquad \mathbb{E}[\phi(t)^2] = 1, \qquad \mathbb{E}[\phi(t)^3] = 0, \qquad \mathbb{E}[\phi(t)^4] = 3$$

and $\Delta\mathrm{Z}(t) = \phi(t)\sqrt{\Delta t}$, so

$$\begin{aligned}\mathbb{E}[\Delta\mathrm{Z}(t)^2] &= \mathbb{E}[\phi(t)\sqrt{\Delta t}] = \Delta t \\ \operatorname{var} \Delta\mathrm{Z}(t)^2 &= \mathbb{E}[\Delta\mathrm{Z}(t)^4] - \mathbb{E}[\Delta\mathrm{Z}(t)^2]^2 = 3(\Delta t)^2 - (\Delta t)^2 = \mathsf{O}((\Delta t)^2)\end{aligned}$$

Therefore $\operatorname{var} \Delta\mathrm{Z}(t)^2$ goes to zero quadratically, and so $(\mathrm{dZ}(t))^2$ is deterministic, and thus

$$(\mathrm{dZ}(t))^2 = \mathrm{d}t$$

2.2 Itô's Integral

In classical calculus,

$$\int_0^T f(t)\,\mathrm{d}t = \lim_{N\to+\infty} \sum_{n=0}^{N-1} f(t_n)\Delta t$$

Definition

Let f, Z be Stochastic Processes, the **Itô's Integral** of f w.r.t. Z is

$$\int_0^T \mathrm{f}(t)\,\mathrm{dZ}(t) = \lim_{N\to+\infty} \left(\sum_{n=0}^{N-1} \mathrm{f}(t_n) \cdot (\mathrm{Z}(t_{n+1}) - \mathrm{Z}(t_n)) \right)$$

where the limit is taken in the mean square sense, i.e.

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{n=0}^{N-1} \mathrm{f}(t_n) \cdot (\mathrm{Z}(t_{n+1}) - \mathrm{Z}(t_n)) \right) - \int_0^T \mathrm{f}(t)\,\mathrm{dZ}(t) \right] = 0$$

Note that $\sum_{n=0}^{N-1} \mathrm{f}(t_n) \cdot (\mathrm{Z}(t_{n+1}) - \mathrm{Z}(t_n))$ is random.

The integral can also be written as $\sum_{n=0}^{N-1} \mathrm{f}(t_{n+1}) \cdot (\mathrm{Z}(t_{n+1}) - \mathrm{Z}(t_n))$, but this sum is no longer causal and requires knowledge of the future.

How does the option value $\mathrm{V}(\mathrm{S}, t)$ change if

$$\mathrm{dS}(t) = a\mathrm{S}(t)\,\mathrm{d}t + \sigma\mathrm{S}(t)\,\mathrm{dZ}(t)$$

Assume that

$$\mathrm{dS}(t) = a(\mathrm{S}, t)\,\mathrm{d}t + b(\mathrm{S}, t)\,\mathrm{dZ}(t)$$

For Black-Scholes Model,

$$a(\mathrm{S}(t), t) = \mu\mathrm{S}(t), \qquad b(\mathrm{S}(t), t) = \sigma\mathrm{S}(t)$$

2.2.1 Itô's Lemma.

Let $G(s,t) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and $\mathrm{Y}(t) = G(\mathrm{S}, t)$ an Itô process.

Then

$$\mathrm{dY}(t) = \left(\frac{\partial G}{\partial t} + a(\mathrm{S}, t)\frac{\partial G}{\partial \mathrm{S}} + \frac{1}{2}\left(b(\mathrm{S}, t)^2 \frac{\partial^2 G}{\partial \mathrm{S}^2} \right) \right) \mathrm{d}t + b(\mathrm{S}, t)\frac{\partial G}{\partial \mathrm{S}}\,\mathrm{dZ}(t)$$

Proof. Using Taylor expansion, for $\Delta t > 0$,

$$\begin{aligned}
\Delta \mathrm{Y}(t) &= G(\mathrm{S}(t) + \Delta\mathrm{S}(t), t + \Delta t) - G(\mathrm{S}(t), t) \\
&= \frac{\partial G}{\partial \mathrm{S}} \Delta\mathrm{S}(t) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial \mathrm{S}^2}\Delta\mathrm{S}(t)^2 + \frac{\partial^2 G}{\partial \mathrm{S}\partial t}(\Delta S(t))\Delta t + \mathsf{O}(\Delta t^2)
\end{aligned}$$

Notice that

$$\begin{aligned}
(\mathrm{dS}(t))^2 &= (a\,\mathrm{d}t + b\mathrm{Z}(t))^2 \\
&= a^2(\mathrm{d}t)^2 + b^2(\mathrm{dZ}(t))^2 + 2ab\,\mathrm{d}t\,\mathrm{dZ}(t) \\
&= b^2\,\mathrm{d}t
\end{aligned}$$

Let $\Delta t \to 0$ and substitute into the Taylor expansion above,

$$\begin{aligned}
\Delta \mathrm{Y}(t) &= \frac{\partial G}{\partial \mathrm{S}} \Delta\mathrm{S}(t) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial \mathrm{S}^2}\Delta\mathrm{S}(t)^2 + \frac{\partial^2 G}{\partial \mathrm{S}\partial t}(\Delta S(t))\Delta t + \mathsf{O}(\Delta t^2) \\
&= \left(\frac{\partial G}{\partial t} + a(\mathrm{S}, t)\frac{\partial G}{\partial t} + \frac{1}{2}b(\mathrm{S}, t)^2\frac{\partial^2 \mathrm{G}}{\partial \mathrm{S}^2}\right)\mathrm{d}t + b(\mathrm{S}, t)\frac{\partial G}{\partial S}\cdot \mathrm{dZ}(t)
\end{aligned}$$

□

2.3 Monte-Carlo Option Pricing

Recall for no-arbitrage binomial model,

$$\begin{aligned}
S(t_n) &= \mathrm{e}^{-r\Delta t}\mathbb{E}_Q[\mathrm{S}(t_{n+1})] \\
V(S(t_n), t_n) &= \mathrm{e}^{-r\Delta t}\mathbb{E}_Q[V(S(t_{n+1}), t_{n+1})]
\end{aligned}$$

Thus

$$\mathbb{E}_Q\left[\frac{S(t_{n+1})}{S(t_n)}\right] = \mathrm{e}^{r\Delta t} = 1 + r\Delta t + \mathsf{O}(\Delta t^2)$$

This shows

$$\mathbb{E}_Q\left[\frac{S(t_{n+1}) - S(t_n)}{S(t_n)}\right] = \mathrm{e}^{r\Delta t} = r\Delta t + \mathsf{O}(\Delta t^2)$$

and therefore

$$\mathbb{E}_Q\left[\frac{\mathrm{dS}}{\mathrm{S}}\right] = r\,\mathrm{d}t$$

Linear convergence: It can be shown that

$$V_0^{\mathsf{tree}}(\Delta t) = V_0^{\mathsf{exact}} + (\text{const.})\cdot\Delta t + \mathsf{o}(\Delta t)$$

Then

$$\lim_{\Delta t \to 0}\frac{V_0^{\mathsf{tree}}(\frac{\Delta t}{2}) - V_0^{\mathsf{tree}}(\Delta t)}{V_0^{\mathsf{tree}}(\frac{\Delta t}{4}) - V_0^{\mathsf{tree}}(\frac{\Delta t}{2})} = 2$$

In quadratic convergence,

$$V_0^{\mathsf{tree}}(\Delta t) = V_0^{\mathsf{exact}} + (\text{const.})\cdot\Delta t + \mathsf{o}(\Delta t^2)$$

In Black-Scholes Model,

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = \mu\,\mathrm{d}t + \sigma\,\mathrm{dZ}(t)$$

Define $G(\mathrm{S}, t) \coloneqq \log \mathrm{S}$. Then

$$\begin{aligned}
\mathrm{d}\log\mathrm{S} &= \left(\frac{\partial \mathrm{G}}{\partial t} + a(\mathrm{S}, t)\frac{\partial \mathrm{G}}{\partial \mathrm{S}} + \frac{1}{2}\left(b(\mathrm{S}, t)^2\frac{\partial^2 \mathrm{G}}{\partial \mathrm{S}^2}\right)\right)\mathrm{d}t + b(\mathrm{S}, t)\frac{\partial \mathrm{G}}{\partial \mathrm{S}}\,\mathrm{dZ}(t) \\
&= (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{dZ}
\end{aligned}$$

Integrating on both sides,

$$\begin{aligned}\log \mathrm{S}(t) - \log \mathrm{S}(0) &= \int_0^t (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \int_0^t \sigma\,\mathrm{dZ} \\ &= (\mu - \frac{1}{2}\sigma^2)t + \sigma(\mathrm{Z}(t) - \mathrm{Z}(0))\end{aligned}$$

Hence

$$\mathrm{S}(t) = \mathrm{S}(0)\mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \mathrm{Z}(t)}$$

Note that the only source of randomness in this stochastic representation is $\mathrm{Z}(t)$.

Approximating using tree model:

$$\begin{aligned}\mathrm{V}(\mathrm{S}(0), 0)) &= \mathrm{e}^{-rT}\mathbb{E}_Q[\mathsf{payoff}(\mathrm{S}_T)] \\ &\simeq \mathrm{e}^{-rT}\frac{1}{M}\sum_{j=1}^{M}\mathsf{payoff}(S_T^j)\end{aligned}$$

Under risk-neutral probability, $\mu = r$

$$\boldsymbol{S}(T) = \mathrm{S}(0)\mathrm{e}^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\boldsymbol{z}}$$

2.4 Numerical Solutions to SDE

Consider an Itô's Process:

$$\mathrm{dX}(t) = a(\mathrm{X}, t)\cdot \mathrm{d}t + b(\mathrm{X}, t)\cdot \mathrm{dZ}(t)$$

This means

$$\mathrm{X}(t + \Delta t) = \mathrm{X}(t) + \int_t^{t+\Delta t} a(\mathrm{X}(t), t)\,\mathrm{d}t + \int_t^{t+\Delta t} b(\mathrm{X}(t), t)\,\mathrm{dZ}(t)$$

Assume $0 = t_0 < \cdots < t_N = T$, where $t_n = n\cdot\Delta t$, in $[t_n, t_n + \Delta t]$:

$$\int_{t_n}^{t_n+\Delta t} a(\mathrm{X}(t), t)\,\mathrm{d}t \simeq a(\mathrm{X}(t_n), t_n)\cdot\Delta t$$

$$\int_{t_n}^{t_n+\Delta t} b(\mathrm{X}(t), t)\,\mathrm{dZ}(t) \simeq b(\mathrm{X}(t_n), t_n)\cdot\Delta\mathrm{Z}(t_n)$$

where $\Delta\mathrm{Z}(t_n) = \mathrm{Z}(t_n + \Delta t) - \mathrm{Z}(t_n)$.

Euler's method:

$$\tilde{\mathrm{X}}(t_{n+1}) = \tilde{\mathrm{X}}(t_n) + a(\tilde{\mathrm{X}}(t_n), t_n)\Delta t + b(\tilde{\mathrm{X}}(t_n), t_n)\sqrt{\Delta t}\cdot \mathrm{z}(t), \qquad (\mathrm{z}(t)\sim \mathrm{N}(0,1))$$

has **weak convergence** of order $\beta = 1$:

$$\left|\mathbb{E}[g(\mathrm{X}(t))] - \mathbb{E}[g(\tilde{\mathrm{X}}(t))]\right| = \mathsf{O}((\Delta t)^\beta)$$

This is known as a weak convergence since it is convergence w.r.t. the functional $\mathrm{X}\mapsto\mathbb{E}[g(\mathrm{X})]$.

Strong convergence of order γ is,

$$\mathbb{E}\left[\left|\mathrm{X}(t) - \tilde{\mathrm{X}}(t)\right|\right] = \mathsf{O}((\Delta t)^\gamma)$$

Strong convergence is much more stringent. Euler's method has strong convergence with $\gamma = \frac{1}{2}$.

Milstein's method is a *higher order* method which is better than Euler's method:

$$\tilde{\mathrm{X}}(t_{n+1}) = \tilde{\mathrm{X}}(t_n) + a(\tilde{\mathrm{X}}(t_n), t_n)\Delta t + b(\tilde{\mathrm{X}}(t_n), t_n)\Delta\mathrm{Z}(t_n) + \frac{1}{2}b(\tilde{\mathrm{X}}(t_n), t_n)\frac{\partial b(\mathrm{X}, t)}{\partial \mathrm{X}}(\tilde{\mathrm{X}}(t_n), t_n)\left((\mathrm{Z}(t_n))^2 - \Delta t\right)$$

The second order term is directly derived from Itô's Lemma. Milstein's method has weak and strong convergence of order $\beta = \gamma = 1$.

Caput 3

Hedging and Risk

3.1 Barrier Options

Definition

A **Barrier option** has a barrier b. If the stock price rises to barrier b makes the barrier option worthless, the option is a **out option**. If the stock price rises to barrier b makes the barrier worthwhile, the option is a **in option**.

The Barrier option is **up** if $b > \mathrm{S}(0)$.

Definition

An **Asian option** has a time T and the option's value depends on the average price $\frac{1}{T}\int_0^T \mathrm{S}(t)\,\mathrm{d}t$ with payoff

$$\max(\frac{1}{T}\int_0^T \mathrm{S}(t)\,\mathrm{d}t, 0)$$

Let $t_n = t_{n-1} + \Delta t$, $\Delta t = \frac{T}{N}$. Then

$$A_n = \frac{1}{n}\sum_{k=1}^{n} \mathrm{S}(t_k)$$

procedure Price-AsianOption
 $\boldsymbol{S}_0 = S_0 \cdot \boldsymbol{1}$ ▷ Spawn m trajectories
 for $n = 0, \ldots, N-1$
  $\boldsymbol{A}_{n+1} \leftarrow \frac{n}{n+1}\boldsymbol{A}_n + \frac{1}{n+1}\boldsymbol{S}_n$
  $\boldsymbol{z} \sim \mathrm{N}(\boldsymbol{0}, \mathbf{I})$
  $\boldsymbol{S}_{n+1} \leftarrow \boldsymbol{S}_n \mathrm{e}^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\boldsymbol{z}}$
 end for
 $\boldsymbol{p} \leftarrow \max(\boldsymbol{A}_N - K\boldsymbol{1}, \boldsymbol{0})$ ▷ Payoff
 $\hat{V}_0 \leftarrow \mathrm{e}^{-rT \cdot \operatorname{avg} \boldsymbol{p}}$
end procedure

Using Central Limit Theorem,

$$V_0 - \hat{V}_0 \sim \mathrm{N}(0, \frac{1}{m}\sigma^2(\hat{V}_0))$$

and

$$\hat{\sigma} := \left(\frac{\sum_{j=1}^{m}(Y^j - V^m)^2}{m-1}\right)^{1/2}$$

⚠. Do not use the t-distribution here since normality is approximate.

3.2 Delta Hedging and Greeks

In option heding:

- Rebalancing position dynamically as discrete time
- Important risk management strategy

We conduct a model based heding analysis, which corresponds to the dynamic hegding strategy

$$\{ \underbrace{-V(t)}_{\text{Short option}}, \underbrace{\delta(t) \cdot S(t)}_{\delta(t) \text{ units of underlying}}, \underbrace{\beta(t)}_{\text{Cash}} \}$$

Simulate $S(t_n)$ based on the Black-Scholes model

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = \mu \cdot \mathrm{d}t + \sigma \cdot \mathrm{dZ}(t)$$

Note. Not risk neutral.

If a binomial model, we can solve the replicating equation

$$\delta_j^n = \frac{V_{j+1}^{n+1} - V_j^{n+1}}{(u-d)S_j} \simeq \frac{\partial V}{\partial S}(S_j^n, t_n)$$

The quantity δ is the number of shares we need to hold during $[t_n, t_{n+1}[$ to replicate the option, it is the sensitivity of V w.r.t. S:

$$\delta(t) := \frac{\partial V(S,t)}{\partial S}$$

Initially,

$$-V_0 = -V(S(0), 0)$$

We balance with cash account

$$\beta(0) = V(0) - \delta(0) \cdot S(0)$$

The portfolio is

$$\pi = -V + \delta \cdot S + \beta$$

- At 0: π is balanced at 0: $\pi(0) = 0$.
- At t_n: $\pi(n) = -V(S_n, t_n) + \delta_n S_n + \beta_n$
- At t_{n+1}: Rebalancing position in shares and updating cash account so that

  $$\pi(t_{n+1}^-) = \pi(t_{n+1}^+)$$

  (Self-financing)

This requires

$$-V(S_{n+1}, t_{n+1}) + \delta_n S_{n+1} + \beta_n \mathrm{e}^{r\Delta t} = -V(S_{n+1}, t_{n+1}) + \delta_{n+1} S_{n+1} + \beta_{n+1}$$

so

$$\beta_{n+1} = \beta_n \mathrm{e}^{r\Delta t} + (\delta_n - \delta_{n+1}) S_{n+1}$$

- If $\delta_{n+1} > \delta_n$, buy additional $\delta_{n+1} - \delta_n$ units.
- If $\delta_{n+1} < \delta_n$, sell additional $-(\delta_{n+1} - \delta_n)$ units.

At expiry $T = t_N$, liquid the portfolio formed at t_{N-1}, which has value π_N. Note that π_N is random.

- $\pi_N = 0$: Perfect hedge
- $\pi_N > 0$: Profit for writer
- $\pi_N < 0$: Loss

π_N is **heding error**. We often consider the following quantity

$$\mathsf{P\&L} := \frac{\mathrm{e}^{-rT} \pi_N}{V(S_0, 0)}$$

3.2.1 Computing Delta

The above way of calculating δ_n is the **finite difference approximation**:

$$\frac{\partial V(S_0,0)}{\partial S} \simeq \frac{V(S_0+\Delta S,0)-V(S_0,0)}{\Delta S}$$

$V(S_0+\Delta S,0)$ and $V(S_0,0)$ are computed using Monte-Carlo simulation.
What is the problem with the approximation below?

$$\frac{\partial V(S_0,0)}{\partial S} \simeq \mathrm{e}^{-rT}\frac{\mathsf{mean}(\mathsf{payoff}(S(T))|S_0+\Delta S)-\mathsf{mean}(\mathsf{payoff}(S(T))|S_0)}{\Delta S}$$

The error in each mean is $\mathsf{O}(1/\sqrt{M})$. Dividing this by ΔS generates an error of

$$\mathsf{O}(\frac{1}{\sqrt{M}\Delta S})$$

so the number of trajectories M must grow $\Omega(\Delta S^{-2})$.
Solution: Use the same set of sample paths to cancel sample error. Example:

$$\begin{aligned}
\boldsymbol{z} &\sim \mathrm{N}(\boldsymbol{0},\mathbf{I})\\
S_T &\coloneqq S_0\mathrm{e}^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\boldsymbol{z}}\\
\hat{S}_T &\coloneqq (S_0+\Delta S)\mathrm{e}^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\boldsymbol{z}}\\
V_0 &= \mathrm{e}^{rT}\mathsf{mean}(\mathsf{payoff}(S(T)))\\
\hat{V}_0 &= \mathrm{e}^{rT}\mathsf{mean}(\mathsf{payoff}(\hat{S}(T)))\\
\delta_0 &= \frac{\hat{V}_0-V_0}{\Delta S}
\end{aligned}$$

3.2.2 Delta and Gamma Hedging

Delta Neutral: If $\delta(t)=\frac{\partial V(S,t)}{\partial S}$, the hedged portfolio $\pi=\{-V,\delta\cdot S,\beta\}$ satisfies

$$\frac{\partial \pi}{\partial S} = -\frac{\partial V}{\partial S}+\delta\cdot 1+0 = -\frac{\partial V}{\partial S}+\frac{\partial V}{\partial S}=0$$

Delta hedging *eliminates the sensitivity* of option value to underlying, up to 1st order.
Gamma is the second derivative. We may wish to eliminate 2nd order sensitivities.

$$\frac{\partial \pi}{\partial S}=\frac{\partial^2\pi}{\partial S^2}=0$$

Consider an additional instrument, e.g. liquid option on S. We want this instrument I to have $\frac{\partial^2 I}{\partial S^2}\neq 0$.
e.g. Hedge a long-term option using a short-term option.
Let

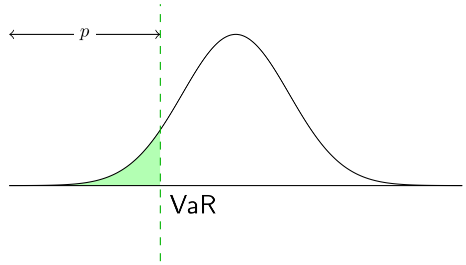
$$\pi \coloneqq \{-V,\delta_I\cdot I,\delta_S\cdot S,B\}$$

This portflio needs to satisfy

$$\begin{aligned}
\frac{\partial \pi}{\partial S} &= -\frac{\partial V}{\partial S}+\delta_I\frac{\partial I}{\partial S}+\delta_S\cdot 1=0\\
\frac{\partial^2 \pi}{\partial S^2} &= -\frac{\partial^2 V}{\partial S^2}+\delta_I\frac{\partial^2 I}{\partial S^2}=0
\end{aligned}$$

Vega is $\frac{\partial V}{\partial \sigma}$.

Descriptio 3.1: Left tail of the distribution of P&L represents VaR.



3.3 Risk Valuation

The Black-Scholes model can be used to assess the risk of weekly hedging compared to daily hedging.

For stocks, volatility σ measures risk.

How to quantify risk? Suppose we have a portfolio $\pi = \{S, C, P\}$.

1. Generate m trajectories of stock prices at time T.
2. Calculate option prices.
3. Calculate the distribution of $\pi(T) - \pi(0)$.

The standard deviation of $\pi(T) - \pi(0)$ does not really convey the risk of the portfolio. The market tend to periodically crash.

Note. In 1987, Alan Greenspan proposed that the worst case measures the risk.

(from Weatherstone, JP Morgan's CEO): At market close (4:15pm), generate a report which captures the loss

Definition

Value-at-Risk (VaR) is the predicted maximum loss with a particular probability.

VaR is usually communicated as "With $100(1-\alpha)\%$ confidence, the minimum P&L in time T is ...". i.e.

$$\int_{-\infty}^{\mathsf{VaR}} p(x)\,\mathrm{d}x = \alpha$$

where $\pi(T) - \pi(0) \sim p$.

For discrete scenarios, VaR is the maximum P&L value x such that

$$\mathsf{VaR} \coloneqq \arg\max_x P(\pi(T) - \pi(0) \leq x) \leq \alpha$$

VaR is an extension to the notion of minimum P&L since minimum often does not exist.

One problem with VaR is that it does not provide information on the extent of the loss beyond the quantile.

Definition

Assume P&L distribution is continuous. The **Conditional Value at Risk** (CVaR) is

$$\mathsf{CVaR} \coloneqq \mathbb{E}[\pi(T) - \pi(0) | \pi(T) - \pi(0) \leq \mathsf{VaR}]$$

Interpretation: The expected P&L, given that we are expecting the worst α probability scenarios.

The advantage of CVaR is CVaR is a *coherent risk measure* it satisfies the properties

1. Normalised: $\varrho(0) = 0$, risk of holding nothing is zero.

2. Monotonicity: If π_1, π_2 are portfolios with $\pi_1 \leq \pi_2$, then $\varrho(\pi_1) \geq \varrho(\pi_2)$: If π_2 has better values than π_1, it almost always has lesser risk.

3. Translational invariance: $\varrho(\pi + A) = \varrho(\pi) - a$, where portfolio A is adding cash a to portfolio π.

4. Convexity: If π_1, π_2 are portfolios and $\lambda \in [0, 1]$,

$$\varrho((1-\lambda)\pi_1 + \lambda\pi_2) \leq (1-\lambda)\varrho(\pi_1) + \lambda\varrho(\pi_2)$$

VaR does not satisfy convexity.

3.3.1 The Greeks

The following Greeks were defined:

$$\delta := \frac{\partial V}{\partial S}$$
$$\gamma := \frac{\partial^2 V}{\partial S^2}$$
$$\mathcal{V} := \frac{\partial V}{\partial \sigma}$$

It can be shown that for call options

$$\mathcal{V} = \frac{\partial V}{\partial \sigma} = \frac{\Phi'(d_1)}{\sigma S_0 \sqrt{T}}$$

where

$$\Phi'(d) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}d^2}$$

the $\mathcal{V}$ for put options is the same via Put-Call Parity.

Since $\mathcal{V} > 0$, the option value $C(\sigma; S(0))$ is strictly monotonically increasing w.r.t. $\sigma > 0$.

3.4 Volatility Surface

Assume today's market price $\hat{V}$ of an option is observed. We can invert a volatility $\hat{\sigma}$ from the market such that

$$V_{\mathsf{BS}}(\hat{\sigma}) = \hat{V}$$

This is the **implied volatility**. Implied volatility tells a trader the expected volatility over the life of the option, from the option market's perspective. Implied volatility cannot always be solved. A unique positive implied volatility exists only if the market price exists in the range which can be generated by the Black-Scholes formula, i.e.

$$\max(0, S(0) - K\mathrm{e}^{-rT}) \leq \hat{V} \leq S_0$$

If Black-Scholes formula is entirely correct, the implied volatility w.r.t. the strike price and expiry time should be constant $\hat{V}(K, T)$, but Black-Scholes formula is not entirely correct so $\hat{V}(K, T)$ forms a volatility surface.

The implies volatility is a root of

$$f(\sigma) = V_{\mathsf{BS}}(S_0, \sigma) - \hat{V} = 0$$

If Black-Scholes model is correct, $\hat{\sigma}$ is the same for options on the same underlying asset. Empirical evidence indicates that the implied volatility changes with strike K and expiry T. This is the **implied volatility surface**.

Usually $\hat{\sigma}$ is descending w.r.t. K.

Manaster and Koehler in 1982 shows that, let

$$\hat{\sigma}^2 := \left|\log \frac{S_0}{K} + rT\right| \frac{2}{T}$$

starting for σ, Newton's method converges to implied volatility quadratically if it exists.

3.4.1 Alternative Models

1. Local Volatility Function Model (LVF) by Dupire 1994,

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = \mu\,\mathrm{d}t + \sigma(\mathrm{S}(t), t)\,\mathrm{dZ}(t)$$

   where σ is a deterministic function.

   In risk neutral worls,

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = r\,\mathrm{d}t + \sigma(\mathrm{S}(t), t)\,\mathrm{dZ}(t)$$

2. Stochastic Volatility Model, Heston's in risk neutral:

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = r\,\mathrm{d}t + \sqrt{\mathrm{V}(t)}\,\mathrm{dZ}^{(1)}(t)$$

   where V (variance) is also a Brownian process:

$$\mathrm{dV}(t) = -\lambda(\mathrm{V}(t) - \bar{V})\,\mathrm{d}t + \eta\sqrt{\mathrm{V}(t)}\,\mathrm{dZ}^{(2)}(t)$$

   This process is a **mean reverting process** since the λ (**speed of mean reverting**) term forces the random variable to return to the mean $\bar{V}$.

   The two Brownian motions $\mathrm{Z}^{(1)}, \mathrm{Z}^{(2)}$ are not completely independent, but

$$\mathbb{E}[\mathrm{dZ}^{(1)} \cdot \mathrm{dZ}^{(2)}] = \rho \cdot \mathrm{d}t$$

3. Jump Model, by Merton's

$$\frac{\mathrm{dS}(t)}{\mathrm{S}(t)} = (r - k\lambda)\,\mathrm{d}t + \sigma\,\mathrm{dZ}(t) + (J - 1)\,\mathrm{dq}(t)$$

   where $\mathrm{q}(t)$ is a Poisson counting process with

$$\begin{cases} P(\mathrm{dq}(t) = 1) & = \lambda\,\mathrm{d}t \\ P(\mathrm{dq}(t) = 0) & = 1 - \lambda\,\mathrm{d}t \end{cases}$$

   λ is the **jump intensity**.

   Assume $\mathrm{dq}(t) = 1$ and let $\mathrm{S}(t^-)$ be the price immediately before the jump. Then

$$\mathrm{S}(t^+) = \mathrm{S}(t^-) \cdot J$$