1. If $m(A) = 1$ and $f \in L^\infty(A)$ show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

💡Let $\epsilon > 0$ and

$$E = \{x \in A : |f(x)| \ge (1-\epsilon)\|f\|_\infty\}.$$

Then $m(E) > 0$ (otherwise esssup$|f|$ would be smaller) and $\|f\|_p \ge (\int_E |f|^p)^{1/p}$.

2. Let $0 < \lambda_1 < \cdots < \lambda_N$ be real numbers. Show that the functions $\mathbb{R} \to \mathbb{C}$

$$x \to e^{i\lambda_j x}, \quad j = 1, 2, \ldots, N,$$

are linearly independent.

💡Take n-th derivative of the function $f(x) = \sum_{j=1}^N c_j e^{i\lambda_j x}$ for very large n. If f is identically 0 on $\mathbb{R}$ then $f^{(n)}$ is also identically zero. Show that this can happen only with all c_j equal to 0.

3. Let $G \subseteq \mathbb{R}$ be an additive subgroup. If G has an accumulation point in $\mathbb{R}$ show that it is dense in $\mathbb{R}$, i.e. that you can find an element of G in any interval.

4. (i) If $a_n, b_n, n = 1, 2, \ldots, N$ are complex numbers and $B_k = \sum_{n=1}^k b_n$ show the very useful formula of summation by parts (which is the analogue of intergration by parts for sums)

$$\sum_{n=M}^N a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

(ii) If $a_n \to 0$ is a decreasing sequence and the partial sums of the series $\sum_n b_n$ are bounded then the series $\sum_n a_n b_n$ converges.