Solutions

1. $C^1(\mathbb{T})$ is the space of functions on $\mathbb{T}$ that have a continuous derivative. Show that the quantity

$$\|f\|_{C^1} := |f(0)| + \|f'\|_\infty$$

is a norm on this space and that with this norm $C^1(\mathbb{T})$ is a Banach space.

Show also that the following quantity is also a norm (on the same function space)

$$\|f\|' := |f(0)| + \|f'\|_{L^2(\mathbb{T})},$$

but that the space is not complete with this norm.

Do we have convergence of the partial sums of the Fourier series on $C^1(\mathbb{T})$ (with the first norm)? Namely, is it true that for every $f \in C^1(\mathbb{T})$

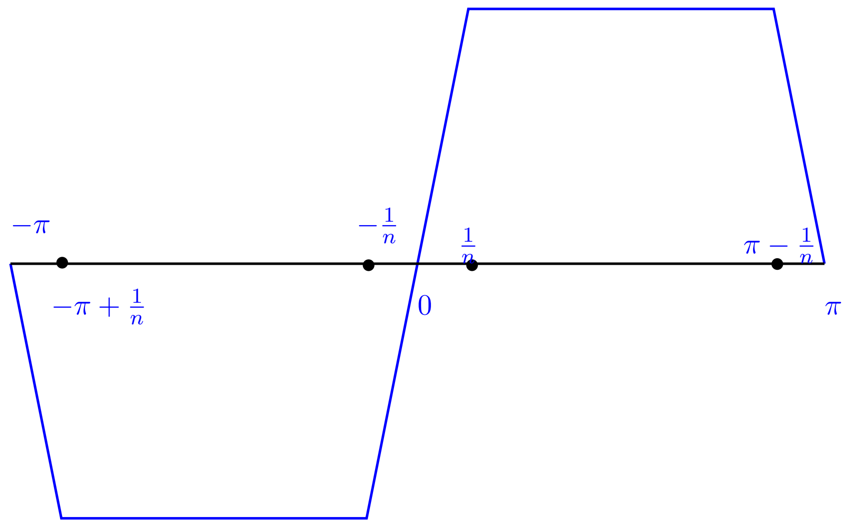
$$\|S_N f - f\|_{C^1} \xrightarrow{N} 0?$$

The same question for the second norm.

Solution: For the first part look at Problem Set 14 where the question appears verbatim.

For the norm $\|\cdot\|'$ the proof of the norm property is also easy. Again, the only property that needs some thought is the property $\|f\|' = 0 \implies f \equiv 0$. If $\|f\|' = 0$ we get that $f(0) = 0$ and $\|f'\|_2 = 0$, so that $f' = 0$ almost everywhere. But $f \in C^1(\mathbb{T})$ which implies that f' is continuous, hence it is 0 *everywhere*.

To show that $C^1(\mathbb{T})$ is not complete under this norm we must find a Cauchy sequence f_n which does not converge in this norm. We define f_n by setting $f_n(0) = 0$ and specifying f_n', a continuous function, taking care that $\int_0^{2\pi} f_n' = 0$ (otherwise f_n will not be periodic). Define f_n' on $[-\pi, \pi)$ to be the following function (where the height is ± 1):



Then f_n' converges (in L^2) to the function that is -1 on $(-\pi, 0)$ and +1 on $(0, \pi)$. The idea is that this is not a continuous function. Suppose then that $f_n \to f \in C^1(\mathbb{T})$ in the $\|\cdot\|'$ norm. This implies that $f_n' \to f'$ in the L^2 norm. But L^2 limits are unique, hence f', which is a continuous function, must be equal a.e. to the function $g(x) = -\mathbb{1}(-\pi < x < 0) + \mathbb{1}(0 < x < \pi)$. But no continuous function f' can do this: suppose $f' = g$ a.e. To be precise let us say that $f = g'$ except on a set $E \subseteq \mathbb{T}$ of measure 0. Then $(f')^{-1}((-1, 1))$ is an open set and, by the indermediate value theorem, it is not empty, hence it is of positive measure. Therefore $(f')^{-1}((-1, 1)) \setminus E$ is also of positive measure, hence nonempty, and $f' = g$ on this set. This is impossible since g only takes the values ± 1.

We now prove that we do not have convergence of the partial sums in the norm $\|\cdot\|$. The trigonometric polynomials are dense in $C^1(\mathbb{T})$ and for every trigonometric polynomial its partial sums are eventually identical with it, so we have convergence at the trigonometric polynomials in any norm. Therefore, to have $\|S_N f - f\| \to 0$ for all $f \in C^1(\mathbb{T})$ it is necessary and sufficient that the operator norms

$$\|S_N\|_{C^1(\mathbb{T}) \to C^1(\mathbb{T})}$$

form a bounded sequence. In other words there must exist a finite constant M such that

$$\|S_n f\| \le M \|f\|, \quad \text{for all } f \in C^1(\mathbb{T}).$$

This is the same as

$$|f(0)| + \|f'\|_\infty \le M(|D_N * f(0)| + \|D_N * f'\|_\infty),$$

where D_N is the usual Dirichlet kernel.

We have seen in the lectures that there exists $\phi \in C(\mathbb{T})$ with $\|\phi\|_\infty \le 1$ such that

$$D_N * \phi(0) = \fint D_N\phi \ge C\log N, \qquad \text{where } C \text{ is a positive constant.}$$

Define then $\psi(x) = \phi(x) - \fint \phi$, so that $|\psi(x)| \le 2$. Define

$$f(x) = \int_0^x \psi(t)\,dt,$$

which implies that $f \in C^1(\mathbb{T})$ with $f' = \psi$ and $f(0) = 0$. We have

$$\|f\| = \|\psi\|_\infty \le 2.$$

We also have

$$\fint D_N f' = \fint D_N(\phi - \fint \phi) = \fint D_N\phi - \fint D_N \fint \phi \ge C\log N - 1,$$

which implies

$$\|S_N f\| \ge \|D_N * f'\|_\infty \ge |D_N * f'(0)| = \left|\fint D_N f'\right| \ge C\log N - 1,$$

thus $\|S_N\| \ge \frac{C\log N - 1}{2}$ and it is not a bounded sequence.

Changing to the $\|\cdot\|'$ norm we will now prove that we do have convergence of the partial sums, which is equivalent to the boundedness of the sequence

$$\|S_N\|_{C^1(\mathbb{T})\to C^1(\mathbb{T})}$$

where now $C^1(\mathbb{T})$ is equipped with the $\|\cdot\|'$ norm. This, in turn, will follows if we prove, for some positive constant M, the bounds

$$|D_N * f(0)| \le M(|f(0)| + \|f'\|_2)$$

and

$$\|D_N * f'\|_2 \le M(|f(0)| + \|f'\|_2).$$

The second bound is easier. By Parseval we have

$$\|D_N * f'\|_2^2 = \sum_{n=-N}^{N} \left|\widehat{f'}(n)\right|^2 \le \sum_{n=-\infty}^{\infty} \left|\widehat{f'}(n)\right|^2 = \|f'\|_2^2.$$

For the first bound we have

$$\begin{aligned}
|D_N * f(0)| &= \left|\fint D_N(x) f(x)\,dx\right| \\
&= \left|\sum_n \widehat{D_N}(n)\widehat{f}(n)\right| \\
&= \left|\sum_{n=-N}^{N} \widehat{f}(n)\right| \\
&\le \left|\widehat{f}(0)\right| + \sum_{n\ne 0} \frac{\widehat{f'}(n)}{|n|} \\
&\le \|f\|_\infty + \sqrt{\sum_{n\ne 0}\frac{1}{n^2}}\sqrt{\sum_{n\ne 0}\left|\widehat{f'}(n)\right|^2} \quad \text{(Cauchy-Schwarz)} \\
&\le \|f\|_\infty + C_1\|f'\|_2,
\end{aligned}$$

where $C_1 = \sqrt{\sum_{n\ne 0}\frac{1}{n^2}}$. We also have, again by the Cauchy-Schwarz inequality,

$$|f(x)| = \left|\int_0^x f'(t)\,dt\right| \le \sqrt{2\pi}\|f'\|_2,$$

so that $\|f\|_\infty$ is also bounded by a multiple of $\|f'\|_2$.