1. If $f \in C^1(\mathbb{T})$ show that $\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| < \infty$ (and thus that the Fourier series of f converges uniformly to f).

💡 $\sum_{n\neq 0} \left|\widehat{f}(n)\right| = \sum_{n\neq 0} \frac{1}{|n|}\left|in\widehat{f}(n)\right|$.

Solution: We know that $\widehat{f'}(n) = in\widehat{f}(n)$, for all $n \in \mathbb{Z}$. So we have

$$\begin{aligned}
\sum_{n\neq 0}\left|\widehat{f}(n)\right| &= \sum_{n\neq 0}\frac{1}{|n|}\left|in\widehat{f}(n)\right| \\
&= \sum_{n\neq 0}\frac{1}{|n|}\left|\widehat{f'}(n)\right| \\
&\le \left(\sum_{n\neq 0}\frac{1}{|n|^2}\right)^{1/2}\cdot\left(\sum_{n\neq 0}\left|\widehat{f'}(n\right|^2\right)^{1/2} \quad \text{by the Cauchy-Schwarz inequality} \\
&= \left(\sum_{n\neq 0}\frac{1}{|n|^2}\right)^{1/2}\cdot\|f'\|_2^2 \quad \text{by the Parseval identity and since } \widehat{f'}(0)=0.
\end{aligned}$$

Since $f' \in C(\mathbb{T})$ we also have that $f' \in L^2(\mathbb{T})$, so the upper bound we found above is a finite number.

Since $\sum_n \left|\widehat{f}(n)\right| < \infty$ the Fourier Series of f converges absolutely and uniformly on $\mathbb{T}$.

2. Compute, as a function of $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, a formula for the series

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^2}.$$

💡 Let $f(x) = \frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha}$. Show that $\widehat{f}(n) = \frac{1}{n+\alpha}$ $(n\in\mathbb{Z})$ and use Parseval's formula.

Solution: You can verify the identity $\widehat{f}(n) = \frac{1}{n+\alpha}$ suggested in the hint by evaluating carefully the integral

$$\widehat{f}(n) = \frac{\pi}{\sin(\pi\alpha)}\frac{1}{2\pi}\int_0^{2\pi} e^{i((\pi-x)\alpha-nx}\,dx.$$

Then, by Parseval's identity, we have

$$\sum_n\frac{1}{(n+\alpha)^2} = \|f\|_2^2 = \frac{\pi^2}{\sin^2(\pi\alpha)}\frac{1}{2\pi}\int_0^{2\pi}dx = \frac{\pi^2}{\sin^2(\pi\alpha)}.$$

3. If $f(x) = x$, for $x \in [0, 2\pi]$, compute the Fourier coefficients of f and use Parseval's formula to compute the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: We have $\widehat{f}(0) = \pi$ and for $n \neq 0$ we can calculate (using integration by parts) that $\widehat{f}(n) = \frac{i}{n}$. By Parseval's identity we obtain

$$\pi^2 + \sum_{n\neq 0}\frac{1}{n^2} = \sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right|^2 = \|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}x^2\,dx = \frac{4}{3}\pi^2,$$

which implies

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$