Solutions

1. In the lecture we showed that if $f \in C(\mathbb{T})$ has non-zero Fourier coefficients only on the powers of 3 then $S_N f$ converges to f uniformly on $\mathbb{T}$.

Prove the same if the Fourier coefficients of f are non-zero only at the locations $\pm n_1, \pm n_2, \pm n_3, \ldots$, with $1 \le n_1 < n_2 < n_3 < \cdots$, where $\frac{n_{k+1}}{n_k} \ge \rho > 1$, for $k \ge 1$.

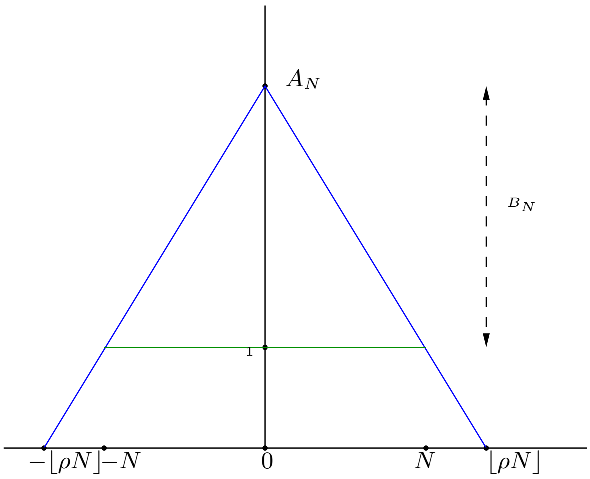
Solution:

The only thing that needs to change in our proof of the case with frequencies at $\pm 3^n$ is that we shall need a different kernel in place of the de la Vallée Poussin kernel. We need a kernel R_N whose Fourier coefficients (on the positive axis, and symmetrically on the negative frequency axis) up to N are equal to 1, its Fourier coefficients from N to ρN are bounded by 1 and its coefficients from ρN and beyond are 0.

For this we define

$$R_N(x) = A_N K_{\lfloor \rho N \rfloor}(x) - B_N K_N(x)$$

for two appropriate numbers A_N, B_N. The Fourier coefficients of R_N are shown below.



The two isosceles triangles $(-\lfloor \rho N \rfloor, 0) - (\lfloor \rho N \rfloor, 0) - (0, A_N)$ and $(-N, 1) - (N, 1) - (0, A_N)$ are similar. We clearly have $B_N = A_N - 1$ and we can find A_N by the similarity of the two triangles

$$\frac{A_N - 1}{A_N} = \frac{N}{\lfloor \rho N \rfloor} \sim \frac{1}{\rho}$$

which leads to $A_N \sim \frac{\rho}{\rho-1}$, $B_N \sim \frac{\rho}{\rho-1} - 1$. It is important that these quantities converge to constants ($\rho/(\rho-1)$ and $1/(\rho-1)$ respectively).

Hence $R_N * f = A_N K_{\lfloor \rho N \rfloor} * f(x) - B_N K_N * f(x)$. Since $K_{\lfloor \rho N \rfloor} * f$ and $K_N * f$ both converge uniformly to f and $A_N \to \rho/(\rho-1)$, $B_N \to 1/(\rho-1)$, it follows that $R_N * f$ converges uniformly to f.

Now, as in the original proof, we observe that $R_N * f$ and $S_N * f$ differ only by a quantity that tends to 0 uniformly in x and this completes the proof.

2. Define the function $f : \mathbb{R} \to \mathbb{R}$ to be 0 on the irrationals and at 0 and to be equal to $1/n$ on every rational of the form m/n with $(m, n) = 1$. Show that f is continuous exactly on the irrationals and at 0.

Solution:

Assume $x = \frac{m}{n}$ is a non-zero rational, with $(m, n) = 1$. Then $f(x) = 1/n$. If we approach x by any sequence of irrationals we get 0 which is $\neq f(x)$, so x is not a point of continuity.

If $x = 0$ then $f(0) = 0$. If $x_n \to x$ then we break up the sequence x_n in its rational members, call it r_n, and its irrational members, call it q_n. Then $f(q_n) = 0$ so the limit is 0 on that sequence and since $r_n \to 0$ it follows that the denominators of r_n (we may assume $r_n \neq 0$) tend to infinity, so $f(r_n) \to 0$, and this shows that 0 is a point of continuity.

If x is irrational, so $f(x) = 0$, and $x_n \to x$ again we break up the sequence x_n in its rational members, call it r_n, and its irrational members, call it q_n. Along q_n the limit is 0 and along r_n the limit is again 0 as, again, the denominators must tend to infinity. No irrational number can be the limit of a sequence of rationals unless the denominators of that sequence tend to infinity. If the denominators did not tend to infinity we could find a subsequence with bounded denominators. But such a convergent sequence must eventually be constant, so our number x would be rational.

3. The function $f : \mathbb{R} \to \mathbb{R}$ is increasing. Show that there exists a countable set $E \subseteq \mathbb{R}$, possibly empty, such that f is continuous on $\mathbb{R} \setminus E$.

Solution:

At any point $x \in \mathbb{R}$ the side limits $L(x) = \lim_{t \to x-} f(t)$ and $R(x) = \lim_{t \to x+} f(t)$ exist and are real numbers because of the monotonicity of f. The function f is continuous at x if and only if $L(x) = R(x)$ (and then $f(x)$ is also forced to have the same value). So if x is a point of discontinuity we have $L(x) < R(x)$.

If $x < y$ are two different points of discontinuity we have $L(x) < R(x) \le L(y) < R(y)$, because f is increasing. If we map each point of continuity to any rational number in the nonempty open interval $(L(x), R(x))$ it follows, from the above remark, that two different discontinuities x and y get mapped to different rational numbers. This mapping is therefore a 1-1 map from the set of discontinuities into the rational numbers. Since the rational numbers are countable then so are the discontinuities of f.