University of Crete – Department of Mathematics and Applied Mathematics
Problem Set No 10

Solutions

1. In our lectures about Weyl's theorem on equidistribution (but also in our notes and in the book of Stein and Shakarchi) Weyl's theorem refers to continuous functions which have period 1. This is natural because the trigonometric polynomials of the form

$$\sum_{k=-N}^{N} p_k e^{2\pi i k x}$$

which are used in the proof (but also in the statement of the theorem) approximate uniformly (according to Fejér's theorem) only functions which have period 1.

But this restriction is unnecessary. Show that if for the sequence $a_n \in [0,1)$ we have

$$\frac{1}{N}\sum_{k=1}^{N} f(a_k) \to \int_0^1 f \tag{1}$$

for every continuous and 1-periodic function f then (1) holds for every continuous, not necessarily periodic, function.

In the proof we gave we showed (1) for every step function, not necessarily periodic.

Solution:

In our proof we showed (1) for all step functions on $[0,1]$. But these step functions can approximate uniformly any continuous function on $[0,1]$, not just periodic continuous functions. By the same argument then, that we transfered (1) from step functions to periodic continuous functions, we can transfer (1) to all continuous functions.

2. If $a \in \mathbb{R} \setminus \{0\}$ and $0 < \sigma < 1$ show that the sequence $\{an^\sigma\}$ is uniformly distributed in $[0,1]$. ($\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.)

Use Weyl's criterion. Approximate the sum $\sum_{n=1}^{N} e^{2\pi i k \{an^\sigma\}} = \sum_{n=1}^{N} e^{2\pi i k a n^\sigma}$ by the integral $\int_1^N e^{2\pi i k a x^\sigma}\,dx$ and bound their difference using the Mean Value Theorem in every interval of the form $[i, i+1]$.

Solution:

Define $f_k(x) = e^{2\pi i k a x^\sigma}$. Then $f_k'(x) = 2\pi i k a \sigma x^{\sigma-1} e^{2\pi i k a x^\sigma}$.

We must show that for all integers $k \neq 0$ we have

$$\sum_{n=1}^{N} e^{2\pi i k \{an^\sigma\}} = o(N), \text{ as } N \to \infty.$$

Since $\{t\} = t - \lfloor t \rfloor$ for any t it follows that the above sum is the same (we ignore the last term which is at most 1, so it cannot affect the conclusion) as

$$S_{k,N} = \sum_{n=1}^{N-1} e^{2\pi i k a n^\sigma} = \sum_{n=1}^{N-1} f_k(n)$$

We will compare this sum to the integral

$$I_{k,N} = \int_1^N e^{2\pi i k a x^\sigma}\,dx = \int_1^N f_k(x)\,dx.$$

First we compute $I_{k,N}$. It differs by a bounded quantity from $\int_0^N e^{2\pi i k a x^\sigma}\,dx$ so we compute the latter integral which is easier and we will show that it is $o(N)$ as $N \to \infty$. After the change of variable

$$y = \frac{2\pi}{N^\sigma} x^\sigma,$$

designed to lead to the interval of integration $[0, 2\pi]$, we get

$$\int_0^N e^{2\pi i k a x^\sigma}\,dx = \frac{N}{\sigma (2\pi)^{1/\sigma}} \int_0^{2\pi} e^{ikaN^\sigma y} y^{\frac{1-\sigma}{\sigma}}\,dy.$$

The function $y^{\frac{1-\sigma}{\sigma}}$ is in $L^1([0,2\pi])$ (in fact, it is even continuous), so the last integral is the Fourier coefficient of this function evaluated at the frequency kaN^σ, which tends to $+\infty$ with N. By the Riemann-Lebesgue lemma this Fourier coefficient is $o(1)$ (tends to 0) so our integral divided by N is clearly $o(N)$. (Strictly speaking, the fact that the frequency kaN^σ is not an integer does not allow us to call this a "Fourier coefficient", but the Riemann-Lebesgue lemma still holds, with the same proof.)

Therefore it is enough to show that

$$|I_{k,N} - S_{k,N}| = o(N).$$

We have

$$\begin{aligned}|I_{k,N} - S_{k,N}| &\le \sum_{n=1}^{N-1} \left| \int_n^{n+1} f_k(x)\,dx - f_k(n) \right| \\ &= \sum_{n=1}^{N-1} \left| \int_n^{n+1} (f_k(x) - f_k(n))\,dx \right| dx\end{aligned}$$

Using the mean value theorem on f_k we have

$$f_k(x) - f_k(n) = f_k'(\xi)(x-n), \text{ for some } \xi \in (n,x),$$

so

$$|f_k(x) - f_k(n)| \le |f_k'(\xi)| = 2\pi|ka|\sigma\frac{1}{\xi^{1-\sigma}} \le \frac{C}{n^{1-\sigma}}.$$

Substituting in the inequality above we get

$$|I_{k,N} - S_{k,N}| \le C\sum_{n=1}^{N-1}\frac{1}{n^{1-\sigma}} = O(N^\sigma) = o(N),$$

as we had to show.

3. Working as in Exercise 2 show that the sequence $\{a\log n\}$ is not uniformly distributed in $[0,1]$ for any $a \in \mathbb{R}$.

Solution:

According to Weyl's theorem it is enough to show that $\sum_{n=1}^N e^{2\pi i k a \log n}$ is not $o(N)$ for some non-zero integer k. We will actually show that this is so for every non-zero integer k. As in the previous exercise, we can show, using the Mean Value Theorem, that the difference of this sum with the integral $\int_1^N e^{2\pi i k a \log x}\,dx$ is $o(N)$. So it is enough to show that this integral is *not* $o(N)$.

Assume it is, i.e. assume that $\frac{1}{N}\int_1^N e^{2\pi i k a \log x}\,dx \to 0$ as $N \to \infty$. Writing $f(N) = \int_1^N e^{2\pi i k a \log x}\,dx$ we have $f(N)/N \to 0$. If $\epsilon > 0$ is a small constant this implies that

$$\frac{f(N) - f((1-\epsilon)N)}{N} \to 0. \tag{2}$$

But

$$f(N) - f((1-\epsilon)N) = \int_{(1-\epsilon)N}^N e^{2\pi i k a \log x}\,dx \tag{3}$$

and in the interval $[(1-\epsilon)N, N]$ we have

$$\log N - \log\frac{1}{1-\epsilon} \le \log x \le \log N.$$

Notice that we can make the positive constant $\log\frac{1}{1-\epsilon}$ as small as we want by choosing $\epsilon > 0$ small enough. In that interval then we have, for the angle appearing in the exponent of the exponential in (3), the bounds

$$2\pi ka\log N - \underbrace{2\pi ka\log\frac{1}{1-\epsilon}}_{\theta_\epsilon} \le 2\pi ka \log x \le 2\pi ka\log N,$$

so that this angle varies by at most θ_ϵ. We choose $\epsilon > 0$ small enough to make $\theta_\epsilon < \frac{2\pi}{1000}$. This implies that in the integral we are integrating complex numbers of modulus 1 whose angles are all within $\frac{2\pi}{1000}$ of each other. This implies that the integral (over an interval of lentgh ϵN) is of modulus $\ge C_\epsilon \epsilon N$. As $N \to \infty$ this contradicts (2).