$f, g \in L^1(\mathbb{T})$

$$f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, g(x-t)\,dt = \int f(t)\, g(x-t)\,dt$$

$f * g \in L^1$

$\Rightarrow \widehat{f*g}(n) = \hat{f}(n)\,\hat{g}(n)$

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)\, e^{ikx}$$

$\downarrow f(x)$

$S_N f$ τριγ. πολυώνυμο

$$\widehat{S_N f}(k) = \begin{cases} \hat{f}(k) & -N \leq k \leq +N \\ 0 & \text{αλλιώς} \end{cases}$$

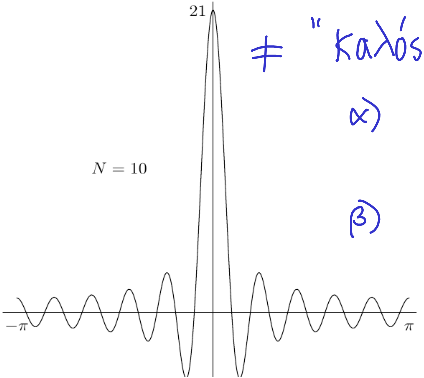
$$\widehat{S_N f}(k) = \hat{f}(k) \cdot \mathbb{1}_{\{-N, \dots, N\}}(k)$$

$$\widehat{S_N f}(k) = \hat{f}(k)\cdot \mathbb{1}_{\{-N,\dots,N\}}(k) = \hat{f}(k)\,\widehat{D_N}(k)$$

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

$$S_N f(x) = f * D_N(x)$$

$$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}}.$$

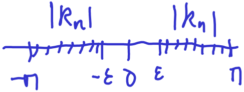


$\neq$ "καλός πυρήνας" k_n

α) $\int_{\mathbb{T}} k_n = 1$

β) $\|k_n\|_1 = \int |k_n| \leq M$

γ)



$\int_{|x|\geq\varepsilon} |k_n|$ οσοδήποτε μικρό για μεγ. n

$$\sigma_N f(x) = \left(S_0 f(x) + S_1 f(x) + \dots + S_N f(x)\right) / N+1$$ Cesaro μέσοι όροι της f

$$\alpha_n \to \alpha \implies \frac{\alpha_1 + \dots + \alpha_n}{n} \to \alpha$$
$$\not\Leftarrow$$

$$\sigma_N f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k) e^{ikx}$$ Fejér

$$\widehat{\sigma_N f}(k) = \hat{f}(k) \left(1 - \frac{|k|}{N+1}\right) \qquad K_N(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{ikx}$$

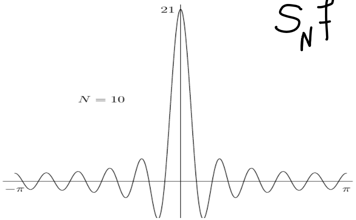
$$K_N(x) = \frac{1}{N+1} \frac{\sin^2 \frac{(N+1)x}{2}}{\sin^2 \frac{x}{2}} \geq 0 \implies$$ καλός πυρήνας $\quad f * K_n \xrightarrow{\text{ομ.}} f$

Θ. Fejér $\quad \sigma_N f \to f$ ομοιόμ. $\quad (f \in C(\mathbb{T}))$

Πυρήνας Dirichlet $D_N(x)$

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

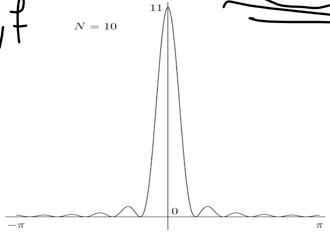
$S_N f$



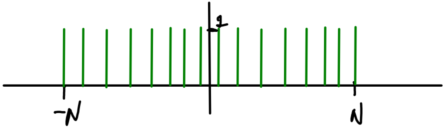
Πυρήνας Fejér $K_N(x)$

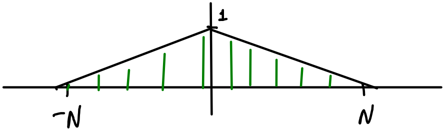
$$K_N(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{ikx}$$

$\sigma_N f$



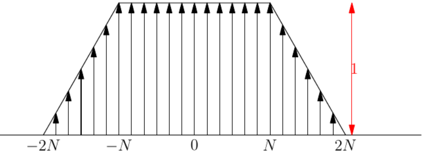
Συντ. Fourier

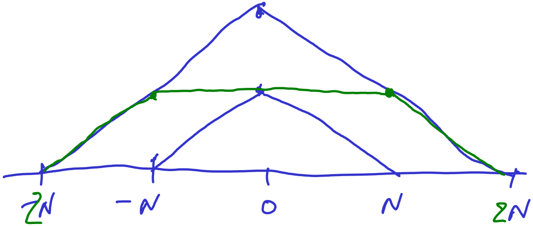




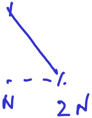
$V_N(x) = 2K_{2N}(x) - K_N(x)$

Πυρήνας de la Vallee Poussin





$$\widehat{V}_N(k) = \begin{cases} 0 & |k| > 2N \\ 1 & |k| \le N \\ & N < |k| \le 2N \end{cases}$$



$$V_N(x) = 2K_{2N}(x) - K_N(x)$$

1) καλός πυρήνας

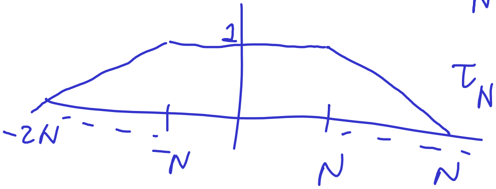
$$\int V_N = 2\int K_{2N} - \int K_N = 1$$

2) "Μοιάζει" στο D_N

$$\|V_N\|_1 \le 2\|K_N\|_1 + \|K_N\|_1 = 3\|K_N\|_1 = 3$$

$$\int_{|x|>\varepsilon} |V_N| \xrightarrow{N} 0 \quad \checkmark$$

$S_N f$



$$\tau_N f = f * V_N$$

$\downarrow$ ομ.

$f \in C(\mathbb{T})$

Μια εφαρμογή του πυρήνα de la Vallée Poussin

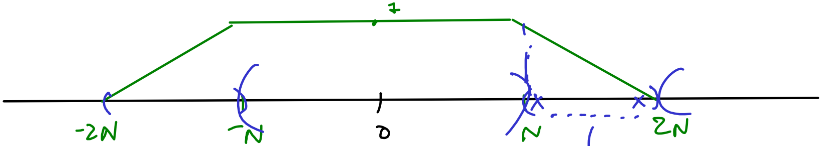
Έστω $f \in C(\mathbb{T})$ με $\hat{f}(k) = 0$ αν $k \neq \pm 3^n$. Τότε $S_N f \xrightarrow{\text{ομοιόμορφα}} f$

↳ lacunary

$\tau_N f = f * V_N \xrightarrow{\text{ομ.}} f$

$\widehat{\tau_N f}(k) = \hat{f}(k)\, \hat{V}_N(k)$

$\tau_N f \qquad S_N f$



το πολύ ένας μη μηδενικός $\hat{f}(k)$

$$|S_N f - f| \le \overbrace{|S_N f - \tau_N f|}^{\to 0 \text{ ομ.}} + |\tau_N f - f|$$

$$\tau_N f(x) = S_N f(x) + \hat{f}(3^\nu)\, e^{i\nu x}$$

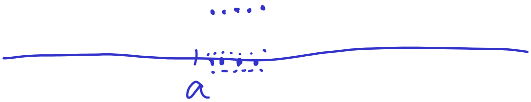
μπορεί

$$|\tau_N f(x) - S_N f(x)| = |\hat{f}(3^\nu)| \text{ ή } 0 \xrightarrow{N} 0$$

(Riemann-Lebesgue $\hat{f}(n) \to 0$ $\quad |n| \to \infty$)

Εφαρμογή: Μια <u>συνεχής</u> συνάρτηση <u>πουθενά παραγωγίσιμη</u>

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{αν } x \in \mathbb{Q} \\ 0 & \text{αν } x \notin \mathbb{Q} \end{cases}$$



$$C(\mathbb{T})$$

$$f(x) = \sum_{n=0}^{\infty} 2^{-\alpha n} e^{i 2^n x} \qquad (0 < \alpha < 1)$$

$$\sum_{n=1}^{\infty} 2^{-\alpha n} < \infty \longrightarrow \text{απόλυτη σύγκλιση}$$

$\longrightarrow$ συγκλίνει ομοιόμορφα $\Longrightarrow f \in C(\mathbb{T})$

Παραγωγίζουμε τη σειρά κατά όρους

$$f'(x) \text{ "=" } \sum_{n=0}^{\infty} i \, \boxed{2^{(1-\alpha)n}} e^{i 2^n x}$$

$\to \infty$

δε συγκλίνει

Λήμμα: Αν $g \in C(\mathbb{T})$ είναι παραγωγίσιμη στο x_0 τότε $\quad T_N g = g * V_N$

$(\sigma_N g)'(x_0) = O(\log N) \Rightarrow (T_N g)'(x_0) = O(\log N)$ $\quad T_N g = 2\sigma_{2N} g - \sigma_N g$

$|(\sigma_N g)'(x_0)| \leq C \log N$

$\|(\sigma_N g)'\|_\infty \leq C \log N$

$(\sigma_N g)' = (K_N * g)' = K_N' * g \qquad \int K_N' = 0 \qquad |\cdot| \leq C|t|$

$$(\sigma_N g)'(x_0) = (K_N' * g)(x_0) = \int K_N'(t)\, g(x_0 - t)\, dt = \int K_N'(t) \underbrace{\left(g(x_0-t) - g(x_0)\right)}_{|\cdot| \leq C|t|} dt$$

$$|(\sigma_N g)'(x_0)| \leq C \int |K_N'(t)| \cdot |t|\, dt$$

$$K_N(x) = \frac{1}{N+1}\frac{\sin^2\frac{(N+1)x}{2}}{\sin^2\frac{x}{2}}.$$

$$K'_N(t) = \frac{\sin\frac{(N+1)t}{2}\cos\frac{(N+1)T}{2}}{\sin^2(t/2)} - \frac{1}{N+1}\frac{\cos\frac{t}{2}\sin^2\frac{(N+1)t}{2}}{\sin^3(t/2)}$$

$$\int |K_N'(t)|\cdot|t|\,dt$$

$c|x| \le |\sin x| \le |x|$

$|\cos x| \le 1$

$x \in (-\pi, \pi)$

$$\Longrightarrow \boxed{|K_N'(t)| \le A \min\{N^2, \frac{1}{t^2}\}}$$

stad.

$$|K_N'(t)| \le \frac{A}{t^2}$$

$$|K_N'(t)| \le AN^2$$

$\le |k| \le N$

$$K_N'(x) = \sum_{k=-N}^{N} ik\left(1-\frac{|k|}{N+1}\right)e^{ikx} \le (2N+1)N$$

$$\le 3N^2$$

$$\boxed{\left| K_N'(t) \right| \le \underset{\text{бтад.}}{A} \min\left\{ N^2, \ \frac{1}{t^2} \right\}}$$

$$\left| (\sigma_N g)'(x_0) \right| \le C \int \left| K_N'(t) \right| \cdot |t| \, dt$$

$$= C \int_{|t| \ge \frac{1}{N}} \left| K_N'(t) \cdot t \right| dt + C \int_{|t| < \frac{1}{N}} \left| K_N'(t) \right| \cdot |t| \, dt$$

$$= I + \underline{II}$$

$$= O(\log N) + O(1) = O(\log N)$$

$$I \le C \cdot A \int_{|t| \ge \frac{1}{N}} \frac{dt}{|t|} = C A \log N$$

$$= O(\log N)$$

$$\underline{II} \le C A \int_{|t| < \frac{1}{N}} N^2 \frac{1}{N} \, dt = C A = O(1)$$

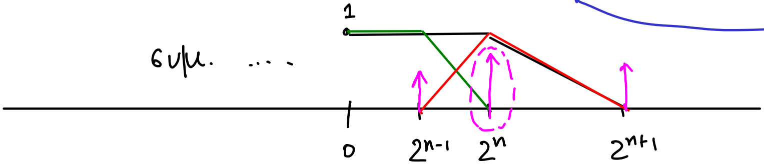
$$f(x) = \sum_{n=0}^{\infty} 2^{-\alpha n} e^{i\,\underline{2^n}\,x} \qquad (0 < \alpha < 1)$$

$$=$$

$$\underline{\tau_{2^n} f(x) - \tau_{2^{n-1}} f(x)} = 2^{-\alpha n} e^{i 2^n x}, \quad \forall n$$

$\boxed{f'(x_0) \text{ υπάρχει}}$

$$\boxed{\left|(\tau_N f)'(x_0)\right| = O(\log N)}$$



$$\left|(\tau_{2^n} f)'(x) - (\tau_{2^{n-1}} f)'(x)\right| = \left|i\, 2^{(1-\alpha)n} e^{i 2^n x}\right| = 2^{(1-\alpha)n}$$

αντίφαση

$$N = 2^n \qquad \longleftrightarrow \qquad \leq O(\log 2^n) = O(n)$$