The goal of the following problems is to help you aquaint yourselves with using complex numbers. Try to avoid breaking up a complex number to its real and imaginary parts.

1. $|z+w| = |z| + |w| \iff \mathrm{Re}\,(z\overline{w}) = |z||w|$.

2. $|z| = |w| = 1,\ zw \neq -1 \implies \dfrac{z+w}{1+zw} \in \mathbb{R}$.

3. if $|z| = |w| = 1, \quad |z+w| = \sqrt{3}$ find $|z-w|$.

4. If $|z| = |w| = |u|, \quad z+w+u = 0$ then there exists $\phi \in [0, 2\pi)$ so that $\{z, w, u\} = \left\{e^{i\phi}, e^{i\phi}e^{2\pi i/3}, e^{i\phi}e^{4\pi i/3}\right\}$.

5. Find all integers n such that

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2.$$

6. $\mathrm{Re}\, z > 1 \implies \left|\dfrac{1}{z} - \dfrac{1}{2}\right| < \dfrac{1}{2}$

7. The general equation of a straight line in the plane is

$$Ax + By + C = 0,$$

where $A, B, C \in \mathbb{R}$ are constants. Write this equation in terms of $z, \overline{z}$ (without x, y).

8. Show that the equation

$$z\overline{z} + az + \overline{az} + b = 0,$$

with $a \in \mathbb{C}, b \in \mathbb{R}$ is satisfied by the z that belong to a circle.

9. If $z = x + iy$ with x constant and $y \to +\infty$ show that

$$|\cos z| = \left(\frac{1}{2} + o(1)\right)e^y, \quad \mathrm{Arg}\cos z = -x + o(1).$$

Here $o(1)$ is a quantity that tends to 0 for $y \to +\infty$.