

1. If f is analytic at z and $f(z) \neq 0$ show that there is $r > 0$ such that the value $|f(z)|$ is not the minimum value of $|f(w)|$ for $|w - z| \leq r$. *Hint:* Consider $g(z) = 1/f(z)$.

(We have solved this in class.)

2. Using the function $e^{f(z)}$ show that the maximum principle holds also for the real part of an analytic function f , and, of course, for the imaginary part as well: none of these two functions can have a local maximum. Show that they cannot have a local minimum either.

(We have solved this in class.)

3. Assume f is analytic in the annulus $1 \leq |z| \leq 2$. Assume also that $|f(z)| \leq 3$ for $|z| = 1$ and that $|f(z)| \leq 12$ for $|z| = 2$. Show that

$$|f(z)| \leq 3|z|^2, \quad (1 \leq |z| \leq 2).$$

Hint: The function $f(z)/z^2$ is analytic in this annulus. Use the maximum modulus principle.

4. Assume that f is analytic on the simple closed curve C and in its interior. Assume also that $|f(z) - 1| < 1$ for all $z \in C$. Show that f does not vanish in the interior of C .

Hint: Consider the function $g(z) = f(z) - 1$ and show that it does take the value 1 in the interior of C . Use the maximum modulus principle.

5. Find the maximum and the minimum (and where these values are taken) for $\operatorname{Re} f$ where $f(z) = e^z$ in the region

$$[0, 1] \times [0, \pi].$$

6. Find the maximum and the minimum absolute value of the function $f(z) = (z - 2i)^3$ for $|z| \leq 1$.

7. Assume that f is analytic in $|z| < 1$ and that

$$|f(z)| \leq \frac{1}{1 - |z|}, \quad (|z| < 1).$$

Show that

$$|f^{(n)}(0)| \leq \frac{n!}{R^n(1 - R)}, \quad (0 < R < 1).$$

Which value should we choose for R so as to get the smallest upper bound?

8. Assume that f is analytic in $|z| < r$ and that there we have $|f(z)| \leq M$. Prove that

$$|f^{(n)}(z)| \leq \frac{n!M}{(r - |z|)^n}, \quad (|z| < r).$$

9. Assume that f is entire and that

$$|f(z)| \leq 1 + |z|^5,$$

for all z with $|z| \geq 1$. Show that f is a polynomial of degree up to 5.

10. Show that for every polynomial $p(z)$ of degree n

$$p(z) = p_0 + p_1z + p_2z^2 + \cdots + p_nz^n,$$

we have

$$p_j = \frac{p^{(j)}(0)}{j!}, \quad j = 0, 1, \dots, n.$$

For every such polynomial and with $M = \max_{|z|=1} |p(z)|$ show that $|p_j| \leq M$, for $j = 0, 1, \dots, n$.