All curves are positively oriented unless otherwise noted.

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1. If f is analytic in $\mathbb{C} \setminus \{0, 1\}$ and $\sum_{n=-\infty}^{\infty} a_n z^n$ is its Laurent series for $|z| > 1$ show that the series is not finite towards $-\infty$, i.e., there are arbitrarily large $n > 0$ such that $a_{-n} \neq 0$.

2. Find the singularities and the corresponding residues for the functions:

$$(a)\ \frac{e^3 z}{z-2}, \quad (b)\ \frac{z+1}{z^2-3z+2}, \quad (c)\ \frac{\cos z}{z^2}, \quad (d)\ \left(\frac{z-1}{z+1}\right)^3.$$

3. If f has an isolated singularity at z_0 show that $\mathrm{Res}(f'; z_0) = 0$.

4. If f has a pole of order m at z_0 show that

$$\mathrm{Res}\left(\frac{f'}{f}; z_0\right) = m.$$

5. Find the number of roots of the following polynomials in the region $|z| < 1$.

$$(a)\ z^6 - 5z^4 + z^3 - 2z, \quad (b)\ 2z^4 - 2z^3 + 2z^2 - 2z + 9.$$

6. Find the number of roots of the function $2z^5 - 6z^2 + z + 1$ in the region $1 \leq |z| < 2$.

7. If $c \in \mathbb{C}$, $|c| > e$, show that the equation $cz^n = e^z$ has n roots in the region $|z| < 1$.

8. Compute the following improper integrals:

$$(a)\ \int_0^\infty \frac{dx}{x^2+1}, \quad (b)\ \int_0^\infty \frac{dx}{(x^2+1)^2}, \quad (c)\ \int_0^\infty \frac{dx}{x^4+1}, \quad (d)\ \int_0^\infty \frac{\cos x\, dx}{x^2+1},$$

Notice that the functions are all even, so $\int_0^\infty \ldots = \frac{1}{2}\int_{-\infty}^\infty \ldots$.

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