1. Prove again the uniqueness of the coefficients of trigonometric polynomials without the Vandermonde matrix. If $p(x), q(x)$ are two trigonometric polynomials which are identical *in the entire interval* $[0, 2\pi]$ (not just at $2N+1$ points, where N is the degree of the polynomials) then they have the same coefficients.

Prove that the coefficients of a trigonometric polynomial are given by the formula

$$p_k = \langle p(x), e^{ikx} \rangle = \frac{1}{2\pi} \int_0^{2\pi} p(x) e^{-ikx}\, dx, \quad (k \in \mathbb{Z}).$$

2. Let $0 < \lambda_1 < \cdots < \lambda_N$ be real numbers. Show that the functions $\mathbb{R} \to \mathbb{C}$

$$x \to e^{i\lambda_j x}, \quad j = 1, 2, \ldots, N,$$

are linearly independent.

Take n-th derivative of the function $f(x) = \sum_{j=1}^N c_j e^{i\lambda_j x}$ for very large n. If f is identically 0 on $\mathbb{R}$ then $f^{(n)}$ is also identically zero. Show that this can happen only with all c_j equal to 0.

3. Let $G \subseteq \mathbb{R}$ be an additive subgroup. If G has an accumulation point in $\mathbb{R}$ show that it is dense in $\mathbb{R}$, i.e. that you can find an element of G in any interval.