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1. Assume f is analytic in the annulus $1 \le |z| \le 2$. Assume also that $|f(z)| \le 3$ for $|z| = 1$ and that $|f(z)| \le 12$ for $|z| = 2$. Show that

$$|f(z)| \le 3|z|^2, \quad (1 \le |z| \le 2).$$

2. Assume that f is analytic on the simple closed curve C and in its interior. Assume also that $|f(z) - 1| < 1$ for all $z \in C$. Show that f does not vanish in the interior of C.

3. Show that for every polynomial $p(z)$ of degree n

$$p(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n,$$

we have

$$p_j = \frac{p^{(j)}(0)}{j!}, \quad j = 0, 1, \ldots, n.$$

For every such polynomial and with $M = \max_{|z|=1} |p(z)|$ show that $|p_j| \le M$, for $j = 0, 1, \ldots, n$.

4. Assume that f is analytic in $|z| < 1$ and that

$$|f(z)| \le \frac{1}{1 - |z|}, \quad (|z| < 1).$$

Show that

$$\left|f^{(n)}(0)\right| \le \frac{n!}{R^n(1 - R)}, \quad (0 < R < 1).$$

Which value should we choose for R so as to get the smallest upper bound?

5. Assume that f is analytic in $|z| < r$ and that there we have $|f(z)| \le M$. Prove that

$$\left|f^{(n)}(z)\right| \le \frac{n!M}{(r - |z|)^n}, \quad (|z| < r).$$

6. Assume that f is entire and that

$$|f(z)| \le |z|^5,$$

for all z with $|z| \ge 1$. Show that f is a polynomial of degree up to 5.