These problems are to be solved by you as part of your preparation for the final exam. You will also be examined on the first part of the course for which you already have training material. You must also do the problems that were assigned during the semester.

1. Compute $\oint_C z^5\,dz$, where C is the open polygonal curve that joins the point $-i$ (first), 1, i (last) with straight line segments.

2. Compute $\oint_C \frac{1}{z}\,dz$, where C is the straight line segment joining the points $1, 1+i$.

3. Let C be a simple closed curve and $w \in \mathbb{C}\setminus C$. We define

$$g(w) = \oint_C \frac{z^3+2z}{(z-w)^3}dz.$$

Show that if w is in the interior of C then $g(w) = 6\pi i w$ while if not then $g(w) = 0$.

4. Show that for all $a \in \mathbb{R}$ we have $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$. *Hint:* Use the integral $\oint_{\{|z|=1\}} \frac{e^{az}}{z}dz$.

5. If f is entire show that the maximum of $\operatorname{Re} f(z)$ for $|z| \le 1$ is taken on the circle $|z| = 1$.

6. If f is entire and $|f(z)| \le 10|z|$ for $z \in \mathbb{C}$ show that $f(z) = az$ for some $a \in \mathbb{C}$.

7. If $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ find all Laurent expansions of f centered at 0 and explain in which region each one holds (do not bother with the region boundaries).

8. As in question 7 but for the function $f(z) = \frac{1}{z(1+z^2)}$.

9. If $c \in \mathbb{C}$ then what kind of singularity does $(e^{cz}-1)/z$ have at 0?

10. If f, g analytic at a, $f(a) = g(a) = 0$ and $g'(a) \ne 0$ show that $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}$.

11. For the following functions find their singularities and the kind of each.

$$(a)\ ze^{1/z},\quad (b)\ z^2/(1+z),\quad (c)\ \sin z/z,\quad (d)\ \cos z/z,\quad (e)\ 1/(2-z)^3.$$

12. For the following functions find their residue at 0.

$$(a)\ (z+z^2)^{-1},\quad (b)\ z\cos\frac{1}{z},\quad (c)\ \frac{z-\sin z}{z},\quad (d)\ \frac{\sinh z}{z^4(1-z^2)}.$$

13. If C is the circle $|z| = 1$ and f is analytic on C and in the exterior of C show that

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \operatorname{Res}\left(\frac{1}{z^2}f(1/z); 0\right).$$

14. If C is the circle $|z| = 3$ compute the integral of the following functions on C with the positive orientation

$$(a)\ \frac{e^{-z}}{z^2},\quad (b)\ z^2e^{1/z},\quad (c)\ \frac{z+1}{z^2-2z}.$$

15. Compute the integrals

$$(a)\ \int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx,\quad (b)\ \int_0^\infty \frac{\cos 3x}{1+x^2}dx,\quad (c)\ \int_{-\infty}^\infty \frac{x\sin x}{x^2+2x+2}dx.$$