

## THE ISOTROPIC CONSTANT IN THE THEORY OF HIGH-DIMENSIONAL CONVEX BODIES

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**ABSTRACT.** The investigation of geometric properties of log-concave probability measures, in parallel with the study of volume distribution in high-dimensional convex bodies, has produced a sequence of profound results in asymptotic convex geometry. These developments have culminated in a definitive affirmative solution to Bourgain’s slicing problem, established in recent works of Q. Guan and of B. Klartag and J. Lehec. In this survey, we present the principal tools and methods underlying these breakthroughs and review several of their main consequences for the asymptotic theory of convex bodies.

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### 1. INTRODUCTION

Bourgain’s *slicing problem* asks whether there exists an absolute constant  $c > 0$  such that

$$\max\{\text{vol}_{n-1}(K \cap \xi^\perp) : \xi \in S^{n-1}\} \geq c$$

for every convex body  $K \subset \mathbb{R}^n$  of volume 1 whose barycenter is at the origin. A natural framework for studying this problem is provided by the *isotropic position* of a convex body.

A convex body  $K \subset \mathbb{R}^n$  is called *isotropic* if  $\text{vol}_n(K) = 1$ , its barycenter is at the origin, and its inertia matrix is a multiple of the identity. Equivalently, there exists

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a constant  $L_K > 0$  such that

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2$$

for every  $\xi \in S^{n-1}$ . The constant  $L_K$  is called the *isotropic constant* of  $K$ . Every convex body admits an affine image that is isotropic, which is unique up to orthogonal transformations; this affine image is called the *isotropic position* of  $K$ .

It is known that an affirmative answer to the slicing problem is equivalent to the following statement:

*There exists an absolute constant  $C > 0$  such that*

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leq C. \quad (1.1)$$

In view of this equivalence, we shall use the terms *isotropic constant conjecture*, *Bourgain's slicing problem*, and *hyperplane conjecture* interchangeably throughout this article.

The notion of the isotropic constant extends naturally to the setting of finite log-concave measures. A finite log-concave measure  $\mu$  on  $\mathbb{R}^n$  is called *isotropic* if it is a probability measure, its barycenter is at the origin, and its covariance matrix  $\text{Cov}(\mu)$  equals the identity  $I_n$ . The isotropic constant  $L_\mu$  is defined appropriately so that uniform measures on isotropic convex bodies are recovered as a special case. A theorem of K. Ball shows that, in fact, for some absolute constant  $c > 1$ ,

$$L_n \leq \tilde{L}_n := \sup\{L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leq cL_n.$$

Around 1985-6 (published in 1990), Bourgain [28] obtained the upper bound  $L_n \leq c\sqrt[4]{n} \ln n$  and, in 2006, this estimate was improved by Klartag [72] to  $L_n \leq c\sqrt[4]{n}$ . Actually, Klartag obtained a solution to the "isomorphic slicing problem", by showing that, for every convex body  $K$  in  $\mathbb{R}^n$  and any  $\varepsilon \in (0, 1)$ , one can find a convex body  $T \subset \mathbb{R}^n$  with barycenter at the origin and a point  $x \in \mathbb{R}^n$  such that  $(1 + \varepsilon)^{-1}T \subseteq K + x \subseteq (1 + \varepsilon)T$  and  $L_T \leq C/\sqrt{\varepsilon}$  for some absolute constant  $C > 0$ . Later, Dafnis and Paouris [40] developed a different approach based on small-ball probability estimates, obtaining the slightly weaker bound  $L_n \leq c\sqrt[4]{n}(\ln n)^2$ .

A second main question in this area is the *Kannan-Lovász-Simonovits conjecture* about the isoperimetric constant  $\chi_\mu$  of an isotropic log-concave probability measure  $\mu$ , defined as the largest constant  $\chi \geq 0$  such that

$$\mu^+(A) \geq \chi \min\{\mu(A), 1 - \mu(A)\}$$

for every Borel subset  $A$  of  $\mathbb{R}^n$ , where  $\mu^+(A)$  is the Minkowski content of  $A$ . We shall consistently refer to the Kannan-Lovász-Simonovits conjecture as the *KLS conjecture*.

If we set  $\psi_\mu = 1/\chi_\mu$  then the KLS conjecture becomes the question if there exists an absolute constant  $C > 0$  such that

$$\psi_n := \sup\{\psi_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leq C.$$

An equivalent way to formulate the KLS conjecture is to ask that the Poincaré inequality holds for every isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  with a constant that does not depend on the measure or the dimension  $n$ .

It is now known that Bourgain's slicing problem has an affirmative answer, and it is also known that  $\psi_n \leq c\sqrt{\ln n}$ . Moreover, an affirmative answer has been given

for the *thin-shell conjecture*, which concerns the concentration of an isotropic log-concave random vector  $X$  around the sphere of radius  $\sqrt{n}$ . More precisely, the thin-shell conjecture asks if there exists an absolute constant  $C > 0$  such that, for any  $n \geq 1$  and any isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ , one has

$$\int_{\mathbb{R}^n} (|x| - \sqrt{n})^2 d\mu(x) \leq C^2.$$

This statement implies that most of the mass of  $\mu$  is concentrated on a thin spherical cell whose width  $t$  is much smaller than the central radius  $\sqrt{n}$ . It also implies that high-dimensional log-concave distributions have approximately Gaussian marginals. If we define  $\sigma_\mu^2 = \frac{1}{n} \text{Var}_\mu(|x|^2)$ , where  $\text{Var}_\mu(f)$  is the variance of  $f$  with respect to  $\mu$ , then the thin-shell conjecture is equivalent to the question if

$$\sigma_n := \sup\{\sigma_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leq C.$$

Note that  $\sigma_\mu \leq C\psi_\mu$  for every Borel probability measure  $\mu$  on  $\mathbb{R}^n$ , and hence  $\sigma_n \leq C\psi_n$ . On the other hand, Eldan and Klartag [46] showed that there exists an absolute constant  $C > 0$  such that, for every  $n \geq 1$ ,

$$L_n \leq C\sigma_n,$$

therefore the KLS conjecture implies that  $L_n$  is bounded; in fact, any upper bound for  $\psi_n$  is an upper bound for  $L_n$ , up to an absolute constant.

A key tool underlying the recent progress on these conjectures is *stochastic localization*, introduced by Eldan in his PhD thesis. This method allows one to decompose a high-dimensional probability measure into a mixture of simpler, randomly localized measures, and to study their evolution via stochastic analysis. Using this approach, Eldan [45] proved that

$$\psi_n^2 \leq C \ln n \sum_{k=1}^n \frac{\sigma_k^2}{k},$$

showing that the thin-shell conjecture implies the KLS conjecture up to polylogarithmic factors. Combining this result with work of Guédon and E. Milman [66], one obtains the bound  $\psi_n \leq C \sqrt[3]{n} \ln n$ . Later, Lee and Vempala [93] developed a variant of Eldan's stochastic localization and obtained the bound  $\psi_n \leq c \sqrt[4]{n}$ . In a breakthrough work, Chen [37] proved that one has  $\psi_n \leq \exp(C \sqrt{\ln n (\ln \ln n)})$ , and hence  $\psi_n \leq n^\varepsilon$  for any  $\varepsilon > 0$  and all large enough  $n$ . This development was the starting point for a series of important works, that led to the estimate  $\psi_n \leq c \sqrt{\ln n}$  by Klartag [78]. This is currently the best known result on the KLS conjecture.

A recent technical breakthrough by Guan [65], who also obtained  $\sigma_n = O(\ln \ln n)$ , was used by Klartag and Lehec [82] to prove the isotropic constant conjecture (1.1). Soon thereafter, Bizeul [16] gave an alternative proof. More recently, Klartag and Lehec [83] confirmed the thin-shell conjecture.

The resolution of these three conjectures shows that the isotropic position provides a canonical regular position for convex bodies. There are several other well-known positions of a convex body in  $\mathbb{R}^n$ , such as the John position, the  $\ell$ -position, and the  $M$ -position. As we explain in Section 3, the solution of the slicing problem

implies that the isotropic position is an  $M$ -position. Moreover, as shown in [70], if  $K$  is isotropic then

$$\sqrt{\frac{n+2}{n}} B_2^n \subseteq K \subseteq \sqrt{n(n+2)} B_2^n,$$

so the geometric distance to the Euclidean ball is at most  $n$ , as in John's position. This naturally leads to the question of whether the isotropic position is as effective as the  $\ell$ -position for bounding the parameter  $MM^*$ . Defining the Minkowski functional  $p_K(x) = \inf\{t > 0 : x \in tK\}$  and the support function  $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$  of  $K$ , we set

$$M(K) = \int_{S^{n-1}} p_K(x) d\sigma(x) \quad \text{and} \quad M^*(K) = \int_{S^{n-1}} h_K(x) d\sigma(x),$$

where  $\sigma$  is the rotationally invariant probability measure on  $S^{n-1}$ . It is always true that  $M(K)M^*(K) \geq 1$ .

A classical result due to Figiel–Tomczak–Jaegermann [48], Lewis [95], and Pisier [121] asserts that for any symmetric convex body  $K \subset \mathbb{R}^n$  there exists  $T \in GL_n$  such that

$$M(TK)M^*(TK) \leq c \ln n,$$

where  $c > 0$  is an absolute constant. Without assuming symmetry, it is natural to consider the parameter

$$E(K) = \inf M(TK)M^*(TK),$$

where the infimum runs over all invertible affine transformations  $T$  of  $\mathbb{R}^n$  satisfying  $0 \in \text{int}(TK)$ . It turns out that the isotropic position yields a sharp upper bound for  $E(K)$ .

E. Milman proved in [108] that if  $K$  is isotropic in  $\mathbb{R}^n$ , then

$$M^*(K) \leq C \sqrt{n} (\ln n)^2 L_K \leq c_1 \sqrt{n} (\ln n)^2, \tag{1.2}$$

where the second inequality follows from the boundedness of  $L_n$ . The dependence on  $n$  is optimal up to the logarithmic factor. The dual estimate for  $M(K)$  in the isotropic position was obtained recently by Bizeul and Klartag [17], who showed that

$$M(K) \leq c_2 \frac{\ln n}{\sqrt{n}}. \tag{1.3}$$

Combining (1.2) and (1.3), we obtain

$$E(K) \leq c_3 (\ln n)^3 \tag{1.4}$$

for every convex body  $K \subset \mathbb{R}^n$ , which is optimal up to logarithmic factors. Thus, the isotropic position may be viewed as a natural  $\ell$ -position, up to polylogarithmic losses.

We present the main ideas behind these results in Section 10, and illustrate the significance of (1.4) through several applications to fundamental problems in asymptotic convex geometry.

We conclude this introduction with an overview of the article's structure.

- Section 2 introduces isotropic position, the slicing problem, Bourgain's isotropic constant conjecture, and their connections with asymptotic versions of other classical questions in convex geometry.
- Sections 3 and 5 discuss consequences of the slicing theorem in the asymptotic theory of high-dimensional convex bodies.

- Sections 4–6 develop tools from asymptotic convex geometry and log-concave measures, leading to a reduction of the isotropic constant conjecture to a suitable small-ball estimate.
- Sections 7–9 present stochastic localization method, Klartag’s estimate towards the KLS conjecture, the proof of the slicing theorem and some of the key ideas underlying the proof of the thin-shell conjecture.
- Section 10 is devoted to the  $MM^*$ -estimate for isotropic convex bodies and some of its applications.
- Section 11 provides a detailed discussion of the technical covariance estimates that play a crucial role in the developments described above.

We refer to Schneider [127] for classical convex geometry, to [2, 3] for asymptotic theory, and to [32] for isotropic convex bodies and log-concave measures.

## 2. ISOTROPIC POSITION AND THE SLICING PROBLEM

We start with basic notation and definitions from convex geometry. We work in  $\mathbb{R}^n$ , which is equipped with the standard inner product  $\langle \cdot, \cdot \rangle$ . We denote the corresponding Euclidean norm by  $|\cdot|$ , and write  $B_2^n$  for the Euclidean unit ball, and  $S^{n-1}$  for the unit sphere. Volume in  $\mathbb{R}^n$  is denoted by  $\text{vol}_n$ . We write  $\omega_n$  for the volume of  $B_2^n$  and  $\sigma$  for the rotationally invariant probability measure on  $S^{n-1}$ . We also denote the Haar measure on  $O(n)$  by  $\nu$ . The Grassmann manifold  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  is equipped with the Haar probability measure  $\nu_{n,k}$ . For any integer  $1 \leq k \leq n-1$  and any  $F \in G_{n,k}$  we denote the orthogonal projection from  $\mathbb{R}^n$  onto  $F$  by  $P_F$ . We also define  $B_F = B_2^n \cap F$  and  $S_F = S^{n-1} \cap F$ .

The letters  $c, c', c_1, c_2$  etc. denote absolute positive constants whose value may change from line to line. Whenever we write  $a \approx b$ , we mean that there exist absolute constants  $c_1, c_2 > 0$  such that  $c_1 a \leq b \leq c_2 a$ . Also if  $K, C \subseteq \mathbb{R}^n$  we will write  $K \approx C$  if there exist absolute constants  $c_1, c_2 > 0$  such that  $c_1 K \subseteq C \subseteq c_2 K$ .

A convex body in  $\mathbb{R}^n$  is a compact convex subset  $K$  of  $\mathbb{R}^n$  with non-empty interior. We say that  $K$  is symmetric if  $K = -K$ , and that  $K$  is centered if its barycenter  $\text{bar}(K)$  is at the origin, i.e. if

$$\int_K \langle x, \xi \rangle dx = 0$$

for every  $\xi \in S^{n-1}$ . If  $K$  is a centered convex body in  $\mathbb{R}^n$  then  $\{T(K) : T \in GL_n\}$  is the family of positions of  $K$ .

The radial function  $\rho_K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^+$  of a convex body  $K$  with  $0 \in \text{int}(K)$  is the function  $\rho_K(x) = \max\{t > 0 : tx \in K\}$ , and the support function of  $K$  is defined for every  $y \in \mathbb{R}^n$  by  $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$ . The mean width of  $K$  is the expectation

$$w(K) = \int_{S^{n-1}} h_K(\xi) d\sigma(\xi)$$

of  $h_K$  over the sphere. The radius of  $K$  is

$$R(K) = \max\{|x| : x \in K\}$$

and the volume radius of  $K$  is the quantity

$$\text{vrad}(K) = \left( \frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)} \right)^{1/n}.$$

The polar body  $K^\circ$  of a convex body  $K$  with  $0 \in \text{int}(K)$  is the convex body

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

For every convex body  $K \subseteq \mathbb{R}^n$  we write  $\bar{K}$  for the multiple of  $K$  that has volume 1; in other words,  $\bar{K} := \text{vol}_n(K)^{-1/n}K$ .

A closed bounded set  $K \neq \{0\}$  in  $\mathbb{R}^n$  is called a star body if for every  $x \in K \setminus \{0\}$  we have that the interval  $[0, x)$  is contained in the interior of  $K$  (thus, every straight line passing through the origin crosses the boundary of  $K$  at exactly two points different from the origin), and the Minkowski functional of  $K$  defined by  $p_K(x) = \inf\{t > 0 : x \in tK\}$  is a continuous function on  $\mathbb{R}^n$ .

**§ 2.1. Isotropic convex bodies.** A convex body  $K$  in  $\mathbb{R}^n$  is called isotropic if it has volume  $\text{vol}_n(K) = 1$ , it is centered, and there is a constant  $\alpha > 0$  such that

$$\int_K \langle x, y \rangle^2 dx = \alpha^2 |y|^2 \quad (2.1)$$

for all  $y \in \mathbb{R}^n$ . It is useful to note that the isotropic condition (2.1) is equivalent to the fact that

$$\int_K x_i x_j dx = \alpha^2 \delta_{ij} \quad (2.2)$$

for every  $i, j = 1, \dots, n$ , where  $x_j = \langle x, e_j \rangle$  are the coordinates of  $x$  with respect to any given orthonormal basis  $\{e_1, \dots, e_n\}$  of  $\mathbb{R}^n$ . This is in turn equivalent to the fact that for every linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (we write  $T \in L(\mathbb{R}^n)$ ),

$$\int_K \langle x, Tx \rangle dx = \alpha^2 \text{tr}(T). \quad (2.3)$$

Another consequence of the isotropic condition (2.1) is that

$$\int_K |x|^2 dx = \sum_{i=1}^n \int_K \langle x, e_i \rangle^2 dx = n\alpha^2.$$

Note that if  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then  $U(K)$  is also isotropic for every  $U \in O(n)$ .

The next proposition shows that every centered convex body has a linear image which satisfies the isotropic condition. Moreover, this isotropic position of  $K$  is unique up to an orthogonal transformation.

**Proposition 2.1.** *Let  $K$  be a centered convex body in  $\mathbb{R}^n$ . There exists  $T \in GL_n$  such that  $T(K)$  is isotropic.*

To see this, note that the operator  $M_K \in L(\mathbb{R}^n)$  defined by  $M_K(y) = \int_K \langle x, y \rangle x dx$  is symmetric and positive definite; therefore, it has a symmetric and positive definite square root  $S$ . Consider the linear image  $\tilde{K} = S^{-1}(K)$  of  $K$ . Then, for every  $y \in \mathbb{R}^n$  we

have

$$\begin{aligned} \int_{\tilde{K}} \langle x, y \rangle^2 dx &= |\det S|^{-1} \int_K \langle S^{-1}x, y \rangle^2 dx = |\det S|^{-1} \int_K \langle x, S^{-1}y \rangle^2 dx \\ &= |\det S|^{-1} \left\langle \int_K \langle x, S^{-1}y \rangle x dx, S^{-1}y \right\rangle = |\det S|^{-1} \langle M_K S^{-1}y, S^{-1}y \rangle = |\det S|^{-1} |y|^2. \end{aligned}$$

Normalizing the volume of  $\tilde{K}$  we obtain an isotropic convex body.

We can now show that the isotropic position of a convex body is uniquely determined up to orthogonal transformations, and arises as a solution of a minimization problem. Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Define

$$\Delta(K) = \inf \left\{ \int_{TK} |x|^2 dx : T \in SL_n \right\}. \quad (2.4)$$

We shall show that a position  $K_1$  of  $K$ , of volume 1, is isotropic if and only if

$$\int_{K_1} |x|^2 dx = \Delta(K). \quad (2.5)$$

Fix an isotropic position  $K_1$  of  $K$ . We know that there exists  $\alpha > 0$  such that

$$\int_{K_1} \langle x, Tx \rangle dx = \alpha^2 \text{tr}(T)$$

for every  $T \in L(\mathbb{R}^n)$ . Then, for every  $T \in SL_n$  we have

$$\int_{TK_1} |x|^2 dx = \int_{K_1} |Tx|^2 dx = \int_{K_1} \langle x, T^*Tx \rangle dx = \alpha^2 \text{tr}(T^*T) \geq n\alpha^2 = \int_{K_1} |x|^2 dx, \quad (2.6)$$

where we have used the arithmetic-geometric means inequality in the form  $\text{tr}(T^*T) \geq n(\det(T^*T))^{1/n}$ . This shows that  $K_1$  satisfies (2.5). In particular, the infimum in (2.4) is a minimum.

Note also that if we have equality in (2.6) then  $T^*T = I_n$ , the identity operator, and hence  $T \in O(n)$ . This shows that any other position  $\tilde{K}$  of  $K$  which satisfies (2.5) is an orthogonal image of  $K_1$ , therefore it is isotropic. Finally, if  $K_2$  is some other isotropic position of  $K$  then the first part of the proof shows that  $K_2$  satisfies (2.5). By the previous step, we must have  $K_2 = U(K_1)$  for some  $U \in O(n)$ .

Based on the above we may define the isotropic constant of any convex body  $K$  in  $\mathbb{R}^n$  by

$$L_K^2 = \frac{1}{n} \min \left\{ \frac{1}{\text{vol}_n(T\tilde{K})^{1+\frac{2}{n}}} \int_{T\tilde{K}} |x|^2 dx : T \in GL_n \right\},$$

where  $\tilde{K} = K - \text{bar}(K)$  is the centered translate of  $K$ . Note that  $L_K$  depends only on the affine class of  $K$ . Note also that if  $K$  is isotropic then for all  $\xi \in S^{n-1}$  we have

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2.$$

The isotropic constant conjecture, which is now a theorem, is the following statement.

**Theorem 2.2** (isotropic constant problem). *There exists an absolute constant  $C > 0$  such that for any  $n \geq 1$  and any convex body  $K$  in  $\mathbb{R}^n$  we have*

$$L_K \leq C.$$

A strong formulation of the isotropic constant problem asks whether equality in

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \quad (2.7)$$

is attained when the convex body  $K \subset \mathbb{R}^n$  is a simplex. An affirmative answer to this question, together with the equality  $\sup_n L_n = 1/e$ , would have far-reaching consequences. In particular, it would imply Mahler's conjecture concerning the product of the volume of a convex body and the volume of its polar body; see [77].

There is an analogous strong version of the slicing problem in the class of centrally symmetric convex bodies. In this setting, one asks whether the supremum in (2.7), when restricted to centrally symmetric convex bodies (that is, the ones satisfying  $K = -K$ ), is attained by the cube. If this conjecture were true, it would imply the Minkowski lattice conjecture; see, for instance, Magazinov [103].

As we will see, the isotropic constant problem is equivalent to Bourgain's slicing problem.

**§ 2.2. Bourgain's slicing problem.** The isotropic constant problem was stated explicitly as a question in the article of V. Milman and Pajor [111] and in the PhD Thesis of K. Ball [6]. The question appears for the first time in the work of Bourgain [25] on high-dimensional maximal functions associated with arbitrary convex bodies. Bourgain was interested in bounds for the  $L_p$ -norm of the maximal function

$$\mathcal{M}_K f(x) = \sup \left\{ \frac{1}{\text{vol}_n(tK)} \int_{tK} |f(x+y)| dy : t > 0 \right\}$$

of  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ , where  $K$  is a symmetric convex body in  $\mathbb{R}^n$ . Let  $C_p(K)$  denote the best constant such that

$$\|\mathcal{M}_K f\|_p \leq C_p(K) \|f\|_p$$

and  $C_{1,1}(K)$  the best constant so that the weak type inequality

$$\|\mathcal{M}_K f\|_{1,\infty} \leq C_{1,1}(K) \|f\|_1$$

is satisfied (recall that  $\|g\|_{1,\infty} := \sup_{t>0} t g^*(t)$ , where  $g^*$  is the decreasing rearrangement of  $g$ , is the Lorentz norm of  $g$  with indices 1 and  $\infty$ ). Stein proved in [130] that if  $K = B_2^n$  is the Euclidean unit ball then  $C_p(B_2^n)$  is bounded independently of the dimension for all  $p > 1$ . Bourgain showed that there exists an absolute constant  $C > 0$  (independent of  $n$  and  $K$ ) such that

$$\|\mathcal{M}_K\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C.$$

Around the same time, the result for  $\|\mathcal{M}_K\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}$  was generalized to all  $p > 3/2$  by Bourgain [26] and, independently, by Carbery [35]. By the definition of  $\mathcal{M}_K$  it is clear that in order to obtain a uniform bound on  $\|\mathcal{M}_K\|_{2 \rightarrow 2}$  one can start with any suitable position  $T(K)$  of  $K$ . Bourgain used the isotropic position; the property that played an important role in his argument was that when  $K$  is isotropic then  $L_K |K \cap \xi^\perp| \approx 1$  for all  $\xi \in S^{n-1}$ . We shall explain this fact in this subsection. Bourgain [25] mentions the fact that  $L_K \geq c$  and asks whether a reverse inequality might hold true.

The slicing problem, which is now a theorem, can be formulated as follows.

**Theorem 2.3** (slicing problem). *There exists an absolute constant  $c > 0$  such that*

$$\max\{\text{vol}_{n-1}(K \cap \xi^\perp) : \xi \in S^{n-1}\} \geq c \quad (2.8)$$

for every centered convex body  $K$  of volume 1 in  $\mathbb{R}^n$ .

To see the connection of Bourgain's slicing problem with the isotropic constant problem, we shall exploit the fact that the moments of inertia of a centered convex body are closely related with the volume of its hyperplane sections that pass through the origin. In the isotropic case this relation takes the following form.

**Theorem 2.4.** *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . For every  $\xi \in S^{n-1}$  we have*

$$\frac{c_1}{\text{vol}_{n-1}(K \cap \xi^\perp)} \leq \left( \int_K \langle x, \xi \rangle^2 dx \right)^{1/2} \leq \frac{c_2}{\text{vol}_{n-1}(K \cap \xi^\perp)}, \quad (2.9)$$

where  $c_1, c_2 > 0$  are absolute constants. In particular, if  $K$  is isotropic then for every  $\xi \in S^{n-1}$  we have

$$\frac{c_1}{L_K} \leq \text{vol}_{n-1}(K \cap \xi^\perp) \leq \frac{c_2}{L_K}. \quad (2.10)$$

For the proof, given  $\xi \in S^{n-1}$  we consider the function  $f(t) = f_{K,\xi}(t) = \text{vol}_{n-1}(K \cap \{x : \langle x, \xi \rangle = t\})$ ,  $t \in \mathbb{R}$ . It is a consequence of the Brunn-Minkowski inequality (see [2, Section 1.2]) that  $\ln f$  is a concave function on its support.

We restrict our attention to the symmetric case. Then,  $f$  is even and  $\|f\|_\infty = f(0)$ . For the proof in the general case, which is more or less the same, we need an additional fact (that can be found in Fradelizi [50]) which shows that hyperplane sections through the center of mass are, up to an absolute constant, maximal: If  $K$  is a centered convex body of volume 1 in  $\mathbb{R}^n$  then, for every  $\xi \in S^{n-1}$ ,

$$\|f\|_\infty \leq e f(0) = e \text{vol}_{n-1}(K \cap \xi^\perp).$$

*Proof of Theorem 2.4 (symmetric case).* Let  $f := f_{K,\xi}$ . To prove the left-hand side of (2.9) we set  $b = \int_0^{+\infty} f(t) dt = \frac{1}{2}$  and define

$$g(t) = \|f\|_\infty \mathbb{1}_{[0, b/\|f\|_\infty]}(t).$$

Since  $g \geq f$  on the support of  $g$ , we have

$$\int_0^s f(t) dt \leq \int_0^s g(t) dt$$

for every  $0 \leq s \leq b/\|f\|_\infty$ . The integrals of  $f$  and  $g$  on  $[0, +\infty)$  are both equal to  $b$ . It follows that

$$\int_s^\infty g(t) dt \leq \int_s^\infty f(t) dt$$

for every  $s \geq 0$ . Then,

$$\begin{aligned} \int_0^\infty t^2 f(t) dt &= \int_0^\infty \int_0^t 2s f(t) ds dt = \int_0^\infty 2s \left( \int_s^\infty f(t) dt \right) ds \\ &\geq \int_0^\infty 2s \left( \int_s^\infty g(t) dt \right) ds = \int_0^\infty t^2 g(t) dt \\ &= \int_0^{b/\|f\|_\infty} t^2 \|f\|_\infty dt = \frac{b^3}{3\|f\|_\infty^2}. \end{aligned}$$

This shows that

$$\int_K \langle x, \xi \rangle^2 dx = 2 \int_0^\infty t^2 f(t) dt \geq \frac{2b^3}{3\|f\|_\infty^2} = \frac{1}{12f(0)^2}.$$

To prove the right-hand side inequality of (2.9) we distinguish two cases. Assume first that there exists  $s > 0$  such that  $f(s) = \frac{1}{2}f(0)$ . Then,

$$\frac{1}{2} = \int_0^\infty f(t) dt \geq \int_0^s f(t) dt \geq s f(s) = \frac{1}{2} s f(0),$$

because, since  $\ln f$  is even and concave, we have that  $f(t) \geq f(s)$  on  $[0, s]$ . On the other hand, if  $t > s$ , then

$$f(s) \geq (f(0))^{1-\frac{s}{t}} (f(t))^{\frac{s}{t}},$$

which implies that  $f(t) \leq f(0)2^{-t/s}$ . We now write

$$\begin{aligned} \int_0^\infty t^2 f(t) dt &= \int_0^s t^2 f(t) dt + \int_s^\infty t^2 f(t) dt \leq f(0) \int_0^s t^2 dt + \int_s^\infty t^2 f(0)2^{-t/s} dt \\ &= f(0) \left( \frac{s^3}{3} + s^3 \int_1^\infty u^2 2^{-u} du \right) \leq c_0 f(0) s^3 \leq c_0 / (f(0))^2. \end{aligned}$$

Now, assume that, for every  $s > 0$  on the support of  $f$ , we have  $f(s) > \frac{1}{2}f(0)$ . Then, the role of  $s$  is played by  $s_0 = \sup\{s > 0 : f(s) > 0\}$ . We have  $\frac{1}{2} \geq \frac{1}{2}f(0)s_0$  and

$$\int_{-\infty}^\infty t^2 f(t) dt = 2 \int_0^\infty t^2 f(t) dt = 2 \int_0^{s_0} t^2 f(t) dt \leq \frac{2f(0)s_0^3}{3} \leq \frac{2}{3(f(0))^2}.$$

Thus, we get the same estimate as before, without using the fact that  $\ln f$  is concave.  $\square$

Theorem 2.4 reveals a close connection between the isotropic constant problem and the slicing problem. In fact, one direction is very simple by the previous discussion; assume that the slicing problem has an affirmative answer. If  $K$  is isotropic then Theorem 2.4 shows that *all* sections  $K \cap \xi^\perp$  have volume bounded from above by  $c_2/L_K$ . Since (2.8) must be true for at least one  $\xi \in S^{n-1}$ , we get  $L_K \leq c_2/c$ .

Conversely, we will show in § 2.5 that if there exists an absolute bound  $C$  for the isotropic constant, then the slicing problem has an affirmative answer. We shall first discuss the connection of the isotropic constant problem and of the slicing problem with two other classical problems from convex geometry and then establish the equivalence of all four of them.

**§ 2.3. Busemann-Petty problem.** The Busemann-Petty problem was posed in [34], first in a list of ten problems concerning central sections of symmetric convex bodies in  $\mathbb{R}^n$  and coming from questions in Minkowski geometry. It was originally formulated as follows:

*Assume that  $K_1$  and  $K_2$  are symmetric convex bodies in  $\mathbb{R}^n$  that satisfy*

$$\text{vol}_{n-1}(K_1 \cap \xi^\perp) \leq \text{vol}_{n-1}(K_2 \cap \xi^\perp)$$

*for all  $\xi \in S^{n-1}$ . Does it follow that  $\text{vol}_n(K_1) \leq \text{vol}_n(K_2)$ ?*

The first breakthrough on the Busemann-Petty problem came in 1975. Larman and Rogers [89] chose  $K_2 = B_2^n$  and proved that if  $n \geq 12$  then there exist symmetric convex bodies  $K_1$  which are arbitrarily small perturbations of  $B_2^n$  such that the pair  $(K_1, B_2^n)$  provides a negative answer to the problem. The proof is probabilistic in nature.

Ball [7] proved that if  $Q_n = [-1/2, 1/2]^n$  is the cube of volume 1 in  $\mathbb{R}^n$  then  $\text{vol}_{n-1}(Q_n \cap \xi^\perp) \leq \sqrt{2}$  for all  $\xi \in S^{n-1}$ . Then, he observed in [9] that, if  $n \geq 10$ , and if  $K_1 = Q_n$  and  $K_2$  is the Euclidean ball of volume 1, then

$$\text{vol}_{n-1}(Q_n \cap \xi^\perp) \leq \sqrt{2} < \omega_{n-1} \omega_n^{-\frac{n-1}{n}} = \text{vol}_{n-1}(K_2 \cap \xi^\perp)$$

for all  $\xi \in S^{n-1}$ . Ball's counterexample is essentially different from the one of Larman and Rogers: the cube is far from being a perturbation of a ball. After Ball's example, counterexamples to the Busemann-Petty problem were given independently by Giannopoulos [57] and Bourgain [27] for  $n \geq 7$ , and by Papadimitrakis [120] in dimensions  $n = 6$  and  $n = 5$ .

The key notion that led to the final solution to the Busemann-Petty problem is Lutwak's definition of an intersection body. The intersection body  $IK$  of a symmetric convex body  $K$  in  $\mathbb{R}^n$  is the symmetric convex body whose radial function is defined by

$$\rho_{IK}(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}.$$

Lutwak observed in [100] that intersection bodies are closely connected with the Busemann-Petty problem. Using simple facts from the theory of dual mixed volumes one can see that if  $K_1$  is an intersection body then the answer to the Busemann-Petty problem is affirmative for  $K_1$  and any symmetric convex body  $K_2$ . However, if  $K_1$  is a symmetric convex body that is not an intersection body, then  $K_1$  and a perturbation  $K_2$  of  $K_1$  provide a counterexample to the question. Therefore, the Busemann-Petty problem has an affirmative answer in  $\mathbb{R}^n$  if and only if every symmetric convex body in  $\mathbb{R}^n$  is an intersection body. Using this reduction of the problem, Gardner [52], [53] and Zhang [139] gave a negative answer to the problem for  $n \geq 5$  by providing examples of non-intersection bodies in  $\mathbb{R}^5$ . Around the same time, Gardner [54] proved that every symmetric convex body in  $\mathbb{R}^3$  is an intersection body, and hence the Busemann-Petty problem has an affirmative answer in dimension  $n = 3$ . For a few years it was believed that the problem has a negative answer in the remaining case  $n = 4$ . Through the work of Koldobsky who established a Fourier

analytic characterization of intersection bodies in [86] it was understood that the case  $n = 4$  was still open. Zhang [141] proved that the answer in  $\mathbb{R}^4$  is affirmative, and around the same time, using the Fourier analytic approach, Gardner, Koldobsky and Schlumprecht [56] gave a unified solution to the problem in all dimensions. Thus, the answer to the Busemann-Petty problem is positive if  $n \leq 4$  and negative for all higher dimensions (see the books of Gardner [55] and Koldobsky [87] for a complete discussion of the problem and its history).

The asymptotic version of the Busemann-Petty problem, which is now a theorem, can be formulated as follows.

**Theorem 2.5** (asymptotic Busemann-Petty problem). *There exists an absolute constant  $c > 0$  such that if  $K_1$  and  $K_2$  are centered convex bodies in  $\mathbb{R}^n$  that satisfy  $\text{vol}_{n-1}(K_1 \cap \xi^\perp) \leq \text{vol}_{n-1}(K_2 \cap \xi^\perp)$  for all  $\xi \in S^{n-1}$  then  $\text{vol}_n(K_1)^{\frac{n-1}{n}} \leq c \text{vol}_n(K_2)^{\frac{n-1}{n}}$ .*

In Section 2.5, assuming that the isotropic constant problem has an affirmative answer, we shall give a proof of Theorem 2.5, restricted to the class of symmetric convex bodies, with the help of Busemann's formula. Later on we shall discuss an alternative approach which works for the class of centered convex bodies and also settles the lower-dimensional asymptotic Busemann-Petty problem.

**§ 2.4. Sylvester problem.** Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$  and let  $x_1, \dots, x_{n+1}$  be random points which are independently and uniformly distributed in  $K$ . Their convex hull  $\text{conv}\{x_1, \dots, x_{n+1}\}$  is a random simplex contained in  $K$ . For every  $p > 0$  we define

$$m_p(K) = \left( \int_K \cdots \int_K \text{vol}_n(\text{conv}\{x_1, \dots, x_{n+1}\})^p dx_{n+1} \cdots dx_1 \right)^{1/p}.$$

If we drop the assumption that  $\text{vol}_n(K) = 1$  then we normalize as follows:

$$m_p(K) = \left( \frac{1}{\text{vol}_n(K)^{n+p+1}} \int_K \cdots \int_K \text{vol}_n(\text{conv}\{x_1, \dots, x_{n+1}\})^p dx_{n+1} \cdots dx_1 \right)^{1/p}.$$

Then,  $m_p(K)$  is invariant under non-degenerate affine transformations: If  $T \in GL_n$  and  $u \in \mathbb{R}^n$ , then  $m_p(K) = m_p(TK + u)$  for all  $p > 0$ . The quantity  $m_1(K)$  is the expectation of the normalized volume of a random simplex inside  $K$ .

Sylvester's problem asks for the affine classes of convex bodies for which  $m_p(K)$  is minimized or maximized. It is known that, for every  $p > 0$ ,

$$m_p(K) \geq m_p(B_2^n)$$

with equality if and only if  $K$  is an ellipsoid (see e.g. Groemer [63] and [64] for the case  $p \geq 1$  and Giannopoulos and Tsolomitis [59] for an extension to all  $p > 0$ ). The problem of the maximum is completely open in dimensions  $n \geq 3$ . The simplex conjecture asserts that for every convex body  $K$  in  $\mathbb{R}^n$ ,

$$m_1(K) \leq m_1(\Delta_n)$$

where  $\Delta_n$  is a simplex in  $\mathbb{R}^n$ . This has been verified only in the planar case (see [18] and [19]).

Sylvester's problem is related to the isotropic constant problem: in fact, one can check that the simplex conjecture implies that  $L_n \leq C$ . To see this, we consider the variant of  $m_p(K)$

$$S_p(K) = \left( \frac{1}{\text{vol}_n(K)^{n+p}} \int_K \cdots \int_K \text{vol}_n(\text{conv}\{0, x_1, \dots, x_n\})^p dx_n \cdots dx_1 \right)^{1/p}.$$

It is not hard to compare the quantities  $m_p(K)$  and  $S_p(K)$ .

**Proposition 2.6.** *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Then, for every  $p \geq 1$  we have*

$$S_p(K) \leq m_p(K) \leq (n+1)S_p(K).$$

Now, note that  $\text{vol}_n(\text{conv}\{0, x_1, \dots, x_n\}) = \frac{1}{n!} |\det(x_1, \dots, x_n)|$ . The function  $f_i : K \rightarrow \mathbb{R}$  defined by  $x_i \mapsto |\det(x_1, \dots, x_n)|$  for fixed  $x_j$  in  $K$ ,  $j \neq i$ , is a seminorm, as is the function  $g_i : K \rightarrow \mathbb{R}$  defined by

$$x_i \mapsto \int_K \cdots \int_K |\det(x_1, \dots, x_n)| dx_{i+1} \cdots dx_n$$

for fixed  $x_j$  in  $K$  when  $j < i$ . It is a well-known fact, following from Borell's lemma (see Section 4 below) that

$$\|g\|_{L^q(K)} \leq \frac{cq}{p} \|g\|_{L^p(K)} \quad (2.11)$$

for any seminorm  $g$  and any  $1 \leq p < q$ . With successive applications of Fubini's theorem, using each time (2.11) with  $p = 1$  and  $q = 2$ , we obtain the following.

**Proposition 2.7.** *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Then,*

$$S_2(K) \leq c^n S_1(K),$$

where  $c > 0$  is an absolute constant.

Consider the matrix of inertia  $(M_K)_{i,j} = \int_K x_i x_j dx$  of  $K$  with respect to some fixed orthonormal basis of  $\mathbb{R}^n$ . The connection of  $m_p(K)$  and  $S_p(K)$  with the isotropic constant of  $K$  becomes clear by the next identity, which is known as Blaschke formula.

**Proposition 2.8** (Blaschke formula). *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Then,*

$$S_2^2(K) = \frac{\det(M_K)}{n!}.$$

*Proof.* By definition,

$$S_2^2(K) = \int_K \cdots \int_K \text{vol}_n(\text{conv}\{0, x_1, \dots, x_n\})^2 dx_n \cdots dx_1.$$

We write  $x_i = (x_{ij})$ ,  $j = 1, \dots, n$ . Then,

$$(n!)^2 S_2^2(K) = \int_K \cdots \int_K |\det(x_1, \dots, x_n)|^2 dx_n \cdots dx_1,$$

and expanding the determinant we get

$$\begin{aligned}
(n!)^2 S_2^2(K) &= \int_K \cdots \int_K \left( \sum_{\sigma} \varepsilon_{\sigma} \prod_{i=1}^n x_{i,\sigma(i)} \right) \left( \sum_{\tau} \varepsilon_{\tau} \prod_{i=1}^n x_{i,\tau(i)} \right) dx_n \cdots dx_1 \\
&= \int_K \cdots \int_K \left( \sum_{\sigma,\tau} \varepsilon_{\sigma} \varepsilon_{\tau} \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\tau(i)} \right) dx_n \cdots dx_1 \\
&= \int_K \cdots \int_K \left( \sum_{\sigma,\varphi} \varepsilon_{\varphi} \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\varphi(\sigma(i))} \right) dx_n \cdots dx_1 \\
&= \sum_{\sigma,\varphi} \varepsilon_{\varphi} \prod_{i=1}^n \left( \int_K x_i x_{\varphi(i)} dx \right) \\
&= n! \det(M_K),
\end{aligned}$$

which completes the proof.  $\square$

It is not hard to check that if  $K$  is a centered convex body of volume 1 in  $\mathbb{R}^n$  then for every  $T \in SL_n$  we have that  $M_{T(K)} = TM_K T^*$ , and hence  $\det(M_K) = \det(M_{T(K)})$ . If we choose  $T$  so that  $T(K)$  will be isotropic, then  $M_{T(K)} = L_K^2 I_n$ , and hence  $\det(M_K) = \det(M_{T(K)}) = L_K^{2n}$ . Therefore, we get the next theorem.

**Theorem 2.9.** *Let  $K$  be a centered convex body in  $\mathbb{R}^n$ . Then,*

$$L_K^{2n} = n! S_2^2(K).$$

**Remark 2.10.** Theorem 2.9 already gives the simple bound  $L_n \leq \sqrt{n}$ . Indeed, if  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then, noting that  $S_2(K)$  is obviously bounded by 1, we get

$$L_K \leq \sqrt[2n]{n!} \leq \sqrt{n}.$$

What is more interesting is the next observation.

**Fact 2.11.** *If the simplex conjecture is true, then  $L_K \leq C$  for every  $n \geq 1$  and any convex body  $K$  in  $\mathbb{R}^n$ , where  $C > 0$  is an absolute constant.*

*Proof.* Consider the simplex  $\Delta_n = \left\{ x \in \mathbb{R}^n : -\frac{1}{n+1} \leq x_i \leq \frac{n}{n+1}, \sum_{i=1}^n x_i \leq \frac{1}{n+1} \right\}$ . Then,  $\Delta'_n = (n!)^{1/n} \Delta_n$  has volume 1 and barycenter at the origin. A simple computation shows that

$$\int_{\Delta'_n} x_i^2 dx < \frac{(n!)^{1+\frac{2}{n}}}{(n+2)!},$$

and since  $M_K$  is symmetric and positive definite, Hadamard's inequality gives

$$S_2^2(\Delta'_n) = \frac{\det(M_{\Delta'_n})}{n!} \leq \frac{1}{n!} \left( \frac{(n!)^{1+\frac{2}{n}}}{(n+2)!} \right)^n \leq \frac{1}{n!}.$$

Now, if  $K$  is an isotropic convex body in  $\mathbb{R}^n$  we have assumed that  $m_1(K) \leq m_1(\Delta'_n)$ , and combining Propositions 2.6 and 2.8 with Theorem 2.9 we obtain

$$\begin{aligned}
L_K^n &= \sqrt{n!} S_2(K) \leq \sqrt{n!} c^n S_1(K) \leq \sqrt{n!} c^n m_1(K) \leq \sqrt{n!} c^n m_1(\Delta'_n) \\
&\leq \sqrt{n!} c^n (n+1) S_1(\Delta'_n) \leq \sqrt{n!} c^n (n+1) S_2(\Delta'_n) \leq (n+1) c^n.
\end{aligned}$$

This shows that  $L_K \leq 2c$ . □

Fact 2.11 has probably been, at least for some people, the strongest evidence that Bourgain's slicing problem should have an affirmative answer. Now that this has been confirmed, we can reverse the argument to obtain an affirmative answer to the so-called asymptotic Sylvester problem.

**Theorem 2.12** (asymptotic Sylvester problem). *Let  $K$  be a convex body in  $\mathbb{R}^n$ . Then,*

$$(m_1(K))^{1/n} \approx \frac{1}{\sqrt{n}}.$$

In fact, Blaschke formula shows that Theorem 2.12 is equivalent with an affirmative answer to the isotropic constant problem.

**§ 2.5. Equivalence of the four problems.** We have already seen that Theorem 2.3 implies Theorem 2.2, and that Theorem 2.12 is equivalent to Theorem 2.2.

Next, we show that Theorem 2.2 implies Theorem 2.3. Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Consider the ellipsoid  $\mathcal{E}_B(K)$  defined by

$$\|y\|_{\mathcal{E}_B(K)}^2 = \langle M_K y, y \rangle = \int_K \langle x, y \rangle^2 dx$$

where  $M_K$  is the matrix of inertia of  $K$ . The ellipsoid  $\mathcal{E}_B(K)$  is the Binet ellipsoid of  $K$ . Note that  $K$  is in isotropic position if and only if  $\mathcal{E}_B(K) = L_K^{-1} B_2^n$ . It is not hard to see that the volume of  $\mathcal{E}_B(K)$  is invariant under the action of  $SL_n$ . From the definition of the Binet ellipsoid we check that if  $T \in SL_n$  then  $\|y\|_{\mathcal{E}_B(T(K))} = \|T^* y\|_{\mathcal{E}_B(K)}$  for all  $y \in \mathbb{R}^n$ , therefore

$$\mathcal{E}_B(T(K)) = (T^*)^{-1}(\mathcal{E}_B(K)).$$

It follows that

$$\text{vol}_n(\mathcal{E}_B(T(K))) = \text{vol}_n(\mathcal{E}_B(K))$$

for every  $T \in SL_n$ . Now, we use the fact that  $T_0(K)$  is isotropic for some  $T_0 \in SL_n$  and this implies that

$$\text{vol}_n(\mathcal{E}_B(T(K))) = \text{vol}_n(\mathcal{E}_B(T_0(K))) = \text{vol}_n(L_K^{-1} B_2^n) = \omega_n L_K^{-n}.$$

Assume that the isotropic constant conjecture has an affirmative answer and let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Integration in spherical coordinates shows that

$$L_K^{-n} = \frac{\text{vol}_n(\mathcal{E}_B(K))}{\omega_n} = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}_B(K)}^{-n} d\sigma(\xi),$$

and hence there exists a direction  $\xi \in S^{n-1}$  such that

$$\int_K \langle x, \xi \rangle^2 dx = \|\xi\|_{\mathcal{E}_B(K)} \leq L_K^2 \leq C^2.$$

Then, from (2.9) we see that  $\text{vol}_{n-1}(K \cap \xi^\perp) \geq c_1/C$  and this proves the slicing conjecture.

It remains to show the equivalence of Theorem 2.2 with Theorem 2.5. In the symmetric case, one way to do this is using the Busemann formula.

**Theorem 2.13** (Busemann formula). *If  $K$  is a convex body in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ , then*

$$\text{vol}_n(K)^{n-1} = \frac{n! \omega_n}{2} \int_{S^{n-1}} \text{vol}_{n-1}(K \cap \xi^\perp)^n S_1(K \cap \xi^\perp) d\sigma(\xi).$$

Theorem 2.13 is an example of the so-called Blaschke-Petkantschin formulas which, roughly speaking, allow us to compute the integral of a function of  $k$ -tuples  $(x_1, \dots, x_k)$  of points in  $\mathbb{R}^n$  by first integrating over all such  $k$ -tuples in some  $F \in G_{n,k}$  and then averaging with respect to the Haar measure  $\nu_{n,k}$  on  $G_{n,k}$ .

Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Since all central hyperplane sections  $K \cap \xi^\perp$  are symmetric, and hence centered, from Theorem 2.9 we know that

$$\left(S_1(K \cap \xi^\perp)\right)^{\frac{1}{n-1}} \approx \left(S_2(K \cap \xi^\perp)\right)^{\frac{1}{n-1}} \approx \frac{L_{K \cap \xi^\perp}}{\sqrt{n}}.$$

Therefore, Theorem 2.13 has the following immediate consequence.

**Corollary 2.14.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$\text{vol}_n(K)^{\frac{n-1}{n}} \approx \left( \int_{S^{n-1}} L_{K \cap \xi^\perp}^{n-1} \text{vol}_{n-1}(K \cap \xi^\perp)^n d\sigma(\xi) \right)^{1/n}.$$

Given that the isotropic constant of any convex body is bounded from above and from below by an absolute positive constant, we get:

**Corollary 2.15.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$\text{vol}_n(K)^{\frac{n-1}{n}} \approx \left( \int_{S^{n-1}} \text{vol}_{n-1}(K \cap \xi^\perp)^n d\sigma(\xi) \right)^{1/n}.$$

Now, let  $K_1$  and  $K_2$  be two symmetric convex bodies in  $\mathbb{R}^n$  that satisfy

$$\text{vol}_{n-1}(K_1 \cap \xi^\perp) \leq \text{vol}_{n-1}(K_2 \cap \xi^\perp)$$

for all  $\xi \in S^{n-1}$ . Corollary 2.15 shows that

$$\begin{aligned} \text{vol}_n(K_1)^{n-1} &\leq c_1^n \int_{S^{n-1}} \text{vol}_{n-1}(K_1 \cap \xi^\perp)^n d\sigma(\xi) \leq c_1^n \int_{S^{n-1}} \text{vol}_{n-1}(K_2 \cap \xi^\perp)^n d\sigma(\xi) \\ &\leq c_2^n \text{vol}_n(K_2)^{n-1}, \end{aligned}$$

which gives

$$\text{vol}_n(K_1)^{\frac{n-1}{n}} \leq c_3 \text{vol}_n(K_2)^{\frac{n-1}{n}}$$

for an absolute constant  $c_3 > 0$ . This shows that Theorem 2.2 implies Theorem 2.5 in the symmetric case.

Conversely, assuming Theorem 2.5 we can give a direct proof of Theorem 2.2. Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Choose  $\xi_0 \in S^{n-1}$  so that

$$\text{vol}_{n-1}(K \cap \xi_0^\perp) = \max_{\xi \in S^{n-1}} \text{vol}_{n-1}(K \cap \xi^\perp)$$

and  $r > 0$  so that  $\omega_{n-1} r^{n-1} = \text{vol}_{n-1}(K \cap \xi_0^\perp)$ . Then,

$$\text{vol}_{n-1}(K \cap \xi^\perp) \leq \omega_{n-1} r^{n-1} = \text{vol}_{n-1}((rB_2^n) \cap \xi^\perp)$$

for all  $\xi \in S^{n-1}$ , therefore

$$\text{vol}_n(K)^{n-1} \leq c^n \text{vol}_n(rB_2^n)^{n-1} = \frac{c^n \omega_n^{n-1}}{\omega_{n-1}^n} \text{vol}_{n-1}(K \cap \xi_0^\perp)^n \leq c_1^n \text{vol}_{n-1}(K \cap \xi_0^\perp)^n,$$

for some absolute constant  $c_1 > 0$ . Since  $\text{vol}_n(K) = 1$ , we see that

$$\text{vol}_{n-1}(K \cap \xi_0^\perp) \geq 1/c_1.$$

On the other hand,  $K$  is isotropic and hence we have  $\text{vol}_{n-1}(K \cap \xi^\perp) \approx 1/L_K$  for every  $\xi \in S^{n-1}$ . It follows that  $L_K \leq C$  for some absolute constant  $C > 0$ .

### 3. THE ISOTROPIC POSITION IS AN $M$ -POSITION

Let  $K$  and  $C$  be two convex bodies in  $\mathbb{R}^n$ . The covering number  $N(K, C)$  of  $K$  by  $C$  is the least integer  $N$  for which there exist  $N$  translates of  $C$  whose union covers  $K$ :

$$N(K, C) = \min \left\{ N \in \mathbb{N} : \exists x_1, \dots, x_N \in \mathbb{R}^n \text{ such that } K \subseteq \bigcup_{j=1}^N (x_j + C) \right\}.$$

The next theorem of V. Milman [110] establishes the existence of the so-called  $M$ -position of a convex body.

**Theorem 3.1.** *There exists an absolute constant  $\beta > 0$  such that every centered convex body  $K$  in  $\mathbb{R}^n$  has a linear image  $\tilde{K}$  which satisfies  $\text{vol}_n(\tilde{K}) = \text{vol}_n(B_2^n)$  and*

$$\max \left\{ N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ) \right\} \leq \exp(\beta n). \quad (3.1)$$

We say that a convex body  $\tilde{K}$  which has volume  $\text{vol}_n(\tilde{K}) = \text{vol}_n(B_2^n)$  and satisfies (3.1) is in  $M$ -position with constant  $\beta$ .

Pisier [122] (see also [123]) has proposed a different approach to this result in the symmetric case, which allows one to find a whole family of  $M$ -positions and to give more detailed information on the behavior of the corresponding covering numbers. The precise statement is as follows.

**Theorem 3.2.** *For every  $0 < \alpha < 2$  and every symmetric convex body  $K$  in  $\mathbb{R}^n$  there exists a linear image  $K_\alpha$  of  $K$  such that*

$$\max \left\{ N(K_\alpha, tB_2^n), N(B_2^n, tK_\alpha), N(K_\alpha^\circ, tB_2^n), N(B_2^n, tK_\alpha^\circ) \right\} \leq \exp \left( \frac{c(\alpha)n}{t^\alpha} \right)$$

for every  $t \geq c(\alpha)^{1/\alpha}$ , where  $c(\alpha)$  depends only on  $\alpha$ , and  $c(\alpha) = O((2-\alpha)^{-\alpha/2})$  as  $\alpha \rightarrow 2^-$ .

As we will see in this section, the fact that  $L_n \leq C$  implies that the isotropic position of any convex body is an  $M$ -position with an absolute constant  $\beta$ . This in turn shows that isotropic convex bodies satisfy the reverse Brunn-Minkowski inequality. It also leads to a simple proof of the reverse Santaló inequality of Bourgain and V. Milman [29].

**§ 3.1. Covering estimates.** We assume that  $L_n \leq C$ . Our goal is to show that every isotropic convex body is in  $M$ -position with an absolute constant  $\beta$ .

**Theorem 3.3.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then,*

$$\max \left\{ \ln N(K, D_n), \ln N(D_n, K), \ln N(K^\circ, D_n^\circ), \ln N(D_n^\circ, K^\circ) \right\} \leq \beta n,$$

where  $D_n$  is the centered Euclidean ball of volume 1 in  $\mathbb{R}^n$  and  $\beta > 0$  is an absolute constant.

The proof is based on an observation of V. Milman and Pajor in [111]; they gave an estimate for the covering numbers  $N(K, tB_2^n)$ ,  $t > 0$ , where  $K$  is a convex body in  $\mathbb{R}^n$ , in terms of the quantity

$$I_1(K) = \frac{1}{\text{vol}_n(K)^{1+\frac{1}{n}}} \int_K |x| dx.$$

**Lemma 3.4.** *Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$  such that  $0 \in \text{int}(K)$ . For any  $t > 0$  we have that*

$$\ln N(K, tB_2^n) \leq \frac{c_1 n I_1(K)}{t} + \ln 2, \quad (3.2)$$

where  $c_1 > 0$  is an absolute constant.

*Proof.* Consider the Borel probability measure  $\mu$  on  $\mathbb{R}^n$  defined by

$$\mu(A) = \frac{1}{c_K} \int_A e^{-p_K(x)} dx.$$

where  $p_K(x) = \inf\{t > 0 : x \in tK\}$  is the Minkowski functional of  $K$  and  $c_K = \int_{\mathbb{R}^n} \exp(-p_K(x)) dx$ . A simple computation, based on the fact that  $\{x \in \mathbb{R}^n : p_K(x) \leq t\} = tK$  for any  $t > 0$ , shows that  $c_K = n!$ .

Let  $\{x_1, \dots, x_N\}$  be a subset of  $K$  which is maximal with respect to the condition  $|x_i - x_j| \geq t$  for  $i \neq j$ . Then  $K \subseteq \bigcup_{1 \leq i \leq N} (x_i + tB_2^n)$ , and hence  $N(K, tB_2^n) \leq N$ . Let  $\alpha > 0$ . Note that if we set  $y_i = (2\alpha/t)x_i$ , by the subadditivity and positive homogeneity of  $p_K$  and the fact that  $p_K(x_i) \leq 1$ , we have

$$\mu(y_i + \alpha B_2^n) \geq \frac{1}{c_K} \int_{\alpha B_2^n} e^{-p_K(x)} e^{-p_K(y_i)} dx \geq e^{-2\alpha/t} \mu(\alpha B_2^n).$$

The bodies  $y_i + \alpha B_2^n$  have disjoint interiors, therefore  $N e^{-2\alpha/t} \mu(\alpha B_2^n) \leq 1$ . This shows that

$$N(K, tB_2^n) \leq e^{2\alpha/t} (\mu(\alpha B_2^n))^{-1}.$$

Now, we choose  $\alpha > 0$  so that  $\mu(\alpha B_2^n) \geq 1/2$ . A simple computation shows that

$$\gamma := \int_{\mathbb{R}^n} |x| d\mu(x) = (n+1)I_1(K).$$

By Markov's inequality,  $\mu(2\gamma B_2^n) \geq 1/2$ , so if we choose  $\alpha = 2\gamma$ , we get

$$N(K, tB_2^n) \leq 2 \exp(4(n+1)I_1(K)/t)$$

for every  $t > 0$ . □

If  $K$  is an isotropic convex body then  $I_1(K) \leq \sqrt{n}L_K$ . Let  $D_n = \overline{B}_2^n$  be the centered Euclidean ball of volume 1. Since  $D_n \approx \sqrt{n}B_2^n$ , Lemma 3.4 shows that

$$\ln N(K, tD_n) \leq \frac{c_2 n L_K}{t} \leq \frac{c_3 n}{t} \quad (3.3)$$

for any  $t > 0$  (note that if  $t$  is large then this estimate is trivially true, since every isotropic body  $K$  satisfies the inclusion  $K \subseteq (n+1)L_K B_2^n$ ; see [32, Theorem 3.2.1]).

Knowing that, for any set  $S$ ,

$$N(S - S, 2D_n) = N(S - S, D_n - D_n) \leq N(S, D_n)^2,$$

we can use (3.3) to also get an upper bound for the covering numbers of the difference body of an isotropic convex body  $K$  by  $D_n$ :

$$\ln N(K - K, tD_n) \leq \frac{2c_3 n}{t}.$$

The next lemma allows us to bound the dual covering numbers  $N(B_2^n, tK^\circ)$ .

**Lemma 3.5.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  which contains 0 in its interior. For every  $t > 0$  we set  $A(t) := t \ln N(K, tB_2^n)$  and  $B(t) := t \ln N(B_2^n, tK^\circ)$ . Then, one has*

$$\sup_{t>0} B(t) \leq 16 \sup_{t>0} A(t).$$

*In particular, if  $K$  is isotropic (or a translate of an isotropic convex body which still contains 0 in its interior), then*

$$\ln N(B_2^n, tK^\circ) \leq \ln N(B_2^n, t(K - K)^\circ) \leq \frac{c_2 n^{3/2} L_K}{t}, \quad (3.4)$$

where  $c_2 > 0$  is an absolute constant.

*Proof.* We use a well-known idea from [134] (see also [92, Section 3.3]). For any  $t > 0$  we have  $(t^2 K^\circ) \cap (4K) \subseteq 2tB_2^n$ . Passing to the polar bodies we see that

$$B_2^n \subseteq \text{conv} \left( \frac{t}{2} K^\circ, \frac{2}{t} K \right) \subseteq \frac{t}{2} K^\circ + \frac{2}{t} K.$$

We write

$$\begin{aligned} N(B_2^n, tK^\circ) &\leq N \left( \frac{t}{2} K^\circ + \frac{2}{t} K, tK^\circ \right) \leq N \left( \frac{2}{t} K, \frac{t}{2} K^\circ \right) \leq N \left( \frac{2}{t} K, \frac{1}{4} B_2^n \right) N \left( \frac{1}{4} B_2^n, \frac{t}{2} K^\circ \right) \\ &= N \left( K, \frac{t}{8} B_2^n \right) N(B_2^n, 2tK^\circ). \end{aligned}$$

Taking logarithms we get  $B(t) \leq 8A(t/8) + \frac{1}{2}B(2t)$  for all  $t > 0$ . This implies that  $B := \sup_{t>0} B(t) \leq 16A$ , and the result follows.  $\square$

Since  $D_n^\circ \approx (1/\sqrt{n})B_2^n$ , (3.3) and (3.4) immediately imply the following.

**Proposition 3.6.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then,*

$$\max\{\ln N(K, tD_n), \ln N(D_n^\circ, tK^\circ)\} \leq \frac{cn}{t}$$

for all  $t > 0$ , where  $c > 0$  is an absolute constant.

We shall also use the next covering lemma, which provides some standard entropy estimates that are valid for arbitrary convex bodies in  $\mathbb{R}^n$ .

**Lemma 3.7.** *Let  $K$  and  $L$  be convex bodies in  $\mathbb{R}^n$ . If  $L$  is symmetric, then*

$$N(K, L) \leq \frac{\text{vol}_n(K + L/2)}{\text{vol}_n(L/2)} \leq 2^n \frac{\text{vol}_n(K + L)}{\text{vol}_n(L)}. \quad (3.5)$$

*If  $L$  is arbitrary, then*

$$N(K, L) \leq 4^n \frac{\text{vol}_n(K + L)}{\text{vol}_n(L)}. \quad (3.6)$$

*Moreover,*

$$\frac{\text{vol}_n(K + L)}{\text{vol}_n(L)} \leq 2^n N(K, L). \quad (3.7)$$

*Proof.* The proof of (3.7) is an immediate consequence of the definitions. To prove (3.5), note that if  $N$  is a maximal subset of  $K$  with respect to the property

$$x, y \in N \text{ and } x \neq y \implies \|x - y\|_L \geq 1,$$

then  $K \subseteq \bigcup_{x \in N} (x + L)$ , while every two sets  $x + L/2, y + L/2$  ( $x, y \in N$ ) have disjoint interiors when  $x \neq y$ .

Finally, when  $L$  is not necessarily symmetric, we recall that  $N(K+x, L+y) = N(K, L)$  for every  $x, y \in \mathbb{R}^n$ , and also that the ratio  $\text{vol}_n(K + L)/\text{vol}_n(L)$  obviously remains unaltered if we translate  $K$  or  $L$ . Hence, we can assume that  $L$  is centered, in which case it follows from [112, Corollary 3] that

$$\text{vol}_n(L \cap (-L)) \geq 2^{-n} \text{vol}_n(L).$$

But then, from (3.5) we get that

$$N(K, L) \leq N(K, L \cap (-L)) \leq 2^n \frac{\text{vol}_n(K + (L \cap (-L)))}{\text{vol}_n(L \cap (-L))} \leq 4^n \frac{\text{vol}_n(K + L)}{\text{vol}_n(L)},$$

and this implies (3.6).  $\square$

Let  $K$  be a convex body in  $\mathbb{R}^n$ . The function  $z \mapsto \text{vol}_n(K) \text{vol}_n((K - z)^\circ)$ , defined on  $\text{int}(K)$ , is strictly convex and has a unique point of minimum, the Santaló point  $s(K)$  of  $K$ . The Blaschke-Santaló inequality states that  $\text{vol}_n(K) \text{vol}_n((K - s(K))^\circ) \leq \omega_n^2$ . One can also prove that if  $\text{bar}(K) = 0$  then  $s(K^\circ) = 0$ . Since  $(K^\circ)^\circ = K$ , we see that if  $\text{bar}(K) = 0$  then

$$\text{vol}_n(K) \text{vol}_n(K^\circ) = \text{vol}_n((K^\circ - s(K^\circ))^\circ) \text{vol}_n(K^\circ) \leq \omega_n^2.$$

Therefore, we get:

**Theorem 3.8.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  with either  $\text{bar}(K) = 0$  or  $s(K) = 0$ . Then,*

$$\text{vol}_n(K) \text{vol}_n(K^\circ) \leq \omega_n^2 = \text{vol}_n(B_2^n)^2.$$

Combining Proposition 3.6 with the Blaschke-Santaló inequality and Lemma 3.7, we can now prove Theorem 3.3.

*Proof of Theorem 3.3.* From Proposition 3.6 we already know that

$$\max \{N(K, D_n), N(D_n^\circ, K^\circ)\} \leq \exp(cn).$$

For the other two covering numbers we note that  $N(D_n, K) \leq 8^n N(K, D_n)$  by Lemma 3.7, which means that  $\ln N(D_n, K) \leq (c + \ln 8)n$ . Similarly,

$$N(K^\circ, D_n^\circ) \leq 2^n \frac{\text{vol}_n(K^\circ + D_n^\circ)}{\text{vol}_n(D_n^\circ)} \leq 2^n \frac{\text{vol}_n(K^\circ + D_n^\circ)}{\text{vol}_n(K^\circ)} \leq 4^n N(D_n^\circ, K^\circ),$$

which means that  $\ln N(K^\circ, D_n^\circ) \leq (c + \ln 4)n$ , where we have also used the fact that  $\text{vol}_n(K) = \text{vol}_n(D_n)$  and hence  $\text{vol}_n(K^\circ) \leq \text{vol}_n(D_n^\circ)$  from the Blaschke-Santaló inequality. This completes the proof.  $\square$

**§ 3.2. Bourgain-Milman inequality.** The Bourgain-Milman inequality [29], also known as reverse Santaló inequality, states that there exists an absolute constant  $0 < c < 1$  with the following property: for every  $n \geq 1$  and any convex body  $K$  in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ ,

$$s(K) = \text{vol}_n(K) \text{vol}_n(K^\circ) \geq c^n \omega_n^2 = c^n \text{vol}_n(B_2^n)^2, \quad (3.8)$$

where  $c > 0$  is an absolute constant. The existence of an  $M$ -position for any convex body and the asymptotic form of the Santaló inequality and its inverse are interconnected results. We illustrate this by giving a very simple proof of the Bourgain-Milman inequality starting from the fact that every isotropic convex body is in  $M$ -position.

The discussion about the Blaschke-Santaló inequality in the previous subsection shows that if  $0 \in \text{int}(K)$  then

$$\text{vol}_n(K) \text{vol}_n(K^\circ) \geq \text{vol}_n(K - s(K)) \text{vol}_n((K - s(K))^\circ)$$

therefore we may replace  $K$  by  $K_1 := K - s(K)$ . Then, we know that  $\text{bar}(K_1^\circ) = 0$ . It is also easily checked that

$$\text{vol}_n(K) \text{vol}_n(K^\circ) = \text{vol}_n(T(K)) \text{vol}_n((T(K))^\circ)$$

for every  $T \in GL_n$ , and hence we may assume that  $K_1^\circ$  is in isotropic position. Then, from Theorem 3.3 we know that

$$\max \left\{ \ln N(K_1^\circ, D_n), \ln N(D_n, K_1^\circ), \ln N(K_1, D_n), \ln N(D_n, K_1) \right\} \leq cn,$$

where  $D_n$  is the centered Euclidean ball of volume 1 in  $\mathbb{R}^n$  and  $c > 0$  is an absolute constant. It follows that

$$\begin{aligned} \text{vol}_n(D_n) &\leq N(D_n, K_1^\circ) \text{vol}_n(K_1^\circ) \leq e^{cn} \text{vol}_n(K_1^\circ), \\ \text{vol}_n(D_n^\circ) &\leq N(D_n^\circ, K_1) \text{vol}_n(K_1) \leq e^{cn} \text{vol}_n(K_1), \end{aligned}$$

and hence

$$\omega_n^2 = \text{vol}_n(D_n) \text{vol}_n(D_n^\circ) \leq e^{2cn} \text{vol}_n(K_1) \text{vol}_n(K_1^\circ)$$

which proves (3.8).

**§ 3.3. Reverse Brunn-Minkowski inequality.** As a consequence of Theorem 3.3 and Lemma 3.7, we get the “reverse” Brunn-Minkowski inequality.

**Theorem 3.9.** *Let  $K_1, \dots, K_m$  be isotropic convex bodies in  $\mathbb{R}^n$ . Then, for any  $\lambda_1, \dots, \lambda_m > 0$ ,*

$$\text{vol}_n(\lambda_1 K_1 + \dots + \lambda_m K_m)^{1/n} \leq c_1 m \sum_{j=1}^m \lambda_j \text{vol}_n(K_j)^{1/n} \quad (3.9)$$

where  $c_1 > 0$  is an absolute constant. Moreover, if  $K_1$  and  $K_2$  are isotropic convex bodies in  $\mathbb{R}^n$  then, for any  $\lambda_1, \lambda_2 > 0$ ,

$$\text{vol}_n(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ)^{1/n} \leq c_2 (\lambda_1 \text{vol}_n(K_1^\circ)^{1/n} + \lambda_2 \text{vol}_n(K_2^\circ)^{1/n}). \quad (3.10)$$

where  $c_2 > 0$  is an absolute constant.

*Proof.* From Proposition 3.6 we know that

$$N(\lambda_1 K_1 + \cdots + \lambda_m K_m, t(\lambda_1 + \cdots + \lambda_m) D_n) \leq \prod_{j=1}^m N(K_j, t D_n) \leq \exp(cmn/t),$$

and hence

$$\text{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)^{1/n} \leq e^{cm/t} (t(\lambda_1 + \cdots + \lambda_m))$$

for every  $t > 0$ . Choosing  $t = cm$  we see that

$$\text{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)^{1/n} \leq c_1 m \sum_{j=1}^m \lambda_j = c_1 m \sum_{j=1}^m \lambda_j \text{vol}_n(K_j)^{1/n}$$

with  $c_1 = ec$ . For the second claim, note that Theorem 3.3 implies that

$$N(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ, t(\lambda_1 + \lambda_2) D_n^\circ) \leq \exp(2cn/t)$$

for every  $t > 0$ , and apply the same reasoning as before to write

$$\text{vol}_n(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ)^{1/n} \leq 2c(\lambda_1 + \lambda_2) \text{vol}_n(D_n^\circ)^{1/n}.$$

Since  $K_1$  and  $K_2$  are isotropic, the argument of the previous subsection shows that  $\text{vol}_n(D_n^\circ) \leq e^{cn} \text{vol}_n(K_i^\circ)$  for  $i = 1, 2$ . In this way we obtain (3.10) with  $c_2 = 2ce^c$ .  $\square$

#### 4. LOG-CONCAVE PROBABILITY MEASURES

A Borel measure  $\mu$  on  $\mathbb{R}^n$  is called log-concave if  $\mu(H) < 1$  for every hyperplane  $H$  in  $\mathbb{R}^n$  and

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any pair of compact sets  $A, B$  in  $\mathbb{R}^n$  and any  $\lambda \in (0, 1)$ . Borell [23] has proved that, under these assumptions,  $\mu$  has a log-concave density  $f_\mu$ . Recall that a function  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is called log-concave if its support  $\{f > 0\}$  is a convex set in  $\mathbb{R}^n$  and the restriction of  $\ln f$  to it is concave. The Brunn-Minkowski inequality implies that if  $K$  is a convex body in  $\mathbb{R}^n$  then the indicator function  $\mathbb{1}_K$  of  $K$  is the density of a log-concave measure, the Lebesgue measure on  $K$ .

We say that  $\mu$  is symmetric if  $\mu(-B) = \mu(B)$  for every Borel subset  $B$  of  $\mathbb{R}^n$  and that  $\mu$  is centered if the barycenter  $\text{bar}(\mu) = \int_{\mathbb{R}^n} x d\mu(x)$  of  $\mu$  is at the origin, i.e.

$$\int_{\mathbb{R}^n} \langle x, \xi \rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi \rangle f_\mu(x) dx = 0$$

for all  $\xi \in S^{n-1}$ . If  $\mu$  is symmetric then  $f_\mu$  is even and it follows that  $\|f_\mu\|_\infty = f_\mu(0)$ . On the other hand, Fradelizi [50] has shown that if  $\mu$  is a centered log-concave probability measure then

$$\|f_\mu\|_\infty \leq e^n f_\mu(0). \quad (4.1)$$

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a measurable function. For any  $\alpha \geq 1$  we define the  $\psi_\alpha$ -norm of  $f$  as follows:

$$\|f\|_{\psi_\alpha} := \inf \left\{ t > 0 : \int_{\Omega} \exp((|f|/t)^\alpha) d\mu \leq 2 \right\},$$

provided that the set on the right-hand side is non-empty. Note that the  $\psi_\alpha$ -norm is exactly the Orlicz norm corresponding to the function  $t \in \mathbb{R} \rightarrow e^{|t|^\alpha} - 1$ . An equivalent expression for the  $\psi_\alpha$ -norm in terms of the  $L_p$ -norms is that

$$\|f\|_{\psi_\alpha} \approx \sup_{p \geq \alpha} \frac{\|f\|_{L_p(\mu)}}{p^{1/\alpha}},$$

up to some absolute constants.

A well-known lemma of Borell (see [32, Lemma 2.4.5] for a proof) asserts that if  $\mu$  is a log-concave probability measure on  $\mathbb{R}^n$  then, for any symmetric convex set  $A$  in  $\mathbb{R}^n$  with  $\mu(A) = \alpha \in (0, 1)$  and any  $t > 1$  we have

$$1 - \mu(tA) \leq \alpha \left( \frac{1 - \alpha}{\alpha} \right)^{\frac{t+1}{2}}. \quad (4.2)$$

Using Borell's lemma one can show that there exists an absolute constant  $c_0 > 0$  such that for every log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ , for any seminorm  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and any  $q > p \geq 1$ , we have

$$\left( \int_{\mathbb{R}^n} |f|^p d\mu \right)^{1/p} \leq \left( \int_{\mathbb{R}^n} |f|^q d\mu \right)^{1/q} \leq c_0 \frac{q}{p} \left( \int_{\mathbb{R}^n} |f|^p d\mu \right)^{1/p} \quad (4.3)$$

(a proof is given in [32, Theorem 2.4.6]). In particular,  $\|f\|_{\psi_1} \leq c_1 \|f\|_1$ , where  $c_1 > 0$  is an absolute constant.

**§ 4.1. Isotropic log-concave probability measures.** For any log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  with density  $f_\mu$ , we define the isotropic constant of  $\mu$  by

$$L_\mu := \left( \frac{\sup_{x \in \mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x) dx} \right)^{\frac{1}{n}} \left( \det \text{Cov}(\mu) \right)^{\frac{1}{2n}},$$

where  $\text{Cov}(\mu)$  is the covariance matrix of  $\mu$  with entries

$$\text{Cov}(\mu)_{i,j} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx}.$$

A log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  is called isotropic if it is centered and  $\text{Cov}(\mu) = I_n$ , where  $I_n$  is the identity  $n \times n$  matrix. By the definition of the isotropic constant, if  $\mu$  is isotropic then  $L_\mu = \|f_\mu\|_\infty^{1/n}$ . Note that a convex body  $K$  of volume 1 is isotropic if and only if the log-concave probability measure  $\mu_K$  with density  $L_K^n \mathbb{1}_{K/L_K}$  is isotropic.

Let  $\mu$  and  $\nu$  be two log-concave probability measures on  $\mathbb{R}^n$ . Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a measurable function which is defined  $\mu$ -almost everywhere and satisfies

$$\nu(B) = \mu(T^{-1}(B))$$

for every Borel subset  $B$  of  $\mathbb{R}^n$ . We then say that  $T$  pushes forward  $\mu$  to  $\nu$  and write  $T_*\mu = \nu$ . It is not hard to check that for every log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  there exists an invertible affine transformation  $T$  such that the log-concave probability measure  $T_*\mu$  is isotropic, and  $L_{T_*\mu} = L_\mu$ .

One can prove that the isotropic constants of all log-concave measures are uniformly bounded from below by a constant  $c > 0$  which is independent of the dimension. If  $\mu$  is an isotropic log-concave probability measure, then

$$L_\mu = \|f_\mu\|_\infty^{1/n} \approx (f_\mu(0))^{1/n} \geq c, \quad (4.4)$$

where  $c > 0$  is an absolute constant (see [32, Proposition 2.3.12]). The isotropic constant problem for log-concave measures, which is now a theorem, can be stated as follows:

*Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . Then,  $L_\mu = \|f_\mu\|_\infty^{1/n} \leq C$ , where  $C > 0$  is an absolute constant.*

We can define  $\tilde{L}_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\}$ . Since  $L_K = L_{\mu_K} \leq \tilde{L}_n$  for every centered convex body  $K$  in  $\mathbb{R}^n$ , it is clear that  $L_n \leq \tilde{L}_n$ .

**§ 4.2. K. Ball's bodies.** Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . Note that  $f_\mu(0) > 0$ , where  $f_\mu$  is the density of  $\mu$ . An important family of convex bodies associated with  $\mu$  was introduced by K. Ball, who also established their convexity in [8]: for every  $p > 0$ , we define

$$K_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_0^\infty r^{p-1} f_\mu(rx) dr \geq \frac{f_\mu(0)}{p} \right\}.$$

From the definition it follows that the radial function of  $K_p(\mu)$  is given by

$$\varrho_{K_p(\mu)}(x) = \left( \frac{1}{f_\mu(0)} \int_0^\infty pr^{p-1} f_\mu(rx) dr \right)^{1/p} \quad (4.5)$$

for  $x \neq 0$ . It is easily checked that if  $K$  is a convex body in  $\mathbb{R}^n$  with  $0 \in K$  then  $K_p(\mathbf{1}_K) = K$  for all  $p > 0$ . A very useful observation is that

$$f_\mu(0) \text{vol}_n(K_n(\mu)) = 1.$$

To see this, we write

$$\begin{aligned} \text{vol}_n(K_n(\mu)) &= \int_{K_n(\mu)} \mathbf{1} dx = n\omega_n \int_{S^{n-1}} \int_0^{\rho_{K_n(\mu)}(\xi)} r^{n-1} dr d\sigma(\xi) \\ &= \frac{n\omega_n}{f_\mu(0)} \int_{S^{n-1}} \int_0^\infty r^{n-1} f_\mu(r\xi) dr d\sigma(\xi) = \frac{1}{f_\mu(0)} \int_{\mathbb{R}^n} f_\mu(x) dx = \frac{1}{f_\mu(0)} \end{aligned}$$

using (4.5) and integration in spherical coordinates.

Another important observation is that, for every  $0 < p \leq q$  we have that

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} K_q(\mu) \subseteq K_p(\mu) \subseteq e^{\frac{n}{p}-\frac{n}{q}} K_q(\mu). \quad (4.6)$$

A proof is given in [32, Proposition 2.5.7]. As a consequence we obtain an approximate formula for the volume of  $K_{n+p}(\mu)$ . When  $p > 0$ , we have

$$e^{-1} \leq f_\mu(0)^{\frac{1}{n} + \frac{1}{p}} \text{vol}_n(K_{n+p}(\mu))^{\frac{1}{n} + \frac{1}{p}} \leq e^{\frac{n+p}{n}}, \quad (4.7)$$

while for  $-n < p < 0$  we have

$$e^{-1} \leq f_\mu(0)^{\frac{1}{-p} - \frac{1}{n}} \text{vol}_n(K_{n+p}(\mu))^{\frac{1}{-p} - \frac{1}{n}} \leq e. \quad (4.8)$$

If  $\mu$  is an even log-concave probability measure on  $\mathbb{R}^n$  then  $T = K_{n+2}(\mu)$  is a symmetric convex body that satisfies

$$c_1 L_\mu \leq L_T \leq c_2 L_\mu,$$

where  $c_1, c_2 > 0$  are absolute constants (see [32, Proposition 2.5.9]). Furthermore, if  $\mu$  is isotropic, then  $\bar{T} = \text{vol}_n(T)^{-1/n} T$  is an isotropic convex body. In the case where the measure  $\mu$  is centered, but not necessarily symmetric, we prefer to work with the convex body  $K_{n+1}(\mu)$  instead of  $K_{n+2}(\mu)$ . The reason is that  $T = K_{n+1}(\mu)$  is a centered convex body in  $\mathbb{R}^n$  and we still have that

$$c_1 L_\mu \leq L_T \leq c_2 L_\mu,$$

where  $c_1, c_2 > 0$  are absolute constants.

We briefly sketch the proof (for more details see [32, Proposition 2.5.12]). First we check that  $T$  is centered and satisfies

$$\int_T |\langle x, \xi \rangle| dx = \frac{1}{f_\mu(0)} \int_{\mathbb{R}^n} |\langle x, \xi \rangle| f_\mu(x) dx$$

for all  $\xi \in S^{n-1}$ . Borell's lemma implies that for every  $y \in \mathbb{R}^n$

$$\begin{aligned} \left( \frac{1}{\text{vol}_n(T)} \int_T \langle x, y \rangle^2 dx \right)^{1/2} &\approx \frac{1}{\text{vol}_n(T)} \int_T |\langle x, y \rangle| dx = \frac{1}{f_\mu(0) \text{vol}_n(T)} \int_{\mathbb{R}^n} |\langle x, y \rangle| f_\mu(x) dx \\ &\approx \frac{1}{f_\mu(0) \text{vol}_n(T)} \left( \int_{\mathbb{R}^n} \langle x, y \rangle^2 f_\mu(x) dx \right)^{1/2}, \end{aligned}$$

which, combined with the fact that  $T$  and  $f_\mu$  are both centered, implies that there exist absolute constants  $c_1, c_2 > 0$  such that as positive definite matrices

$$c_1 \text{Cov}(\mathbb{1}_T) \leq (\text{vol}_n(T) f_\mu(0))^{-2} \text{Cov}(\mu) \leq c_2 \text{Cov}(\mathbb{1}_T).$$

Therefore,

$$\left( \det \text{Cov}(\mathbb{1}_T) \right)^{1/n} \approx (\text{vol}_n(T) f_\mu(0))^{-2} \left( \det \text{Cov}(\mu) \right)^{1/n}. \quad (4.9)$$

From the definition of the isotropic constant it follows that

$$\begin{aligned} L_T &= \frac{1}{\text{vol}_n(T)^{1/n}} \left( \det \text{Cov}(\mathbb{1}_T) \right)^{\frac{1}{2n}} \approx \text{vol}_n(T)^{-1/n} (f_\mu(0) \text{vol}_n(T))^{-1} \left( \det \text{Cov}(\mu) \right)^{\frac{1}{2n}} \\ &\approx (f_\mu(0) \text{vol}_n(T))^{-1 - \frac{1}{n}} L_\mu, \end{aligned}$$

where we have also used the fact that  $\|f_\mu\|_\infty^{1/n} \approx f_\mu(0)^{1/n}$  by Fradelizi's inequality (4.1). Finally, applying (4.7) with  $p = 1$  we get that

$$e^{-1} \leq (f_\mu(0)\text{vol}_n(T))^{1+\frac{1}{n}} \leq e \frac{n+1}{n} \leq 2e, \quad (4.10)$$

and this completes the proof. This discussion shows that

$$\tilde{L}_n \approx L_n.$$

In particular, knowing now that  $L_n \leq C$ , we get:

**Theorem 4.1.** *There exists an absolute constant  $C > 0$  such that  $L_\mu \leq C$  for every isotropic log-concave probability measure on  $\mathbb{R}^n$ .*

**§ 4.3.  $L_p$ -centroid bodies.** Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . For any  $p \geq 1$  we define the  $L_p$ -centroid body  $Z_p(\mu)$  of  $\mu$  as the convex body whose support function is

$$h_{Z_p(\mu)}(y) := \left( \int_{\mathbb{R}^n} |\langle x, y \rangle|^p f_\mu(x) dx \right)^{1/p}, \quad y \in \mathbb{R}^n.$$

The convex bodies  $Z_p(\mu)$  are always symmetric, and  $Z_p(T_*\mu) = T(Z_p(\mu))$  for every  $T \in GL_n$  and  $p \geq 1$ . Note that if  $\mu$  is isotropic then  $Z_2(\mu) = B_2^n$ . Also, from (4.3) it follows that, for every  $1 \leq p < q$ ,

$$Z_p(\mu) \subseteq Z_q(\mu) \subseteq \frac{cq}{p} Z_p(\mu), \quad (4.11)$$

where  $c > 0$  is an absolute constant.

If  $\mu = \lambda_K$  is the Lebesgue measure on a centered convex body  $K$  of volume 1 in  $\mathbb{R}^n$  then we denote  $Z_p(K) := Z_p(\lambda_K)$ . In this case we have some additional results. For every  $p \geq n$  we have that

$$Z_p(K) \supseteq c_2 Z_\infty(K) \quad (4.12)$$

where  $c_2 > 0$  is an absolute constant and  $Z_\infty(K) = \text{conv}\{K, -K\}$ . This is a consequence of the inequality

$$\int_K |\langle x, \xi \rangle|^p dx \geq \frac{\Gamma(p+1)\Gamma(n)}{2e\Gamma(p+n+1)} \max\{h_K^p(\xi), h_K^p(-\xi)\},$$

which holds true for all  $\xi \in S^{n-1}$  and  $p \geq 1$ . Therefore, if  $p \geq n$  we see that

$$\|\langle \cdot, \xi \rangle\|_{L_p(K)} \approx \max\{h_K(\xi), h_K(-\xi)\},$$

and hence  $Z_p(K) \supseteq cZ_\infty(K)$ . In particular, the Rogers-Shephard inequality  $\text{vol}_n(K - K) \leq \binom{2n}{n} \text{vol}_n(K)$  for the difference body  $K - K$  of  $K$  (see [2, Theorem 1.5.2]) implies that

$$c \leq \text{vol}_n(Z_n(K))^{1/n} \leq \text{vol}_n(K - K)^{1/n} \leq 4 \quad (4.13)$$

for some absolute constant  $c > 0$ .

A nonsymmetric variant of the  $L_p$ -centroid bodies of  $\mu$  is defined as follows. For every  $p \geq 1$  we consider the convex body  $Z_p^+(\mu)$  with support function

$$h_{Z_p^+(\mu)}(y) = \left( \int_{\mathbb{R}^n} \langle x, y \rangle_+^p f_\mu(x) dx \right)^{1/p}, \quad y \in \mathbb{R}^n,$$

where  $a_+ = \max\{a, 0\}$ . When  $f_\mu$  is even, we have that  $Z_p^+(\mu) = 2^{-1/p} Z_p(\mu)$ . In any case, it is clear that  $Z_p^+(\mu) \subseteq Z_p(\mu)$ . One can also check that if  $1 \leq p < q$  then

$$\left(\frac{4}{e}\right)^{\frac{1}{p}-\frac{1}{q}} Z_p^+(\mu) \subseteq Z_q^+(\mu) \subseteq c_1 \left(\frac{4(e-1)}{e}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{q}{p} Z_p^+(\mu). \quad (4.14)$$

(for a proof see [66], where the family of bodies  $\tilde{Z}_p^+(\mu) = 2^{1/p} Z_p^+(\mu)$  is considered).

The  $L_p$ -centroid bodies were introduced, under a different normalization, by Lutwak and Zhang in [101], while in [117] for the first time, and in [118] later on, Paouris used geometric properties of them to acquire detailed information about the distribution of the Euclidean norm with respect to  $\mu$ . A basic observation of Paouris is the next asymptotic formula that appears in [118].

**Theorem 4.2.** *Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . Then,*

$$c_1 \leq \left(f_\mu(0) \text{vol}_n(Z_n(\mu))\right)^{1/n} \leq c_2, \quad (4.15)$$

where  $c_1, c_2 > 0$  are absolute constants.

*Proof.* Direct computation shows that, for every  $p \geq 1$ ,

$$\text{vol}_n(K_{n+p}(\mu))^{1+\frac{p}{n}} \int_{\overline{K_{n+p}(\mu)}} |\langle x, \xi \rangle|^p dx = \int_{K_{n+p}(\mu)} |\langle x, \xi \rangle|^p dx = \frac{1}{f_\mu(0)} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p f_\mu(x) dx$$

for all  $\xi \in S^{n-1}$ , or equivalently

$$Z_p(\overline{K_{n+p}(\mu)}) \text{vol}_n(K_{n+p}(\mu))^{\frac{1}{p}+\frac{1}{n}} f_\mu(0)^{\frac{1}{p}} = Z_p(\mu). \quad (4.16)$$

Now, let  $1 \leq p \leq n$ . Using also (4.7) we see that

$$\frac{1}{e} Z_p(\overline{K_{n+p}(\mu)}) \subseteq f_\mu(0)^{1/n} Z_p(\mu) \subseteq e^{\frac{n+p}{n}} Z_p(\overline{K_{n+p}(\mu)}) \subseteq 2e Z_p(\overline{K_{n+p}(\mu)}). \quad (4.17)$$

On the other hand, using (4.6) we can check that

$$h_{Z_p(\overline{K_{n+p}(\mu)})}(\xi) \approx h_{Z_p(\overline{K_{n+1}(\mu)})}(\xi)$$

for every  $\xi \in S^{n-1}$ . This shows that  $Z_p(\overline{K_{n+p}(\mu)}) \approx Z_p(\overline{K_{n+1}(\mu)})$ . Therefore, for all  $1 \leq p \leq n$  we get

$$c_1 f_\mu(0)^{1/n} Z_p(\mu) \subseteq Z_p(\overline{K_{n+1}(\mu)}) \subseteq c_2 f_\mu(0)^{1/n} Z_p(\mu) \quad (4.18)$$

where  $c_1, c_2 > 0$  are absolute constants.

Now recall that, since  $\mu$  is centered, the body  $K_{n+1}(\mu)$  is also centered. Applying (4.13) for the body  $\overline{K_{n+1}(\mu)}$  we see that

$$\text{vol}_n(Z_n(\overline{K_{n+1}(\mu)}))^{1/n} \approx 1$$

and hence, by (4.18),

$$\left(f_\mu(0) \operatorname{vol}_n(Z_n(\mu))\right)^{1/n} \approx \operatorname{vol}_n(Z_n(\overline{K_{n+1}}(\mu)))^{1/n} \approx 1.$$

This completes the proof.  $\square$

**§ 4.4. Marginals and projections.** Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$ . For any  $1 \leq k \leq n-1$  and  $F \in G_{n,k}$  we define the marginal of  $\mu$  with respect to  $F$  setting

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A))$$

for every Borel subset  $A$  of  $F$ . One can check that  $\pi_F(\mu)$  is a log-concave probability measure on  $F$ , whose log-concave density  $f_{\pi_F(\mu)}$  agrees almost everywhere with the function  $\pi_F(f_\mu)$  defined by

$$\pi_F(f_\mu)(x) = \int_{x+F^\perp} f_\mu(y) dy. \quad (4.19)$$

Then, for every measurable function  $g : F \rightarrow \mathbb{R}$  we have

$$\int_{\mathbb{R}^n} g(P_F x) f_\mu(x) dx = \int_F g(x) \pi_F(f_\mu)(x) dx.$$

It follows that if  $f_\mu$  is centered then, for every  $F \in G_{n,k}$  we have that  $\pi_F(f_\mu)$  is also centered, and if  $f$  is isotropic then  $\pi_F(f_\mu)$  is also isotropic.

A basic observation of Paouris from [117] is that any projection of the  $L_p$ -centroid body of  $\mu$  coincides with the  $L_p$ -centroid body of the corresponding marginal of  $\mu$ .

**Theorem 4.3.** *Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . For every  $1 \leq k \leq n$  and any  $F \in G_{n,k}$  and  $p \geq 1$ , we have that*

$$P_F(Z_p(\mu)) = Z_p(\pi_F(\mu)). \quad (4.20)$$

The proof is a direct application of Fubini's theorem.

Applying Theorem 4.2 to  $\pi_F(\mu)$  and taking into account (4.20) we obtain the following result.

**Theorem 4.4.** *Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . Then, for every  $1 \leq k \leq n-1$  and any  $F \in G_{n,k}$  we have*

$$c_1 \leq \left(\pi_F(f_\mu)(0) \operatorname{vol}_k(P_F(Z_k(f)))\right)^{1/k} \leq c_2, \quad (4.21)$$

where  $c_1, c_2 > 0$  are absolute constants.

## 5. GEOMETRY OF ISOTROPIC CONVEX BODIES

In this section we still assume that  $L_n \leq C$ . We shall discuss a few additional important consequences of this fact on the geometry of high-dimensional isotropic convex bodies.

**§ 5.1. The ellipsoid intersection conjecture.** We start with the positive answer to the following conjecture.

**Theorem 5.1** (ellipsoid intersection conjecture). *For every centered convex body  $K$  of volume 1 in  $\mathbb{R}^n$  there exists a centered ellipsoid  $\mathcal{E}$  of volume  $\text{vol}_n(\mathcal{E}) = \text{vol}_n(K) = 1$  such that*

$$\text{vol}_n(K \cap c_1 \mathcal{E}) \geq \frac{1}{2} \text{vol}_n(K)$$

where  $c_1 > 0$  is an absolute constant.

For the proof we may assume that  $K$  is isotropic. Then,  $\int_K |x|^2 dx = nL_K^2 \leq Cn$  for some absolute constant  $C > 0$ . From Markov's inequality we immediately see that

$$\text{vol}_n(K \setminus \sqrt{2Cn}B_2^n) = \text{vol}_n(\{x \in K : |x| > \sqrt{2Cn}\}) \leq \frac{1}{2Cn} \int_K |x|^2 dx \leq \frac{1}{2}.$$

It follows that  $\text{vol}_n(K \cap \sqrt{2C} \sqrt{n}B_2^n) \geq \frac{1}{2} \text{vol}_n(K)$  and since  $D_n \approx \sqrt{n}B_2^n$  it is simple to check that

$$\text{vol}_n(K \cap c_1 D_n) \geq \frac{1}{2} \text{vol}_n(K)$$

for some absolute constant  $c_1 > 0$ .

Let us note that the ellipsoid intersection conjecture is in fact equivalent to the isotropic constant conjecture.

**§ 5.2. Volume of sections and projections.** The next theorem gives a very useful estimate for the volume of sections of an isotropic convex body through its barycenter.

**Theorem 5.2.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then, for every  $1 \leq k \leq n-1$  and any  $F \in G_{n,k}$ ,*

$$\text{vol}_{n-k}(K \cap F^\perp)^{1/k} \approx 1. \quad (5.1)$$

*Proof.* We denote by  $\mu_K$  the isotropic log-concave probability measure with density  $L_K^n \mathbb{1}_{\frac{K}{L_K}}$  and write  $f_K$  for the density of  $\mu_K$ . Fix  $1 \leq k \leq n-1$  and  $F \in G_{n,k}$ . We know that  $\pi_F(\mu_K)$  is isotropic. Using (4.18) with  $p = 2$  we get:

$$\begin{aligned} L_{\overline{K_{k+1}(\pi_F(\mu_K))}} &= \left( \frac{\text{vol}_k(Z_2(\overline{K_{k+1}(\pi_F(\mu_K))}))}{\text{vol}_k(B_F)} \right)^{1/k} \approx \pi_F(f_K)(0)^{1/k} \left( \frac{\text{vol}_k(Z_2(\pi_F(\mu_K)))}{\text{vol}_k(B_F)} \right)^{1/k} \\ &= \pi_F(f_K)(0)^{1/k} \left( \frac{\text{vol}_k(P_F(Z_2(\mu_K)))}{\text{vol}_k(B_F)} \right)^{1/k}, \end{aligned}$$

where we have also used the identity  $Z_2(\pi_F(\mu_K)) = P_F(Z_2(\mu_K))$  from Theorem 4.3. Since  $K$  is isotropic, we get

$$Z_2(\mu_K) = L_K^{-1} Z_2(K) = B_2^n \quad \text{and hence} \quad P_F(Z_2(\mu_K)) = B_F.$$

Moreover, we have

$$\pi_F(f_K)(0) = \int_{F^\perp} f_K(y) dy = L_K^n \text{vol}_{n-k} \left( \frac{1}{L_K} K \cap F^\perp \right) = L_K^k \text{vol}_{n-k}(K \cap F^\perp).$$

Combining the above we conclude that  $L_{\overline{K_{k+1}(\pi_F(\mu_K))}} \approx L_K \text{vol}_{n-k}(K \cap F^\perp)^{1/k}$ , which is assertion of the theorem.  $\square$

The next inequality, due to Rogers-Shephard and Spingarn, estimates the product of the volumes of a projection of a convex body  $K$  in  $\mathbb{R}^n$  and of the section of  $K$  with the orthogonal subspace (see [124] for the first and [129] for the second claim).

**Theorem 5.3.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ . Then, for any  $1 \leq k \leq n-1$  and any  $F \in G_{n,k}$  we have that*

$$\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \leq \binom{n}{k} \text{vol}_n(K).$$

If  $\text{bar}(K) = 0$  then we also have that

$$\text{vol}_n(K) \leq \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp).$$

In the case where  $K$  is isotropic, these estimates lead to the next fact.

**Theorem 5.4.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then, for any  $1 \leq k \leq n-1$  and any  $F \in G_{n,k}$  we have that*

$$c_1 \leq \text{vol}_k(P_F(K))^{1/k} \leq c_2 \frac{n}{k}$$

where  $c_1, c_2 > 0$  are absolute constants.

This follows from Theorem 5.3 and from the fact that if  $K$  is isotropic then  $\text{vol}_{n-k}(K \cap F^\perp)^{1/k} \approx 1$  by Theorem 5.2.

**§ 5.3. Volume of the centroid bodies.** A lower bound for the volume of the  $L_p$ -centroid bodies  $Z_p(K)$  of a star body  $K$  in  $\mathbb{R}^n$  has been given by Lutwak, Yang and Zhang in [102], and later Haberl and Schuster in [67] obtained a similar lower bound for the nonsymmetric  $L_p$ -centroid bodies  $Z_p^+(K)$  of  $K$ . If  $K$  is a star body, with respect to the origin, in  $\mathbb{R}^n$  then, for every  $1 \leq p < \infty$ , the body  $M_p^+(K)$  is defined through its support function

$$h_{M_p^+(K)}(y) = \left( c_{n,p}(n+p) \int_K \langle x, y \rangle_+^p dx \right)^{1/p},$$

where

$$c_{n,p} = \frac{\Gamma\left(\frac{n+p}{2}\right)}{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}.$$

The normalization of  $M_p^+(K)$  is chosen so that  $M_p^+(B_2^n) = B_2^n$  for every  $p$ . Haberl and Schuster [67, Theorem 6.4] proved that if  $K$  is a star body, with respect to the origin, in  $\mathbb{R}^n$  then, for every  $p \geq 1$ ,

$$\text{vol}_n(K)^{-\frac{n}{p}-1} \text{vol}_n(M_p^+(K)) \geq \text{vol}_n(B_2^n)^{-\frac{n}{p}}$$

with equality if and only if  $K$  is a centered ellipsoid in  $\mathbb{R}^n$ . Since  $M_p^+(K) = (c_{n,p}(n+p))^{1/p} Z_p^+(K)$ , we conclude that if  $\text{vol}_n(K) = 1$  then

$$\text{vol}_n(Z_p^+(K))^{1/n} = (c_{n,p}(n+p))^{-1/p} \text{vol}_n(M_p^+(K))^{1/n} \geq \left( \frac{1}{c_{n,p}(n+p)\omega_n} \right)^{1/p}.$$

Taking into account the value of the constant  $c_{n,p}$  we can formulate this result as follows.

**Proposition 5.5.** *Let  $K$  be a convex body of volume 1 in  $\mathbb{R}^n$ . Then,*

$$\text{vol}_n(Z_p(K))^{1/n} \geq \text{vol}_n(Z_p(\bar{B}_2^n))^{1/n} \geq c \sqrt{p/n}$$

and

$$\text{vol}_n(Z_p^+(K))^{1/n} \geq \text{vol}_n(Z_p^+(\bar{B}_2^n))^{1/n} \geq c \sqrt{p/n}$$

for every  $1 \leq p \leq n$ , where  $c > 0$  is an absolute constant.

Paouris [117] showed that a reverse inequality holds true (up to the isotropic constant). Although the next theorem is stated for  $p \in [2, n]$  in [117], in view of (4.11) it is clear that the result remains true for all  $p \in [1, n]$ .

**Theorem 5.6.** *If  $\mu$  is an isotropic log-concave measure on  $\mathbb{R}^n$ , then for every  $1 \leq p \leq n$  we have that*

$$\text{vol}_n(Z_p(\mu))^{1/n} \leq c \sqrt{p/n}. \quad (5.2)$$

Moreover, if  $K$  is a centered convex body of volume 1 in  $\mathbb{R}^n$ , then for every  $1 \leq p \leq n$  we have that

$$\text{vol}_n(Z_p(K))^{1/n} \leq c \sqrt{p/n} L_K \leq c_1 \sqrt{p/n}, \quad (5.3)$$

where  $c, c_1 > 0$  are absolute constants.

For the proof we shall use a number of classical facts from the Brunn-Minkowski theory (see [127] for a complete exposition). Steiner's formula asserts that for every convex body  $C$  in  $\mathbb{R}^n$  we have

$$\text{vol}_n(C + tB_2^n) = \sum_{k=0}^n \binom{n}{k} W_k(C) t^k$$

for all  $t > 0$ , where  $W_k(C) = V_{n-k}(C) = V(C; n-k, B_2^n; k)$  is the  $k$ -th quermassintegral of  $C$ . Also, the Aleksandrov-Fenchel inequality implies the log-concavity of the sequence  $(W_0(C), \dots, W_n(C))$ , and in particular we have that

$$\left( \frac{W_{n-i}(C)}{\omega_n} \right)^{1/i} \geq \left( \frac{W_{n-j}(C)}{\omega_n} \right)^{1/j}, \quad (5.4)$$

for all  $1 \leq i < j \leq n$ . We shall also use Kubota's integral formula:

$$W_{n-m}(C) = \frac{\omega_n}{\omega_m} \int_{G_{n,m}} \text{vol}_m(P_F(C)) dv_{n,m}(F), \quad 1 \leq m \leq n-1. \quad (5.5)$$

*Proof of Theorem 5.6.* It is enough to prove (5.2) for integer values of  $1 \leq p \leq n-1$ . Observe that for any  $F \in G_{n,p}$  we have

$$\text{vol}_p(P_F(Z_p(\mu)))^{1/p} = \text{vol}_p(Z_p(\pi_F(\mu)))^{1/p} \leq \frac{c_1}{(f_{\pi_F(\mu)}(0))^{1/p}} \leq c_2,$$

where we have used Theorem 4.3, Theorem 4.2 and (4.4) for the isotropic density  $f_{\pi_F(\mu)} = \pi_F(f_\mu)$ . Applying (5.5) we get

$$W_{n-p}(Z_p(\mu)) \leq \frac{\omega_n}{\omega_p} c_2^p.$$

Now, we apply (5.4) for the convex body  $C = Z_p(\mu)$  with  $j = n$  and  $i = p$ ; this gives

$$W_{n-p}^{1/p}(Z_p(\mu)) \geq \text{vol}_n(Z_p(\mu))^{1/n} \omega_n^{1/p-1/n}.$$

Combining the above, we get

$$\text{vol}_n(Z_p(\mu))^{1/n} \leq \frac{\omega_n^{1/n}}{\omega_p^{1/p}} c_2.$$

Since  $\omega_k^{1/k} \approx 1/\sqrt{k}$ , we obtain (5.2). For the second assertion of the theorem we may assume that  $K$  is isotropic (because the volume of  $Z_p(T(K))$  is the same for all  $T \in SL_n$ ). Consider the measure  $\mu_K$  with density  $L_K^n \mathbb{1}_{\frac{K}{L_K}}$ . Then,  $\mu_K$  is isotropic and we easily check that  $Z_p(\mu_K) = L_K^{-1} Z_p(K)$ . Thus, the result follows immediately from (5.2) and the fact that the isotropic constant conjecture is true.  $\square$

**§ 5.4. Lower dimensional Busemann-Petty problem.** In this subsection we discuss a natural generalization of the Busemann-Petty problem, for lower dimensional sections. Let  $1 \leq k \leq n-1$  and let  $\beta_{n,k}$  be the smallest constant  $\beta > 0$  with the property that for every pair of centered convex bodies  $K$  and  $C$  in  $\mathbb{R}^n$  that satisfy

$$\text{vol}_{n-k}(K \cap F) \leq \text{vol}_{n-k}(C \cap F)$$

for all  $F \in G_{n,n-k}$ , one has

$$\text{vol}_n(K)^{\frac{n-k}{n}} \leq \beta^k \text{vol}_n(C)^{\frac{n-k}{n}}.$$

It is known that  $\beta_{n,k} > 1$  if  $k < n-3$ , while for  $k = n-2$  and  $k = n-3$  (two- and three-dimensional sections) it is not known whether  $\beta_{n,k}$  has to be strictly greater than 1. The asymptotic lower dimensional Busemann-Petty problem asks if the constants  $\beta_{n,k}$  are uniformly bounded. The next theorem provides an affirmative answer.

**Theorem 5.7.** *There exists an absolute constant  $C > 0$  such that  $\beta_{n,k} \leq C$  for all  $n$  and  $k$ .*

For the proof we introduce the dual affine quermassintegrals of a convex body. Let  $1 \leq k \leq n-1$ . For every convex body  $K$ , or more generally for any bounded Borel set, in  $\mathbb{R}^n$ , the  $k$ -th dual affine quermassintegral of  $K$  is defined by

$$\Psi_k(K) = \text{vol}_n(K)^{-\frac{n-k}{kn}} \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}}.$$

Grinberg proved in [62] that this is an affinely invariant quantity and that

$$\Psi_k(K) \leq \Psi_k(B_2^n) \leq \sqrt{e} \tag{5.6}$$

for every bounded Borel set  $K$  of positive volume in  $\mathbb{R}^n$ .

Now, let  $K$  be a centered convex body in  $\mathbb{R}^n$ . Since  $\Psi_k(K) = \Psi_k(T(K))$  for every  $T \in GL_n$ , we may assume that  $K$  is isotropic. From Theorem 5.2 we know that  $\text{vol}_{n-k}(K \cap F^\perp)^{1/k} \approx 1$  for every  $F \in G_{n,k}$ , and hence

$$\Psi_k(K) = \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} \approx 1.$$

In other words, for every centered convex body  $K$  in  $\mathbb{R}^n$  and any  $1 \leq k \leq n-1$  we have that

$$c_1 \leq \Psi_k(K) \leq c_2 \tag{5.7}$$

where  $c_1, c_2 > 0$  are absolute constants.

*Proof of Theorem 5.7.* Let  $K, C$  be two centered convex bodies in  $\mathbb{R}^n$ . Assume that for some  $1 \leq k \leq n-1$  we have that

$$\text{vol}_{n-k}(K \cap F) \leq \text{vol}_{n-k}(C \cap F)$$

for all  $F \in G_{n,n-k}$ . Then,

$$\begin{aligned} \Psi_k(K) \text{vol}_n(K)^{\frac{n-k}{kn}} &= \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} \\ &\leq \left( \int_{G_{n,k}} \text{vol}_{n-k}(C \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} = \Psi_k(C) \text{vol}_n(C)^{\frac{n-k}{kn}}. \end{aligned}$$

Taking into account (5.7) we get

$$\text{vol}_n(K)^{\frac{n-k}{n}} \leq \left( \frac{\Psi_k(C)}{\Psi_k(K)} \right)^k \text{vol}_n(C)^{\frac{n-k}{n}} \leq \left( \frac{c_2}{c_1} \right)^k \text{vol}_n(C)^{\frac{n-k}{n}}$$

and the result follows.  $\square$

*Note.* In particular, the case  $k = 1$  of Theorem 5.7 shows that the isotropic constant conjecture implies the asymptotic Busemann-Petty conjecture for the class of centered convex bodies, thus completing the discussion in §2.5.

The affine quermassintegrals of a convex body  $K$  in  $\mathbb{R}^n$  were introduced by Lutwak in [99]. We shall discuss an appropriately normalized variant that was considered by Dafnis and Paouris in [41]. For every convex body  $K$  in  $\mathbb{R}^n$  and every  $1 \leq k \leq n$ , we define the normalized  $k$ -th affine quermassintegral of  $K$  by

$$\Phi_k(K) := \text{vol}_n(K)^{-\frac{1}{n}} \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-n} d\nu_{n,k}(F) \right)^{-\frac{1}{kn}}.$$

Grinberg proved in [62] that these quantities are invariant under volume preserving affine transformations. In this language, Lutwak conjectured in [100] that the affine quermassintegrals satisfy the inequalities

$$\Phi_k(B_2^n) \leq \Phi_k(K). \quad (5.8)$$

Dafnis and Paouris studied in [41] an isomorphic variant of Lutwak's conjecture, which asks if there exist absolute constants  $c_1, c_2 > 0$  such that for every convex body  $K$  in  $\mathbb{R}^n$  and any  $1 \leq k \leq n-1$ ,

$$c_1 \sqrt{n/k} \leq \Phi_k(K) \leq c_2 \sqrt{n/k}. \quad (5.9)$$

Note that in the case  $k = 1$ , (5.9) follows by the Blaschke-Santaló and the Bourgain-Milman inequality, while in the case  $k = n-1$  the conjectured rate of growth for  $\Phi_{n-1}(K)$  is again true, by the Petty projection inequality and its reverse, proved by Zhang [140].

The left-hand side of (5.9) was proved by Paouris and Pivovarov in [119]; it confirms Lutwak's conjecture in an isomorphic sense. The proof relies on a duality argument, that employs the Blaschke Santaló inequality and the Bourgain-Milman inequality, combined with Grinberg's inequality (5.6). Subsequently, E. Milman

and Yehudayoff [109] established the sharp lower bound  $\Phi_k(B_2^n) \leq \Phi_k(K)$  and verified Lutwak's conjecture, including a characterization of the equality cases, for all  $1 \leq k \leq n-1$ : ellipsoids are the only local minimizers with respect to the Hausdorff metric.

Regarding the upper bound in (5.9), an almost optimal estimate (up to a  $\ln n$ -term) was given by Dafnis and Paouris in [41]. Let us briefly recall their argument: The Aleksandrov inequalities (see [127, Section 6.4]) imply that if  $K$  is a convex body in  $\mathbb{R}^n$  then the sequence

$$Q_k(K) = \left( \frac{1}{\omega_k} \int_{G_{n,k}} \text{vol}_k(P_F(K)) dv_{n,k}(F) \right)^{1/k} \quad (5.10)$$

is decreasing in  $k$ . In particular, for any  $1 \leq k \leq n-1$  we have  $Q_k(K) \leq Q_1(K)$ , which may be written in the equivalent form

$$\left( \frac{1}{\omega_k} \int_{G_{n,k}} \text{vol}_k(P_F(K)) dv_{n,k}(F) \right)^{\frac{1}{k}} \leq w(K), \quad (5.11)$$

where  $w(K)$  is the mean width of  $K$ . Then, by Hölder's inequality,

$$\left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-n} dv_{n,k}(F) \right)^{-\frac{1}{kn}} \leq \left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) dv_{n,k}(F) \right)^{\frac{1}{k}} \leq \omega_k^{1/k} w(K).$$

Since the term on the left-hand side of this inequality is invariant under volume preserving affine transformations, we may assume that  $K$  has minimal mean width, and it is known that in this case we have  $w(K) \leq c \sqrt{n} \ln n \text{vol}_n(K)^{1/n}$  for some absolute constant  $c > 0$  (see [2, Chapter 6]). Combining the above we get

$$\Phi_k(K) \leq c_2 \sqrt{n/k} \ln n. \quad (5.12)$$

It was also shown in [41] that

$$\Phi_k(K) \leq c_3 (n/k)^{3/2} \sqrt{\ln(en/k)}.$$

In other words, if  $k$  is proportional to  $n$  then the upper bound for  $\Phi_k(K)$  is of the order of 1. The main question that remains open is whether the  $\ln n$ -term in (5.12) can actually be dropped.

**§ 5.5. The deviation inequality of Paouris.** Our goal in this subsection is to briefly explain the proof of a very useful deviation inequality of Paouris from [117].

**Theorem 5.8.** *Let  $\mu$  be an isotropic log-concave probability measure in  $\mathbb{R}^n$ . Then,*

$$\mu(\{x \in \mathbb{R}^n : |x| \geq ct \sqrt{n}\}) \leq \exp(-t \sqrt{n}) \quad (5.13)$$

for every  $t \geq 1$ , where  $c > 0$  is an absolute constant.

The proof of Theorem 5.8 is reduced to the behavior of the moments of the function  $x \mapsto |x|$ . For every  $p \geq 1$  we define

$$I_p(\mu) = \left( \int_{\mathbb{R}^n} |x|^p d\mu(x) \right)^{1/p}.$$

If  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then we set  $I_p(K) := I_p(\mu_K)$ . From (4.3) we see that for all  $y \in \mathbb{R}^n$  and  $p, q \geq 1$  we have

$$I_{pq}(\mu) \leq c_1 p I_q(\mu).$$

In particular, we have

$$I_p(\mu) \leq c_2 p I_2(\mu)$$

for all  $p \geq 2$ . Paouris proved the following.

**Theorem 5.9.** *There exist absolute constants  $c_1, c_2 > 0$  such that if  $\mu$  is an isotropic log-concave probability measure on  $\mathbb{R}^n$  then*

$$I_p(\mu) \leq c_1 I_2(\mu) \tag{5.14}$$

for all  $p \leq c_2 \sqrt{n}$ .

Assuming that we have proved Theorem 5.9, we obtain Theorem 5.8 as follows. We consider an isotropic log-concave probability measure  $\mu$  in  $\mathbb{R}^n$ . From Markov's inequality, for every  $p \geq 2$  we have

$$\mu(\{|x| \geq e^3 I_p(\mu)\}) \leq e^{-3p}.$$

Then, Borell's lemma gives

$$\mu(\{|x| \geq e^3 I_p(\mu) s\}) \leq (1 - e^{-3p}) \left( \frac{e^{-3p}}{1 - e^{-3p}} \right)^{(s+1)/2} \leq e^{-ps}$$

for every  $s \geq 1$ . Choosing  $p = c_2 \sqrt{n}$ , and using (5.14), we see that

$$\mu(\{|x| \geq c_1 e^3 I_2(\mu) s\}) \leq \exp(-c_2 \sqrt{ns})$$

for all  $s \geq 1$ . Since  $\mu$  is isotropic, we have  $I_2(\mu) = \sqrt{n}$ . This proves Theorem 5.8.

We pass to the proof of Theorem 5.9. We will actually prove a stronger statement.

**Theorem 5.10.** *Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . For every  $p \geq 1$ ,*

$$I_p(\mu) \leq C \left( I_2(\mu) + R(Z_p(\mu)) \right), \tag{5.15}$$

where  $R(Z_p(\mu)) = \max\{|x| : x \in Z_p(\mu)\}$  is the radius of  $Z_p(\mu)$ .

Note that if  $\mu$  is isotropic then  $R(Z_p(\mu)) \leq cp$ , and hence the right-hand side of (5.15) is bounded by  $c_1 \max\{I_2(\mu), p\}$ . Since  $I_2(\mu) = \sqrt{n}$ , for all  $p \leq \sqrt{n}$  we get

$$I_p(\mu) \leq c_1 \max\{I_2(\mu), p\} = c_1 I_2(\mu),$$

which is exactly the statement of Theorem 5.9.

The proof of Theorem 5.10 requires a number of basic results from the asymptotic theory of convex bodies. We write  $k_*(C)$  for the largest integer  $k \leq n$  which satisfies

$$v_{n,k} \left( \left\{ F \in G_{n,k} : \frac{w(C)}{2} |x| \leq h_C(x) \leq 2w(C) |x|, x \in F \right\} \right) \geq \frac{n}{n+k}.$$

In the terminology of [2] this is the so-called Dvoretzky dimension of  $C^\circ$ , i.e. the largest  $k$  for which the majority of  $k$ -dimensional sections of  $C^\circ$  are 4-Euclidean. The next theorem of V. Milman and Schechtman from [114] shows that the dimension  $k_*(C)$  is determined by the parameters  $w(C)$  and  $R(C)$  up to an absolute constant.

**Theorem 5.11.** *There exist absolute constants  $c_1, c_2 > 0$  such that*

$$c_1 n \frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n \frac{w(C)^2}{R(C)^2},$$

for every symmetric convex body  $C$  in  $\mathbb{R}^n$ .

For every  $p \neq 0$  we define

$$w_p(C) = \left( \int_{\mathbb{S}^{n-1}} h_C(\theta)^p d\sigma(\theta) \right)^{1/p}.$$

Note that  $w_1(C) = w(C)$ . The parameters  $w_p$ ,  $p \geq 1$  were studied by Litvak, V. Milman and Schechtman in [97].

**Theorem 5.12.** *Let  $C$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$\max \left\{ w(C), c_1 \frac{R(C) \sqrt{p}}{\sqrt{n}} \right\} \leq w_p(C) \leq \max \left\{ 2w(C), c_2 \frac{R(C) \sqrt{p}}{\sqrt{n}} \right\}$$

for all  $1 \leq p \leq n$ , where  $c_1, c_2 > 0$  are absolute constants.

Note that the behavior of  $w_p(C)$  changes when  $p \approx n(w(C)/R(C))^2$ . This value of  $p$  is roughly equal to the ‘‘dual Dvoretzky dimension’’  $k_*(C)$  of  $C$ . One also has  $w_n(C) \approx R(C)$ , and since  $w_p(C) \leq R(C)$  for every  $p \geq 1$  we conclude that  $w_p(C) \approx R(C)$  for all  $p \geq n$ .

Concerning  $w_p(C)$  for negative values of  $p$ , Klartag and Vershynin [85] established the next important result.

**Theorem 5.13.** *Let  $C$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,  $w_p(C) \approx w_{-p}(C)$  for all  $1 \leq p \leq ck_*(C)$ , where  $c > 0$  is an absolute constant.*

We start with the next lemma, which relates the  $p$ -moment of the Euclidean norm with respect to  $\mu$  with the parameters  $w_p$  and the  $L_p$ -centroid bodies of  $\mu$ .

**Lemma 5.14.** *Let  $\mu$  be a log-concave probability measure in  $\mathbb{R}^n$ . For every  $p \geq 1$  we have*

$$w_p(Z_p(\mu)) = a_{n,p} \sqrt{\frac{p}{p+n}} I_p(\mu)$$

where  $a_{n,p} \approx 1$ .

*Proof.* Direct computation shows that for every  $x \in \mathbb{R}^n$  we have

$$\left( \int_{\mathbb{S}^{n-1}} |\langle x, \xi \rangle|^p d\sigma(\xi) \right)^{1/p} = a_{n,p} \frac{\sqrt{p}}{\sqrt{p+n}} |x|,$$

where  $a_{n,p} \approx 1$ . Since

$$w_p(Z_p(\mu)) = \left( \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p d\mu(x) d\sigma(\xi) \right)^{1/p},$$

the lemma follows.  $\square$

*Proof of Theorem 5.10.* We start with the formula  $I_p(\mu) = c_{n,p} w_p(Z_p(\mu))$  from Lemma 5.14, where  $c_{n,p} \approx \max\{1, \sqrt{n/p}\}$ . Therefore, we need to show that

$$w_p(Z_p(\mu)) \leq C \min\{1, \sqrt{p/n}\} (I_2(\mu) + R(Z_p(\mu))).$$

Since  $w_p(Z_p(\mu)) \leq R(Z_p(\mu))$ , we clearly have the result when  $p \geq n$ , and hence in the sequel we may assume that  $p$  is an integer and  $1 \leq p \leq n$ .

From Theorem 5.12 we have that

$$w_p(Z_p(\mu)) \leq c_1 \max\{w(Z_p(\mu)), \sqrt{p/n} R(Z_p(\mu))\}. \quad (5.16)$$

Therefore, the theorem will follow if we show that, for all  $1 \leq p \leq n$ ,

$$w(Z_p(\mu)) \leq C \sqrt{p/n} (I_2(\mu) + R(Z_p(\mu))). \quad (5.17)$$

If  $p \geq k_*(Z_p(\mu))$  then we have

$$w(Z_p(\mu)) \leq c_2 \sqrt{p/n} R(Z_p(\mu)) \quad (5.18)$$

by Theorem 5.11. If  $p \leq k_*(Z_p(\mu))$  then by the definition of  $k_*(Z_p(\mu))$  we can find some  $F \in G_{n,p}$  that satisfies both

$$\int |P_F(x)|^2 d\mu(x) \leq c_3 (p/n) I_2^2(\mu)$$

(this is justified by averaging over all  $F \in G_{n,k}$  and then applying Markov's inequality) and

$$w(Z_p(\mu)) B_F \subseteq c_4 P_F(Z_p(\mu)). \quad (5.19)$$

Since  $P_F(Z_p(\mu)) = Z_p(\pi_F(\mu))$  and  $\pi_F(\mu)$  is a  $p$ -dimensional centered log-concave probability measure, we get

$$\text{vrad}(Z_p(\pi_F(\mu))) \approx \frac{\sqrt{p}}{\|\pi_F(\mu)\|_\infty^{1/p}} = \sqrt{p} \frac{(\det \text{Cov}(\pi_F(\mu)))^{\frac{1}{2p}}}{L_{\pi_F(\mu)}}. \quad (5.20)$$

Using the fact that  $L_{\pi_F(\mu)} \geq c$  for an absolute constant  $c > 0$ , we see that

$$\text{vrad}(Z_p(\pi_F(\mu))) \leq c_5 \frac{(\int |x|^2 d\pi_F \mu(x))^{1/2}}{L_{\pi_F(\mu)}} \leq c_6 \left( \int |P_F(x)|^2 d\mu(x) \right)^{1/2} \leq c_7 \sqrt{p/n} I_2(\mu). \quad (5.21)$$

Combining (5.19) and (5.21) we see that

$$w(Z_p(\mu)) \leq c_8 \sqrt{p/n} I_2(\mu). \quad (5.22)$$

This completes the proof.  $\square$

In the case where  $\mu = \mu_K$  for an isotropic convex body  $K$  in  $\mathbb{R}^n$ , the inequality (5.13) takes the form

$$\text{vol}_n(\{x \in K : |x| \geq ct \sqrt{n} L_K\}) \leq \exp(-t \sqrt{n}) \quad (5.23)$$

for all  $t \geq 1$ . Taking also into account the fact that  $L_K \approx 1$ , we obtain a much stronger statement than that of the ellipsoid intersection theorem of § 5.1.

## 6. A REDUCTION OF THE SLICING PROBLEM

In this section we do not assume the affirmative answer to the isotropic constant conjecture. We discuss small-ball probability estimates for isotropic log-concave probability measures, starting with the work of Paouris in [118]. In particular, we describe a reduction of the isotropic constant conjecture to such small-ball estimates. An optimal result of this type turns out to be the key fact in the affirmative answer to the problem given by Bizeul in [16].

First we introduce two parameters that are essential for the study of these questions.

**§ 6.1. The parameter  $q_*(\mu)$ .** A parameter which was originally central in the work of Paouris is  $q_*(\mu)$ , which is defined for every centered log-concave probability measure  $\mu$  in  $\mathbb{R}^n$ , as follows:

$$q_*(\mu) = \max\{p \in [2, n] : k_*(Z_p(\mu)) \geq p\}.$$

The next proposition provides a lower bound for  $q_*(\mu)$ .

**Proposition 6.1.** *There exists an absolute constant  $c > 0$  with the following property: if  $\mu$  is a centered log-concave probability measure on  $\mathbb{R}^n$  then*

$$q_*(\mu) \geq c \sqrt{k_*(Z_2(\mu))}.$$

*Proof.* We set  $q_* := q_*(\mu)$ . From Theorem 5.12, Lemma 5.14, Hölder's inequality, and the simple observation that  $I_2(\mu) = \sqrt{n} w_2(Z_2(\mu))$ , we get

$$w(Z_{q_*}(\mu)) \geq c_1 w_{q_*}(Z_{q_*}(\mu)) \geq c_2 \sqrt{q_*/n} I_{q_*}(\mu) \geq c_2 \sqrt{q_*/n} I_2(\mu) = c_2 \sqrt{q_*/n} \sqrt{n} w_2(Z_2(\mu)).$$

In other words,

$$w(Z_{q_*}(\mu)) \geq c_2 \sqrt{q_*} w(Z_2(\mu)). \quad (6.1)$$

Since  $R(Z_{q_*}(\mu)) \leq c_3 q_* R(Z_2(\mu))$ , using the definition of  $q_*$  and Theorem 5.11 we write

$$2q_* \geq k_*(Z_{q_*}(\mu)) \geq c_4 n \left( \frac{w(Z_{q_*}(\mu))}{R(Z_{q_*}(\mu))} \right)^2 \geq c_4 n \frac{c_2^2 q_* w^2(Z_2(\mu))}{c_3^2 q_*^2 R^2(Z_2(\mu))} \geq c_5 \frac{k_*(Z_2(\mu))}{q_*}. \quad (6.2)$$

This shows that  $q_*(\mu) \geq c \sqrt{k_*(Z_2(\mu))}$  for some absolute constant  $c > 0$ .  $\square$

Note that if  $\mu$  is isotropic then  $k_*(Z_2(\mu)) = n$ . Therefore, in the isotropic case we have:

**Corollary 6.2.** *There exists an absolute constant  $c > 0$  with the following property: for every isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ ,*

$$q_*(\mu) \geq c \sqrt{n}.$$

We close this subsection with the following observation.

**Theorem 6.3.** *Let  $\mu$  be an isotropic log-concave probability measure in  $\mathbb{R}^n$ . If  $1 \leq p \leq c\sqrt{n}$ , then*

$$w(Z_p(\mu)) \approx \sqrt{p}. \quad (6.3)$$

For the proof we write  $w(Z_p(\mu)) \approx w_p(Z_p(\mu)) \approx \sqrt{p/n}I_p(\mu) \approx \sqrt{p}$ , where the first equality holds because  $c\sqrt{n} \leq q_*(\mu)$  by Corollary 6.2, the second comes from Lemma 5.14 and the third follows from Theorem 5.9.

**§ 6.2. Small ball probability estimates.** Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . We extend the definition of  $I_p(\mu)$ , allowing negative values of  $p$ , in the obvious way: for every  $p \in (-n, \infty)$ ,  $p \neq 0$ , we define

$$I_p(\mu) := \left( \int_{\mathbb{R}^n} |x|^p d\mu(x) \right)^{1/p}.$$

Also, given any  $\zeta \geq 1$ , we define the parameter

$$q_{-c}(\mu, \zeta) := \max\{p \geq 1 : I_2(\mu) \leq \zeta I_{-p}(\mu)\}. \quad (6.4)$$

The first main result of this section is the next theorem of Paouris from [118].

**Theorem 6.4.** *Let  $\mu$  be a centered log-concave probability measure on  $\mathbb{R}^n$ . For every integer  $1 \leq k \leq q_*(\mu)$  we have*

$$I_{-k}(\mu) \approx I_k(\mu).$$

In particular, Theorem 6.4 shows that for every  $1 \leq k \leq q_*(\mu)$  we have  $I_k(\mu) \leq cI_2(\mu)$ , where  $c > 0$  is an absolute constant. Assuming that  $\mu$  is isotropic and taking into account Corollary 6.2, we immediately obtain the assertion of Theorem 5.9.

The proof of Theorem 6.4 is based on two identities.

**Claim 6.5.** *If  $\mu$  is a centered log-concave probability measure on  $\mathbb{R}^n$  and  $1 \leq k \leq n-1$  is a positive integer, then*

$$I_{-k}(\mu) = c_{n,k} \left( \int_{\mathcal{G}_{n,k}} \pi_F(f_\mu)(0) d\nu_{n,k}(F) \right)^{-1/k}, \quad (6.5)$$

where

$$c_{n,k} = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \approx \sqrt{n}.$$

*Proof.* Let  $1 \leq k \leq n - 1$ . Then, we have

$$\begin{aligned}
\int_{G_{n,k}} \pi_F(f_\mu)(0) dv_{n,k}(F) &= \int_{G_{n,n-k}} \pi_{E^\perp}(f_\mu)(0) dv_{n,n-k}(E) = \int_{G_{n,n-k}} \int_E f_\mu(y) dy dv_{n,n-k}(E) \\
&= \int_{G_{n,n-k}} (n-k)\omega_{n-k} \int_{S_E} \int_0^\infty r^{n-k-1} f_\mu(r\xi) dr d\sigma_E(\xi) dv_{n,n-k}(E) \\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} n\omega_n \int_{S^{n-1}} \int_0^\infty r^{n-k-1} f_\mu(r\xi) dr d\sigma(\xi) \\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} \int_{\mathbb{R}^n} |x|^{-k} f_\mu(x) dx = \frac{(n-k)\omega_{n-k}}{n\omega_n} I_{-k}(\mu).
\end{aligned}$$

It follows that

$$I_{-k}(\mu) = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \left( \int_{G_{n,k}} \pi_F(f_\mu)(0) dv_{n,k}(F) \right)^{-1/k}.$$

Finally, we check that  $c_{n,k} = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \approx \sqrt{n}$ .  $\square$

**Claim 6.6.** *If  $C$  is a symmetric convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq n - 1$  is a positive integer, then*

$$w_{-k}(C) \approx \sqrt{k} \left( \int_{G_{n,k}} \text{vol}_k(P_F(C))^{-1} dv_{n,k}(F) \right)^{-\frac{1}{k}}. \quad (6.6)$$

*Proof.* Using the Blaschke-Santaló and the Bourgain-Milman inequality, we write

$$\begin{aligned}
w_{-k}^{-1}(C) &= \left( \int_{S^{n-1}} \frac{1}{h_C^k(\xi)} d\sigma(\xi) \right)^{1/k} = \left( \int_{G_{n,k}} \int_{S_F} \frac{1}{\|\xi\|_{(P_F C)^\circ}^k} d\sigma(\xi) dv_{n,k}(F) \right)^{1/k} \\
&= \left( \int_{G_{n,k}} \frac{\text{vol}_k(P_F(C))^\circ}{\text{vol}_k(B_2^k)} dv_{n,k}(F) \right)^{1/k} \approx \left( \int_{G_{n,k}} \frac{\text{vol}_k(B_2^k)}{\text{vol}_k(P_F(C))} dv_{n,k}(F) \right)^{1/k},
\end{aligned}$$

and the result follows.  $\square$

Now, consider a centered log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  and an integer  $1 \leq k \leq n - 1$ . Recall that from Theorem 4.4 we also have

$$\frac{1}{\text{vol}_k(P_F(Z_k(\mu)))^{1/k}} \approx \pi_F(f_\mu)(0)^{1/k}.$$

for every subspace  $F \in G_{n,k}$ . Combining Claim 6.5 and Claim 6.6 we get

$$\sqrt{k/n} I_{-k}(\mu) \approx w_{-k}(Z_k(\mu)). \quad (6.7)$$

*Proof of Theorem 6.4.* Let  $1 \leq k \leq n-1$ . We know that  $w_k(Z_k(\mu)) \approx \sqrt{k/n} I_k(\mu)$  and (6.7) shows that we also have  $w_{-k}(Z_k(\mu)) \approx \sqrt{k/n} I_{-k}(\mu)$

We set  $k_0 = \lfloor q_* \rfloor$ , where  $q_* = q_*(\mu)$ . Then,

$$k_*(Z_{k_0}(\mu)) \approx k_*(Z_{q_*}(\mu)) \geq c_1 q_* \geq c_1 k_0. \quad (6.8)$$

From Theorem 5.13 we have

$$w_{-k}(Z_{k_0}(\mu)) \approx w_k(Z_{k_0}(\mu)) \quad (6.9)$$

for every  $1 \leq k \leq c_2 k_*(Z_{k_0}(\mu))$ , and (6.8) shows that (6.9) holds for every  $k \leq c_3 q_*(\mu)$ . Setting  $k_1 = \lfloor c_3 q_*(\mu) \rfloor \approx k_0$ , and using the fact that  $Z_{k_0}(\mu) \approx Z_{k_1}(\mu)$ , we get

$$w_{-k_1}(Z_{k_1}(\mu)) \approx w_{k_1}(Z_{k_1}(\mu)). \quad (6.10)$$

It is now clear that  $I_{-k_1}(\mu) \approx I_{k_1}(\mu)$  and since  $k_1 \approx q_*(\mu)$  we see that  $q \mapsto I_q(\mu)$  is "constant" in the range  $1 \leq |q| \leq c q_*(\mu)$ .  $\square$

Suppose that  $\mu$  is isotropic. Then,  $q_*(\mu) \geq c_1 \sqrt{n}$ , and since  $I_2(\mu) \leq I_p(\mu) \leq c_2 I_{-p}(\mu)$  for all  $1 \leq p \leq c q_*(\mu)$ , we obtain the following.

**Corollary 6.7.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . Then,  $q_{-c}(\mu, \zeta_0) \geq c \sqrt{n}$ , where  $c, \zeta_0 > 0$  are absolute constants.*

Another consequence of Theorem 6.4 is the next small-ball probability estimate:

**Theorem 6.8.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . Then, for every  $0 < \varepsilon < \varepsilon_0$  we have*

$$\mu(\{x \in \mathbb{R}^n : |x| \leq \varepsilon \sqrt{n}\}) \leq \varepsilon^c \sqrt{n}, \quad (6.11)$$

where  $\varepsilon_0, c > 0$  are absolute constants.

*Proof.* We know that  $I_2(\mu) \leq c_1 I_{-p}(\mu)$  for all  $1 \leq p \leq c_2 \sqrt{n}$ . It follows that

$$\begin{aligned} \mu(\{x \in \mathbb{R}^n : |x| \leq \varepsilon I_2(\mu)\}) &\leq \mu(\{x : |x| \leq c_1 \varepsilon I_{-p}(\mu)\}) \\ &\leq (c_1 \varepsilon)^p \leq \varepsilon^{p/2}, \end{aligned}$$

for every  $0 < \varepsilon < c_1^{-2}$  and  $p \leq c_2 \sqrt{n}$ . This gives the result with  $\varepsilon_0 = c_1^{-2}$  and  $c = c_2/2$ .  $\square$

The next theorem of Dafnis and Paouris [40] shows that if the hyperplane conjecture is correct then there are absolute constants  $\tau, \zeta_0 > 0$  such that, for every isotropic convex body  $K$  in  $\mathbb{R}^n$ , one has  $q_{-c}(\mu_K, \zeta_0) \geq \tau n$ .

**Theorem 6.9.** *There exists an absolute constant  $c > 0$  such that, for every  $n$  and every isotropic convex body  $K$  in  $\mathbb{R}^n$ ,*

$$q_{-c}(\mu_K, cL_n) \geq n - 1.$$

*Proof.* Using Theorem 5.2, for every  $1 \leq s \leq n-1$  we write

$$\begin{aligned} I_{-s}(K) &\approx \sqrt{n} \left( \int_{\mathbb{G}_{n,s}} \text{vol}_{n-s}(K \cap F^\perp) dv_{n,s}(F) \right)^{-1/s} \\ &\approx \sqrt{n} \left( \int_{\mathbb{G}_{n,s}} \left( \frac{L_{\overline{K_{s+1}}(\pi_F(\mu_K))}}{L_K} \right)^s dv_{n,s}(F) \right)^{-1/s} \geq \frac{\sqrt{n} L_K}{L_s}. \end{aligned}$$

This shows that

$$L_{-s}(K) \geq \frac{c_1 \sqrt{n} L_K}{L_s} \geq \frac{c_2 \sqrt{n} L_K}{L_n}$$

because it is known that  $L_s \leq c_3 L_n$  for all integers  $1 \leq s \leq n-1$  (see [30]). Since  $I_2(K) = \sqrt{n} L_K$ , we get

$$q_{-c}(K, c_2^{-1} L_n) := \max\{p \geq 1 : I_2(K) \leq c_2^{-1} L_n I_{-p}(K)\} \geq n-1.$$

This is the claim of the theorem.  $\square$

**§ 6.3. A reduction of the isotropic constant problem.** In the previous subsection we introduced the parameter

$$q_{-c}(K, \zeta) := \max\{p \geq 1 : I_2(K) \leq \zeta I_{-p}(K)\}.$$

We shall show that the hyperplane conjecture is equivalent to the following statement:

*There exist absolute constants  $c, \zeta_0 > 0$  such that  $q_{-c}(K, \zeta_0) \geq cn$  for every isotropic convex body  $K$  in  $\mathbb{R}^n$ .*

We already know that there exists a parameter  $q_* := q_*(K)$ , related to the  $L_p$ -centroid bodies of  $K$ , with the following properties:

- (i)  $q_*(K) \geq c \sqrt{n}$ ,
- (ii)  $q_{-c}(K, \zeta_0) \geq q_*(K)$  for some absolute constant  $\zeta_0 \geq 1$ , and hence,  $I_2(K) \leq \zeta_0 I_{-q_*}(K)$ .

What is not clear is the behavior of  $I_{-p}(K)$  when  $p$  lies in the interval  $[q_*, n]$ .

The main idea of Dafnis and Paouris in [40] is to start with an “extremal” isotropic convex body  $K$  in  $\mathbb{R}^n$  with maximal isotropic constant  $L_K \approx L_n$  which is at the same time in  $\alpha$ -regular  $M$ -position. Their starting point, which has a rather technical proof, is the following precise statement.

**Theorem 6.10.** *There exist absolute constants  $\kappa, \tau > 1$  and  $\delta > 0$  such that, for every  $\alpha \in [1, 2)$ , we can find an isotropic convex body  $K_\alpha$  in  $\mathbb{R}^n$  with the following properties:*

- (i)  $L_{K_\alpha} \geq \delta L_n$ ,
- (ii) for every  $t \geq \tau(2-\alpha)^{-3/2}$

$$\ln N(K_\alpha, t \sqrt{n} B_2^n) \leq \frac{\kappa n}{(2-\alpha)^{2\alpha} t^\alpha}. \quad (6.12)$$

*Sketch of the proof.* We shall give a very rough sketch of the proof of Theorem 6.10 for the case  $\alpha = 1$ . We should note here that the idea of considering convex bodies with “maximal” isotropic constant and their  $M$ -ellipsoids had been previously used by Bourgain, Klartag and V. Milman in [30].

We may assume that  $n = 2m$  is even, and we start with an isotropic convex body  $K_0$  that has isotropic constant  $L_{K_0} \geq \delta_0 L_{2m}$ , where  $\delta_0 \in (0, 1)$ . Then, for every  $k$ -codimensional subspace  $F$  of  $\mathbb{R}^{2m}$  we have

$$\text{vol}_{2m-k}(K \cap F^\perp)^{1/k} \approx \frac{L_{\overline{K_{k+1}(\pi_F(\mu_K))}}}{L_K} \leq c_1(\delta_0),$$

where  $c_1(\delta_0)$  is a constant depending only on  $\delta_0$ . If  $\mathcal{E}_1$  is a 1-regular  $M$ -ellipsoid for  $K_0$  then we see that for every  $F \in G_{2m,m}$  we have

$$\text{vol}_m(P_F(K_0))^{1/m} \leq c_1 \text{vol}_m(P_F(\mathcal{E}_1))^{1/m}.$$

From Theorem 5.3 we know that

$$\left(\text{vol}_m(K_0 \cap F)\text{vol}_m(P_{F^\perp}(K_0))\right)^{1/m} \approx \left(\text{vol}_m(\mathcal{E}_1 \cap F)\text{vol}_m(P_{F^\perp}(\mathcal{E}_1))\right)^{1/m} \approx 1.$$

Combining the above with the fact that, for any  $1 \leq k \leq n-1$ , the maximal/minimal volume  $k$ -dimensional section and projection of an ellipsoid coincide, we conclude that

$$c_2(\delta_0) \leq \text{vol}_m(K \cap F^\perp)^{1/m} \leq c_1(\delta_0)$$

for every  $F \in G_{2m,m}$ . Using this fact we can check that if  $\lambda_1 \geq \dots \geq \lambda_{2m} > 0$  are the semi-axes of  $\mathcal{E}_1$  then  $\lambda_m \approx_\delta \sqrt{n}$ . Now, we use the standard fact that there exists  $F_0 \in G_{2m,m}$  such that  $P_{F_0}(\mathcal{E}_1) = \lambda_m B_{F_0}$ , and hence

$$c_3(\delta_0)\overline{B}_{F_0} \subseteq P_{F_0}(\mathcal{E}_1) \subseteq c_4(\delta_0)\overline{B}_{F_0}.$$

Then, we define  $W = \overline{K_{m+1}}(\pi_{F_0}(\mu_{K_0}))$  and  $K_1 = W \times U(W)$ , where  $U \in O(2m)$  satisfies  $U(F_0) = F_0^\perp$ . The convex body  $K_1$  satisfies the assertion of the theorem for  $\alpha = 1$ .  $\square$

Then, Dafnis and Paouris try taking advantage of the fact that small-ball estimates are closely related to estimates for covering numbers. The key lemma is the following.

**Lemma 6.11.** *Let  $K$  be a centered convex body of volume 1 in  $\mathbb{R}^n$ . Assume that, for some  $s > 0$ ,*

$$r_s := \ln N(K, sB_2^n) < n. \quad (6.13)$$

Then,

$$I_{-r_s}(K) \leq 3es.$$

*Proof.* Let  $z_0 \in \mathbb{R}^n$  be such that  $\text{vol}_n(K \cap (-z_0 + sB_2^n)) \geq \text{vol}_n(K \cap (z_0 + sB_2^n))$  for every  $z \in \mathbb{R}^n$ . It follows that

$$\text{vol}_n((K + z_0) \cap sB_2^n) N(K, sB_2^n) \geq \text{vol}_n(K) = 1. \quad (6.14)$$

Let  $q = r_s$ . Then, using Markov's inequality, the definition of  $I_{-q}(K + z_0)$  and (6.13), we get

$$\text{vol}_n((K + z_0) \cap 3^{-1}I_{-q}(K + z_0)B_2^n) \leq 3^{-q} < e^{-q} = e^{-r_s} \leq \frac{1}{N(K, sB_2^n)}.$$

and hence

$$\text{vol}_n((K + z_0) \cap 3^{-1}I_{-q}(K + z_0)B_2^n) < \text{vol}_n((K + z_0) \cap sB_2^n),$$

which implies that

$$3^{-1}I_{-q}(K + z_0) \leq s.$$

Finally, we can check that  $I_{-k}(K + z) \geq e^{-1}I_{-k}(K)$  for any  $1 \leq k \leq n-1$  and  $z \in \mathbb{R}^n$ . Indeed, using Claim 6.5 and the fact that  $K$  is centered (more precisely, Fradelizi's inequality (4.1) for the centered log-concave function  $\pi_{F^\perp}(\mathbb{1}_K)$  where  $F \in G_{n,k}$ ) we write

$$\begin{aligned} I_{-k}(K + z) &= c_{n,k} \left( \int_{G_{n,k}} \text{vol}_{n-k}(K + z) \cap F^\perp) dv_{n,k}(F) \right)^{-1/k} \\ &\geq \frac{c_{n,k}}{e} \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp) dv_{n,k}(F) \right)^{-1/k} = \frac{1}{e} I_{-k}(K). \end{aligned}$$

This proves the lemma.  $\square$

We can now prove the main theorem.

**Theorem 6.12.** *Assume that  $q_{-c}(K, \zeta) \geq \beta n$  for some  $\zeta \geq 1$ , some  $\beta \in (0, 1)$  and every isotropic convex body  $K$  in  $\mathbb{R}^n$ . Then,*

$$L_n \leq \frac{C\zeta}{\sqrt{\beta}} \ln^2(e/\beta),$$

where  $C > 0$  is an absolute constant.

*Proof.* We set  $\alpha := 2 - \ln(e/\beta)^{-1}$  and, for this value of  $\alpha$ , we apply Theorem 6.10 to find an isotropic convex body  $K_\alpha$  which satisfies its conclusion: for some absolute constants  $\kappa, \tau \geq 1$  and  $\delta > 0$  it holds that  $L_{K_\alpha} \geq \delta L_n$  and

$$\ln N(K_\alpha, t \sqrt{n} B_2^n) \leq \frac{\kappa n}{(2 - \alpha)^{2\alpha} t^\alpha} \quad \text{for all } t \geq \tau \ln^{3/2}(e/\beta).$$

We may clearly assume that  $\tau^2 \leq e\kappa$  as well. We choose

$$t_1 = (e\kappa)^{1/\alpha} \frac{1}{\sqrt{\beta}} \ln^2(e/\beta).$$

Note that  $\tau \leq \sqrt{e\kappa} \leq (e\kappa)^{1/\alpha}$ , and hence  $t_1 \geq \tau \frac{1}{\sqrt{\beta}} \ln^2(e/\beta) \geq \tau \ln^{3/2}(e/\beta)$ . Therefore, (6.12) is valid for  $t = t_1$ , and this shows that

$$r_1 := \ln N(K_\alpha, t_1 \sqrt{n} B_2^n) \leq \frac{\kappa n}{(2 - \alpha)^{2\alpha} t_1^\alpha} \leq \frac{1}{e} (\sqrt{\beta})^\alpha n \leq \beta n.$$

Then, from Lemma 6.11 we get

$$I_{-r_1}(K_\alpha) \leq 3et_1 \sqrt{n}.$$

On the other hand, since  $q_{-c}(K_\alpha, \zeta) \geq \beta n$  and  $r_1 \leq \beta n$ , we have that

$$\sqrt{n} L_{K_\alpha} = I_2(K_\alpha) \leq \zeta I_{-r_1}(K_\alpha).$$

It follows that

$$L_{K_\alpha} \leq 3e\zeta t_1 = 3e\zeta (e\kappa)^{1/\alpha} \frac{1}{\sqrt{\beta}} \ln^2(e/\beta) \leq \frac{3e^2\zeta\kappa}{\sqrt{\beta}} \ln^2(e/\beta).$$

Since  $L_{K_\alpha} \geq \delta L_n$ , the result follows.  $\square$

**Remark 6.13.** Since  $q_{-c}(K, \zeta_0) \geq q_*(K) \geq c \sqrt{n}$  for some absolute constants  $\zeta_0 \geq 1$  and  $c > 0$ , we may apply Theorem 6.12 with  $\zeta = \zeta_0$  and  $\beta = c/\sqrt{n}$  to get

$$L_n \leq \frac{C\zeta}{\sqrt{c}} \sqrt[4]{n} \ln^2\left(\frac{e\sqrt{n}}{c}\right) \leq c_1 \sqrt[4]{n} (\ln n)^2,$$

where  $c_1 > 0$  is an absolute constant. This is a non-trivial estimate for  $L_n$ , slightly weaker but close to the estimate  $L_n = O(\sqrt[4]{n} \ln n)$  of Bourgain [28] and the estimate  $L_n = O(\sqrt[4]{n})$  of Klartag [72].

We close this section with a much more direct reduction of the problem but with a much stronger assumption. We shall use the following corollary of Theorem 6.10.

**Proposition 6.14.** *For every  $n \geq 1$  there exists an isotropic convex body  $K$  in  $\mathbb{R}^n$  such that  $L_K \geq c_1 L_n$  and*

$$\text{vol}_n(K \cap \sqrt{c_2 n} B_2^n) > c_0^n$$

where  $c_i > 0$  are absolute constants.

*Proof.* We apply Theorem 6.10 with  $\alpha = 1$  to find an isotropic convex body  $K$  in  $\mathbb{R}^n$  such that  $L_K \geq c_1 L_n$  and

$$\ln N(K, t \sqrt{n} B_2^n) \leq \frac{c_3 n}{t}$$

for all  $t \geq c_3$ . Then, if we set  $c_3 = \sqrt{c_2}$  it is clear that

$$1 = \text{vol}_n(K) \leq e^n N(K, \sqrt{c_2 n} B_2^n) \text{vol}_n(K \cap \sqrt{c_2 n} B_2^n) \leq e^{2n} \text{vol}_n(K \cap \sqrt{c_2 n} B_2^n),$$

and the result follows.  $\square$

From Proposition 6.14 we deduce the following direct reduction of the isotropic constant conjecture to small-ball estimates.

**Theorem 6.15.** *Assume that there exist  $\varepsilon_0, c_0 > 0$  such that for every isotropic convex body  $K$  in  $\mathbb{R}^n$  and any  $0 < \varepsilon \leq \varepsilon_0$  we have that*

$$\text{vol}_n(\{x \in K : |x| \leq \varepsilon \sqrt{n} L_K\}) \leq \varepsilon^{c_0 n}.$$

Then,  $L_n \leq C$  for all  $n \geq 1$ , where  $C = C(\varepsilon_0, c_0)$  depends only on  $\varepsilon_0$  and  $c_0$ .

*Proof.* Proposition 6.14 shows that there exists an isotropic convex body  $K$  in  $\mathbb{R}^n$  with  $L_n \leq C_1 L_K$  and  $\text{vol}_n(K \cap \sqrt{C_2 n} B_2^n) > c_2^n$  for some absolute constants  $C_1, C_2, c_2 > 0$ .

On the other hand, applying the hypothesis we see that for any  $0 < \varepsilon \leq \varepsilon_0$ ,

$$\text{vol}_n(K \cap \varepsilon \sqrt{n} L_K B_2^n) \leq \varepsilon^{c_0 n}.$$

Choosing  $\varepsilon_1 = \min\{\varepsilon_0, c_2^{\frac{1}{c_0}}\}$ , and comparing the above inequalities, we get

$$L_n^2 \leq C_1^2 L_K^2 \leq \frac{C_1^2 C_2}{\varepsilon_1}.$$

$\square$

## 7. ELDAN'S STOCHASTIC LOCALIZATION

Eldan's stochastic localization has been the key for the recent developments in Bourgain's slicing problem and other well-known isoperimetric problems about high-dimensional log-concave probability measures and convex bodies, such as the Kannan-Lovász-Simonovits conjecture that we shall discuss in the next section. In this section we briefly introduce the stochastic localization scheme that we shall use.

First we recall some preliminaries from stochastic calculus. We refer to [116], [125] and [44] for definitions and background on semimartingales and stochastic integration.

Let  $x_t$  and  $y_t$  be real-valued stochastic processes. The quadratic variations  $[x]_t$  and  $[x, y]_t$  are real-valued stochastic processes defined by

$$[x]_t = \lim_{|P| \rightarrow 0} \sum_{i=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})^2 \quad \text{and} \quad [x, y]_t = \lim_{|P| \rightarrow 0} \sum_{n=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})(y_{\tau_n} - y_{\tau_{n-1}}),$$

where  $P = \{0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \uparrow t\}$  is a random partition of the non-negative real line,  $|P| = \max_n(\tau_n - \tau_{n-1})$  is the mesh of  $P$ , and the limits are taken in probability. Note that  $[x]_t$  is non-decreasing in  $t$  and  $[x, y]_t$  satisfies the equation

$$[x, y]_t = \frac{1}{4}([x + y]_t - [x - y]_t).$$

Assume that the processes  $x_t$  and  $y_t$  satisfy the differential equations  $dx_t = a(x_t)dt + \sigma(x_t)dW_t$  and  $dy_t = b(y_t)dt + \eta(y_t)dW_t$  where  $(W_t)_{t \geq 0}$  is a standard Brownian motion. Then we have

$$[x]_t = \int_0^t \sigma^2(x_s)ds \quad \text{and} \quad [x, y]_t = \int_0^t \sigma(x_s)\eta(y_s)ds.$$

Also,  $d[x, y]_t = \sigma(x_t)\eta(y_t)dt$ .

Analogously, for vector valued differential equations  $dx_t = a(x_t)dt + \Sigma(x_t)dW_t$  and  $dy_t = b(y_t)dt + H(y_t)dW_t$  we have that

$$[x^i, x^j]_t = \int_0^t (\Sigma(x_s)\Sigma^T(x_s))_{ij}ds \quad \text{and} \quad d[x^i, y^j]_t = (\Sigma(x_t)H^T(y_t))_{ij}dt.$$

We shall use Itô's formula: If  $(x_t)_{t \geq 0}$  is an  $\mathbb{R}^d$ -valued semimartingale and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is twice continuously differentiable, then

$$df(x_t) = \sum_i \frac{df(x_t)}{dx^i} dx^i + \frac{1}{2} \sum_{i,j} \frac{d^2f(x_t)}{dx^i dx^j} d[x^i, x^j]_t. \quad (7.1)$$

**§ 7.1. Eldan's stochastic localization.** A variant of the stochastic localization scheme that we shall use was introduced by Eldan in [45]. Then, his idea was used in a number of subsequent works [94], [37], [80].

Given a probability measure  $\mu = \mu_0$  on  $\mathbb{R}^n$  with density  $\varrho = \varrho_0$ , the starting idea of localization is to restrict the distribution to a random half-space and then repeat this process. The discrete version of this procedure is to define

$$\varrho_{t+1}(x) = \varrho_t(x) \left(1 + \sqrt{h} \langle x - a_t, w \rangle\right)$$

where  $a_t$  is the barycenter of the measure  $\mu_t$ , the parameter  $h > 0$  is sufficiently small and  $w$  is a *standard* Gaussian random vector. So, each step is a renormalization of the measure with a linear function in a random direction. Using the approximation  $1 + y \sim e^{y - \frac{1}{2}y^2}$  as  $y \rightarrow 0$  we see that this process introduces a Gaussian factor in the exponent, which is more and more concentrated as  $t \rightarrow \infty$ . Eventually,  $\mu_t$  becomes a Dirac measure, and then any set has measure 0 or 1. The idea is to stop at a large enough time  $t$  so that the density would include a strong Gaussian factor but at the same time we would still be able to compare the original measure  $\mu$  with  $\mu_t$ .

In stochastic localization, the discrete steps described above are replaced by infinitesimal steps. Given a probability measure  $\mu$  on  $\mathbb{R}^n$  and a standard Brownian motion  $(W_t)_{t \geq 0}$ , we consider the stochastic differential equation, indexed by  $x \in \mathbb{R}^n$ ,

$$d\varrho_t(x) = \varrho_t(x) \langle x - a_t, dW_t \rangle, \quad \varrho_0(x) = 1,$$

where  $a_t$  is the barycenter of the probability measure with density  $\varrho_t(x) d\mu(x)$ . Note that  $(W_t)_{t \geq 0}$  is *the same* Brownian motion for every  $x$ , so that all the processes  $\varrho_t(x)$  are driven by the same noise. This simplified version of Eldan's process was introduced by Lee and Vempala in [94].

We may assume that  $\varrho_t(x) > 0$  for all  $t$ , almost surely. We have

$$d \left( \int_{\mathbb{R}^n} \varrho_t(x) d\mu(x) \right) = \int_{\mathbb{R}^n} d\varrho_t(x) d\mu(x) = \left\langle \int_{\mathbb{R}^n} (x - a_t) \varrho_t(x) d\mu(x), dW_t \right\rangle = 0$$

by the definition of  $a_t$ , which shows that the total mass of  $\varrho_t d\mu$  remains constant. So,  $\mu_t := \varrho_t d\mu$  is a random probability measure for every  $t > 0$ . We can also check that  $\varrho_t(x)$  is a martingale for all  $x$ . In particular,

$$\mathbb{E}(\varrho_t(x)) = \varrho_0(x) = 1$$

for all  $x$ . This shows that the random measure  $\mu_t$  is equal to  $\mu$  on average:

$$\mathbb{E}(\mu_t) = \mu.$$

Finally, we can solve the equation

$$d\varrho_t(x) = \varrho_t(x) \langle x - a_t, dW_t \rangle$$

explicitly: Applying Itô's formula to  $\ln \varrho_t(x)$  we get

$$d(\ln \varrho_t(x)) = \frac{d\varrho_t(x)}{\varrho_t(x)} - \frac{1}{2} \frac{d[\varrho_t(x)]_t}{\varrho_t^2(x)} = \langle x - a_t, dW_t \rangle - \frac{1}{2} |x - a_t|^2 dt,$$

which gives

$$\varrho_t(x) = \exp \left( \int_0^t \langle x - a_s, dW_s \rangle - \frac{1}{2} \int_0^t |x - a_s|^2 ds \right) = \exp \left( c_t + \langle x, \xi_t \rangle - \frac{t}{2} |x|^2 \right),$$

where  $(c_t)$  and  $(\xi_t)$  are random processes that do not depend on  $x$ . This shows that  $\mu_t$  is obtained from  $\mu$  by a random Gaussian tilt with random linear term and normalization, but deterministic quadratic term  $t|x|^2/2$ .

This means that if the original measure  $\mu$  is log-concave then the measure  $\mu_t$  is  $t$ -uniformly log-concave (i.e. the function  $\varrho_t(x)e^{t|x|^2/2}$  is log-concave) almost surely, and the process becomes more and more peaked as  $t$  grows. In summary, starting with a log-concave probability measure  $\mu$ , this construction yields a family  $(\mu_t)_{t>0}$  of random probability measures such that

- for every  $t > 0$ ,  $\mu_t$  is  $t$ -uniformly log-concave.
- the original measure  $\mu$  is recovered as the mixture of the random measures  $\mu_t$ , in the sense that  $\mu = \mathbb{E}(\mu_t)$ .

Moreover, the process  $(\mu_t)_{t \geq 0}$  is constructed as a solution of a stochastic differential equation, and using Itô's formula enables a quantitative analysis of its evolution in time.

**§ 7.3. Construction of the process.** A rigorous construction of this stochastic localization process is provided in [81] (see also [84]). Consider a standard

$n$ -dimensional Brownian motion  $(\xi_t)$  defined on some probability space  $(\Omega, \mathcal{A}, P)$  equipped with a filtration  $(\mathcal{A}_t)$ . For every  $x \in \mathbb{R}^n$  the process  $(E_t)$  defined by

$$E_t = \exp\left(\langle x, \xi_t \rangle - \frac{t}{2}|x|^2\right)$$

is a martingale. Then, for any test function  $g$ , the process

$$N_t = \int_{\mathbb{R}^n} g(x) \exp\left(\langle x, \xi_t \rangle - \frac{t}{2}|x|^2\right) d\mu(x)$$

is also a martingale, and in particular we have that  $\mathbb{E}(N_t) = N_0 = \mathbb{E}_\mu(g)$ .

We define the random probability measure  $\mu_t$  on  $\mathbb{R}^n$  with

$$d\mu_t(x) = \frac{1}{Z_t} \exp\left(\langle x, \xi_t \rangle - \frac{t}{2}|x|^2\right) d\mu(x) \quad (7.2)$$

where

$$Z_t = \int_{\mathbb{R}^n} \exp\left(\langle x, \xi_t \rangle - \frac{t}{2}|x|^2\right) d\mu(x)$$

is a suitable normalization constant. Then,

$$N_t = Z_t \int_{\mathbb{R}^n} g(x) d\mu_t(x).$$

We fix  $T > 0$ . Since  $(Z_t)_{t \leq T}$  is a positive martingale with expectation equal to 1, if  $Q$  is the probability measure on  $(\Omega, \mathcal{A})$  with density  $Z_T$  with respect to  $P$  we have that the process  $(M_t)$  defined by

$$M_t = \int_{\mathbb{R}^n} g(x) d\mu_t(x)$$

is a martingale with respect to  $Q$ . Using Girsanov's change of measure formula we check that the Itô derivative of  $M_t$  is given by the formula

$$dM_t = \left\langle \int_{\mathbb{R}^n} g(x)(x - a_t) d\mu_t(x), dW_t \right\rangle$$

where  $a_t = \int_{\mathbb{R}^n} x d\mu_t(x)$  is the barycenter of  $\mu_t$ .

One can also obtain a concrete description of the law of  $(\xi_t)$ . It is proved in [84] that  $(\xi_t)$  has the same law as the process  $(tX + W_t)$  where  $(W_t)_{t \geq 0}$  is a standard Brownian motion and  $X$  is a random vector which is independent from  $(W_t)_{t \geq 0}$  and is distributed according to  $\mu$ .

**§ 7.4. Time reversal.** Let  $\varrho$  denote the density of  $\mu$  with respect to the Lebesgue measure. Then, we can reformulate the definition (7.2) of  $\mu_t$  as follows:

$$\int_{\mathbb{R}^n} g d\mu_t = \frac{\int_{\mathbb{R}^n} g(x) \varrho(x) \exp\left(\langle \xi_t, x \rangle - \frac{t}{2}|x|^2\right) dx}{\int_{\mathbb{R}^n} \varrho(x) \exp\left(\langle \xi_t, x \rangle - \frac{t}{2}|x|^2\right) dx} \quad (7.3)$$

for any test function  $g$ . We introduce the heat semi-group

$$P_t f(x) = \mathbb{E}(f(x + W_t)) = f * \gamma_t$$

where  $\gamma_t(x) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$  is the density of the Gaussian measure with mean 0 and covariance matrix  $tI_n$ . Then (7.3) takes the form

$$\int_{\mathbb{R}^n} g d\mu_t = \frac{P_{1/t}(g\varrho)}{P_{1/t}\varrho} \left( \frac{\xi_t}{t} \right).$$

Now set  $s = 1/t$ . Since the process  $(\xi_t)$  has the same law as the process  $(tX + W_t)$  we have

$$\frac{\xi_t}{t} = \frac{tX + W_t}{t} = X + sW_{1/s}$$

in law. Since  $\tilde{W}_s := sW_{1/s}$  is again a standard Brownian motion we conclude that up to the time reversal  $t = 1/s$ , the process  $(\int g d\mu_t)_{t \geq 0}$  has the same distribution as  $(Q_s g(X + W_s))_{s \geq 0}$ , where  $Q_s$  is the operator defined by

$$Q_s g = \frac{P_s(g\varrho)}{P_s\varrho}.$$

Since the heat semigroup is self-adjoint in  $L^2(dx)$  we may also check that

$$Q_s g(X + W_s) = \mathbb{E}(g(X) | X + W_s).$$

Combining the above we see that, up to the time-reversal  $t = 1/s$ , the stochastic localization process  $(\mu_t)_{t > 0}$  initiated from  $\mu$  can be viewed as the evolution of the conditional law of  $X$  given  $X + W_s$ , where  $(W_s)_{s > 0}$  is an independent Brownian motion. In particular, for every test function  $g$  the variable  $\int_{\mathbb{R}^n} g d\mu_t$  has the same law as  $\mathbb{E}(g(X) | X + \sqrt{s}G)$  where  $G$  is a standard Gaussian random vector independent of  $X$ .

## 8. KANNAN-LOVÁSZ-SIMONOVITS CONJECTURE

Let  $\mu$  be a Borel probability measure on  $\mathbb{R}^n$ . For every Borel set  $A \subseteq \mathbb{R}^n$  we define the Minkowski content  $\mu^+(A)$  as follows:

$$\mu^+(A) = \liminf_{t \rightarrow 0^+} \frac{\mu(A_t) - \mu(A)}{t},$$

where  $A_t = \{x \in \mathbb{R}^n : \text{dist}(x, A) < t\}$  is the  $t$ -extension of  $A$ . The isoperimetric ratio of  $A$  is defined by  $\chi_\mu(A) := \mu^+(A) / \min\{\mu(A), 1 - \mu(A)\}$ . Then, the (reciprocal) Cheeger constant  $\psi_\mu$  of  $\mu$  is the quantity

$$\psi_\mu := \sup_A \frac{1}{\chi_\mu(A)} = \sup_A \frac{\min\{\mu(A), 1 - \mu(A)\}}{\mu^+(A)},$$

where the supremum is over all open sets  $A \subset \mathbb{R}^n$  with smooth boundary and  $0 < \mu(A) < 1$ . It follows from results of Rothaus, Cheeger, Maz'ya (see [21]) that if  $\alpha_\mu$  is the smallest constant with the property that for every integrable, locally Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  one has

$$\int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| d\mu(x) \leq \alpha_\mu \int_{\mathbb{R}^n} |\nabla f(x)| d\mu(x),$$

then  $\alpha_\mu/2 \leq \psi_\mu \leq 2\alpha_\mu$ .

**§ 8.1. Isoperimetric profile.** The isoperimetric profile  $I_\mu : [0, 1] \rightarrow [0, +\infty]$  of  $\mu$  is the function

$$I_\mu(t) = \inf\{\mu^+(A) : A \text{ Borel}, \mu(A) = t\}.$$

Note that

$$\psi_\mu = \sup_{0 < t \leq 1/2} \frac{t}{\min\{I_\mu(t), I_\mu(1-t)\}}.$$

The next important result of E. Milman, whose proof employs deep tools from geometric measure theory, is very useful.

**Theorem 8.1.** *Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$ . Then, the isoperimetric profile  $I_\mu$  of  $\mu$  is concave on  $(0, 1)$ , and for every  $t \in (0, 1)$  we have  $I_\mu(t) = I_\mu(1-t)$ . As a consequence,*

$$\psi_\mu = \sup_{0 < t \leq 1/2} \frac{t}{I_\mu(t)} = \frac{1}{2I_\mu(1/2)}.$$

The concavity of the isoperimetric profile on a convex domain was first obtained by Sternberg and Zumbun in [131]. They showed that if  $n \geq 2$  and  $K$  is a convex body in  $\mathbb{R}^n$ , then  $I_{\lambda_K}$  is concave on  $[0, 1]$ , where  $\lambda_K$  is the uniform probability measure on  $K$ . Kuwert later noted in [88] that  $I_{\lambda_K}^{n/(n-1)}$  is also concave on  $[0, 1]$ . This is precisely the correct power to use. Kuwert's result was extended to convex domains in Riemannian manifolds with non-negative Ricci curvature by Bayle and Rosales [14]. E. Milman showed in [106] that the concavity of  $I_\mu$  remains valid for log-concave measures on  $\mathbb{R}^n$ .

Theorem 8.1 asserts that we can calculate the Cheeger constant of a log-concave probability measure  $\mu$  by looking only at Borel sets  $A$  with  $\mu(A) = 1/2$ . In fact, E. Milman proved in [106] that one can determine the order of the constant  $\psi_\mu$  just by testing 1-Lipschitz functions of the form  $x \mapsto d(x, A)$ .

**Theorem 8.2.** *Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$ . Then,*

$$\psi_\mu \approx \sup \left\{ \int_{\mathbb{R}^n} d(x, A) d\mu(x) : \mu(A) \geq \frac{1}{2} \right\}. \quad (8.1)$$

**§ 8.2. Kannan-Lovász-Simonovits conjecture.** Kannan, Lovász and Simonovits conjectured in [70] that the isoperimetric ratio of any Borel set  $A$  with respect to the uniform measure  $\lambda_K$  on a convex body  $K$  in  $\mathbb{R}^n$  should be, up to an absolute constant, at least as large as the minimal isoperimetric ratio over all half-spaces. Recall that when  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then  $1/L_K$  is approximately equal to the  $(n-1)$ -dimensional volume of the section of  $K$  with any hyperplane passing through the origin: we know that  $\text{vol}_{n-1}(K \cap \xi^\perp) \approx 1/L_K$  for every  $\xi \in S^{n-1}$ . On the other hand,  $\text{vol}_{n-1}(K \cap \xi^\perp)$  is the Minkowski content of the intersection of  $K$  with the half-space  $H_\xi^+ := \{x : \langle x, \xi \rangle \geq 0\}$  or  $H_\xi^- := \{x \in K : \langle x, \xi \rangle \leq 0\}$ . Therefore, the KLS conjecture amounts to asking whether  $\psi_{\lambda_K} \leq cL_K$ . Taking into account the isotropic constant conjecture and the different normalization that we have chosen in the definition of isotropic log-concave measures, we can restate the KLS conjecture in the more general setting of isotropic log-concave probability measures as follows:

**Conjecture 8.3** (KLS conjecture). *If  $\mu$  is an isotropic log-concave probability measure on  $\mathbb{R}^n$  then the reciprocal Cheeger constant of  $\mu$  satisfies  $\psi_\mu \leq C$  for some absolute constant  $C > 0$ . In other words,*

$$\psi_n := \sup\{\psi_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leq C.$$

Kannan, Lovász and Simonovits [70] gave an upper bound for  $\psi_{\lambda_K}$  in terms of the quantity

$$J_1(K) := \frac{1}{\text{vol}_n(K)} \int_K |x - \text{bar}(K)| dx,$$

where  $\text{bar}(K)$  is the barycenter of  $K$ .

**Theorem 8.4.** *For every convex body  $K$  in  $\mathbb{R}^n$  one has*

$$\psi_{\lambda_K} \leq cJ_1(K). \tag{8.2}$$

The proof of Theorem 8.4 exploited the localization lemma of Lovász and Simonovits from [98], a very useful tool that allows one to reduce inequalities for log-concave functions in  $\mathbb{R}^n$  to one-dimensional inequalities. Note that, if  $K$  is an isotropic convex body in  $\mathbb{R}^n$  then

$$J_1(K) \leq \left( \int_K |x|^2 dx \right)^{1/2} = \sqrt{n}L_K,$$

and hence (8.2) implies that

$$\psi_{\lambda_K} \leq c \sqrt{n}L_K, \tag{8.3}$$

where  $c > 0$  is an absolute constant.

The KLS conjecture is stronger than the isotropic constant conjecture. Ball and Nguyen [10] showed that  $L_n \leq \exp(c\psi_n^2)$  for an absolute constant  $c > 0$ . In fact, from their work one can conclude that for each individual isotropic log-concave probability measure  $\mu$ , a bound for the KLS constant implies a bound for the isotropic constant: one has that  $L_\mu \leq \exp(c\psi_\mu^2)$ . An important result of Eldan and Klartag [46] establishes a linear dependence of  $L_n$  on  $\psi_n$ .

**Theorem 8.5.** *There exists an absolute constant  $C > 0$  such that, for every  $n \geq 1$ ,*

$$L_n \leq C\psi_n.$$

Theorem 8.5 shows that any upper bound for  $\psi_n$  implies an equivalent, up to an absolute constant, upper bound for  $L_n$ . In fact, as we will see in this section, Klartag has established in [78] the upper bound  $\psi_n \leq c\sqrt{\ln n}$ .

**§ 8.3. Poincaré constant and KLS constant.** The Cheeger constant is closely related to another isoperimetric constant associated to  $\mu$ , the Poincaré constant, defined to be the best, i.e. the smallest, constant  $\vartheta_\mu$  with the property that

$$\text{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2 d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right)^2 \leq \vartheta_\mu^2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

for all locally Lipschitz functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\int |\nabla f|^2 d\mu < \infty$ . We shall use the fact that the Poincaré constant is subadditive: if  $\mu, \nu$  are probability measures then

$\vartheta_{\mu * \nu}^2 \leq \vartheta_\mu^2 + \vartheta_\nu^2$ , where  $\mu * \nu$  is the convolution measure of  $\mu$  and  $\nu$ ; this follows by a classical variance decomposition and the convexity of  $t \mapsto t^2$  (see e.g. [24]).

A theorem of Maz'ya [104], [105] and Cheeger [39] shows that the Poincaré constant  $\vartheta_\mu$  of  $\mu$  is bounded by the reciprocal Cheeger constant  $\psi_\mu$ .

**Theorem 8.6.** *Let  $\mu$  be a Borel probability measure with reciprocal Cheeger constant  $\psi_\mu$ . Then its Poincaré constant  $\vartheta_\mu$  satisfies*

$$\vartheta_\mu \leq 2\psi_\mu. \quad (8.4)$$

On the other hand, the assumption that  $\mu$  is log-concave implies a reverse inequality with a constant that does not depend on the dimension. The next theorem is due to Buser [33] (see also Ledoux [90]).

**Theorem 8.7.** *Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$ . Then,  $\psi_\mu \leq c \vartheta_\mu$ , where  $c > 0$  is an absolute constant.*

**§ 8.4. Improved log-concave Lichnerowicz inequality.** One of the crucial ingredients in Klartag's estimate for  $\psi_n$  is an improved log-concave Lichnerowicz inequality which we now describe.

In what follows we assume that  $\mu$  is a log-concave probability measure with a density  $\varrho = e^{-\varphi}$ , where  $\varphi$  is a convex function on  $\mathbb{R}^n$ . There are some issues that we shall ignore in our discussion. What we will do requires that  $\mu$  is regular. We assume that the density  $\varrho$  of  $\mu$  is smooth and positive on  $\mathbb{R}^n$ , and there exists  $\delta > 0$  such that  $\delta I_n \leq \nabla^2 \varphi \leq \delta^{-1} I_n$ , where  $\nabla^2 \varphi$  denotes the Hessian of  $\varphi$ . Also, we assume that  $\varphi$  and all its partial derivatives grow at most polynomially at infinity. One can obtain a reduction to this case, because of the following lemma: For every  $\mu$  and  $\varepsilon > 0$  we may find a regular  $\nu$  that satisfies  $\vartheta_\nu^2 \geq \vartheta_\mu^2 - \varepsilon$  and  $\|\text{Cov}(\nu) - \text{Cov}(\mu)\|_{\text{op}} \leq \varepsilon$ . A proof of this fact is given in [78, Lemma 2.1].

We consider the class  $\mathcal{F}_\mu$  of all functions  $u$  on  $\mathbb{R}^n$  which are smooth and have subexponential decay relative to  $\varrho$ , i.e. there exist  $C, a > 0$  such that

$$|u(x)| \leq \frac{C}{\sqrt{\varrho(x)}} e^{-a|x|},$$

where  $\varrho$  is the density of  $\mu$ . We also require that the same is true for the partial derivatives of  $u$ . The Laplace-Beltrami operator is defined for  $u \in \mathcal{F}_\mu$  by

$$L_\mu u = \Delta u - \langle \nabla \varphi, \nabla u \rangle.$$

The Laplace-Beltrami operator satisfies the basic identity

$$\int_{\mathbb{R}^n} (L_\mu u) v \, d\mu = - \int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle \, d\mu \quad (8.5)$$

for all  $u, v \in \mathcal{F}_\mu$ . We will also use Bochner's formula

$$\int_{\mathbb{R}^n} (L_\mu u)^2 \, d\mu = \int_{\mathbb{R}^n} \|\nabla^2 u\|_{\text{HS}}^2 \, d\mu + \int_{\mathbb{R}^n} \langle (\nabla^2 \varphi) \nabla u, \nabla u \rangle \, d\mu \quad (8.6)$$

which holds true for every  $u \in \mathcal{F}_\mu$ . Note that  $-L_\mu$  is essentially self-adjoint and positive semi-definite in  $\mathcal{F}_\mu \subset L^2(\mu)$ , and has a discrete spectrum  $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq \dots$ . The minimal eigenvalue  $\lambda_0$  of  $L_\mu$  is 0, a simple eigenvalue with constant

eigenfunctions. Let  $\lambda_\mu = \lambda_1$  denote the smallest positive eigenvalue of  $L_\mu$ . If  $-L_\mu g = \lambda_\mu g$  with  $\int g d\mu = 0$ , then

$$\lambda_\mu \int_{\mathbb{R}^n} g^2 d\mu = \int_{\mathbb{R}^n} (-L_\mu g) g d\mu = \int_{\mathbb{R}^n} |\nabla g|^2 d\mu.$$

This shows that  $1/\lambda_\mu \leq \vartheta_\mu^2$ . In fact, we have equality by the next lemma.

**Lemma 8.8.** *For every log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  we have that*

$$1/\lambda_\mu = \vartheta_\mu^2.$$

*Proof.* Define the energy functional

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \langle \nabla f, \nabla g \rangle d\mu$$

for  $f, g \in \mathcal{F}_\mu$ . Note that  $\mathcal{E}(\cdot, \cdot)$  is linear in each of its arguments and  $\mathcal{E}(f, f) \geq 0$  for all  $f \in \mathcal{F}_\mu$ . We can write any  $f \in \mathcal{F}_\mu$  as

$$f = \langle f, w_0 \rangle w_0 + \sum_{j=1}^{\infty} \langle f, w_j \rangle w_j,$$

where  $-L_\mu w_j = \lambda_j w_j$  and  $w_j$  form an orthonormal sequence in  $L^2(\mu)$ . Note that  $w_0 \equiv 1$ , and hence  $\langle f, w_0 \rangle = \int f d\mu$ . For every  $s \geq 1$  we have that

$$\begin{aligned} 0 &\leq \mathcal{E} \left( f - \sum_{j=1}^s \langle f, w_j \rangle w_j, f - \sum_{k=1}^s \langle f, w_k \rangle w_k \right) \\ &= \mathcal{E}(f, f) - 2 \sum_{j=1}^s \langle f, w_j \rangle \mathcal{E}(w_j, f) + \sum_{j,k=1}^s \langle f, w_j \rangle \langle f, w_k \rangle \mathcal{E}(w_j, w_k). \end{aligned}$$

On observing that

$$\mathcal{E}(w_j, w_k) = \lambda_j \int_{\mathbb{R}^n} w_j w_k d\mu = \lambda_j \delta_{j,k}$$

and

$$\mathcal{E}(w_j, f) = \int_{\mathbb{R}^n} \langle \nabla w_j, \nabla f \rangle d\mu = \int_{\mathbb{R}^n} (-L_\mu w_j) f d\mu = \lambda_j \int_{\mathbb{R}^n} w_j f d\mu = \lambda_j \langle f, w_j \rangle$$

for all  $1 \leq j, k \leq s$ , we conclude that

$$0 \leq \mathcal{E}(f, f) - \sum_{j=1}^s \lambda_j \langle f, w_j \rangle^2,$$

and this shows that

$$\lambda_1 \text{Var}_\mu(f) = \lambda_1 \left( \int_{\mathbb{R}^n} f^2 d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right)^2 \right) \leq \sum_{j=1}^{\infty} \lambda_j \langle f, w_j \rangle^2 \leq \mathcal{E}(f, f) = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

It follows that  $\vartheta_\mu^2 \leq 1/\lambda_1$ , and this completes the proof.  $\square$

**Definition 8.9.** We say that a Borel probability measure  $\mu$  on  $\mathbb{R}^n$  with density  $\varrho = e^{-\varphi}$  is  $t$ -uniformly log-concave if the function  $\varrho(x)e^{t|x|^2/2}$  is log-concave. Then,  $\varphi(x) - \frac{t|x|^2}{2}$  is convex, that is  $\nabla^2\varphi(x) \geq tI_n$  for all  $x \in \mathbb{R}^n$ .

A theorem of Bakry and Ledoux [4] asserts that if  $\mu$  is  $t$ -uniformly log-concave then

$$\vartheta_\mu^2 = \frac{1}{\lambda_\mu} \leq \frac{1}{t}. \quad (8.7)$$

This inequality is known as the log-concave Lichnerowicz inequality, because it is analogous to some investigations of Lichnerowicz [96] in Riemannian geometry (see [5] and [91]). The following stronger “improved log-concave Lichnerowicz inequality” was proved by Klartag in [78].

**Theorem 8.10.** *Let  $t > 0$  and let  $\mu$  be a  $t$ -uniformly log-concave probability measure on  $\mathbb{R}^n$ . Then,*

$$\vartheta_\mu^2 = \frac{1}{\lambda_\mu} \leq \sqrt{\frac{\|\text{Cov}(\mu)\|_{\text{op}}}{t}}. \quad (8.8)$$

Note that (8.8) is stronger than (8.7), since  $\|\text{Cov}(\mu)\|_{\text{op}} \leq 1/\lambda_\mu$  (see Remark 8.12 below), and hence  $\sqrt{\frac{\|\text{Cov}(\mu)\|_{\text{op}}}{t}} \leq \frac{1}{\sqrt{\lambda_\mu t}}$ . So, from (8.8) we get  $\frac{1}{\lambda_\mu} \leq \frac{1}{\sqrt{\lambda_\mu t}}$ , which in turn gives (8.7). On the other hand, the Kannan-Lóvasz-Simonovits conjecture asks for the estimate

$$\frac{1}{\lambda_\mu} \leq c \|\text{Cov}(\mu)\|_{\text{op}}$$

for an absolute constant  $c > 0$ . Therefore, in the context of  $t$ -uniformly log-concave measures, (8.8) stands between the Bakry-Ledoux theorem and the Kannan-Lóvasz-Simonovits conjecture.

Equality in Theorem 8.10 is attained when  $\mu$  is a Gaussian measure, with any covariance matrix. In this case we have that  $\vartheta_\mu^2$  and  $\|\text{Cov}(\mu)\|_{\text{op}}$  coincide, and they also coincide with the inverse lower bound on the Hessian of the potential. If  $\gamma_n^{(s)}$  is the distribution of a Gaussian random vector of mean zero and covariance matrix  $sI_n$  in  $\mathbb{R}^n$  then  $\gamma_n^{(s)}$  satisfies the assumptions of Theorem 8.10 for  $t = 1/s$  while  $\vartheta_{\gamma_n^{(s)}}^2 = \|\text{Cov}(\gamma_n^{(s)})\|_{\text{op}} = s$ .

The proof of Theorem 8.10 is based on the next lemma.

**Lemma 8.11.** *Let  $\mu$  be a regular log-concave probability measure on  $\mathbb{R}^n$  with density  $\varrho = e^{-\varphi}$ , where  $\varphi$  is a convex function. If  $w_1$  is an eigenfunction corresponding to  $\lambda_1 = \lambda_\mu$ , with  $\|w_1\|_{L^2(\mu)} = 1$ , then*

$$\int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1 \rangle d\mu \leq \lambda_1^3 \|\text{Cov}(\mu)\|_{\text{op}}.$$

*Proof.* Applying Bochner’s formula (8.6) for  $w_1$  we get

$$\lambda_1^2 = \int_{\mathbb{R}^n} (L_\mu w_1)^2 d\mu = \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1 \rangle d\mu + \int_{\mathbb{R}^n} \|\nabla^2 w_1\|_{\text{HS}}^2 d\mu.$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^n} \|\nabla^2 w_1\|_{\text{HS}}^2 d\mu &= \sum_{i=1}^n \int_{\mathbb{R}^n} |\nabla \partial^i w_1|^2 d\mu \\
&\geq \lambda_1 \left( \sum_{i=1}^n \int_{\mathbb{R}^n} |\partial^i w_1|^2 d\mu - \sum_{i=1}^n \left| \int_{\mathbb{R}^n} \partial^i w_1 d\mu \right|^2 \right) \\
&= \lambda_1 \left( \int_{\mathbb{R}^n} |\nabla w_1|^2 d\mu - \left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2 \right) \\
&\geq \lambda_1^2 \int_{\mathbb{R}^n} w_1^2 d\mu - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2 \\
&= \lambda_1^2 - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2.
\end{aligned}$$

Therefore,

$$\lambda_1^2 \geq \int_{\mathbb{R}^n} \langle (\nabla^2 \varphi) \nabla w_1, \nabla w_1 \rangle d\mu + \lambda_1^2 - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2,$$

which implies that

$$\int_{\mathbb{R}^n} \langle (\nabla^2 \varphi) \nabla w_1, \nabla w_1 \rangle d\mu \leq \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2. \quad (8.9)$$

Now, let  $\xi \in S^{n-1}$  and consider the function  $g_\xi(x) = \langle x, \xi \rangle$ . Set  $E_\xi = \int g_\xi d\mu$ . We may write

$$\begin{aligned}
\left| \left\langle \int_{\mathbb{R}^n} \nabla w_1 d\mu, \xi \right\rangle \right|^2 &= \left| \int_{\mathbb{R}^n} \langle \nabla w_1, \nabla g_\xi \rangle d\mu \right|^2 = \left| \int_{\mathbb{R}^n} (-L_\mu w_1) g_\xi d\mu \right|^2 \\
&= \lambda_1^2 \left| \int_{\mathbb{R}^n} w_1 g_\xi d\mu \right|^2 = \lambda_1^2 \left| \int_{\mathbb{R}^n} w_1 (g_\xi - E_\xi) d\mu \right|^2 \\
&\leq \lambda_1^2 \int_{\mathbb{R}^n} w_1^2 d\mu \cdot \int_{\mathbb{R}^n} (g_\xi - E_\xi)^2 d\mu = \lambda_1^2 \langle \text{Cov}(\mu) \xi, \xi \rangle,
\end{aligned}$$

taking into account the fact that  $\int w_1 d\mu = 0$ . It follows that

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \nabla w_1 d\mu \right|^2 &= \sup_{\xi \in S^{n-1}} \left| \left\langle \int_{\mathbb{R}^n} \nabla w_1 d\mu, \xi \right\rangle \right|^2 \leq \lambda_1^2 \sup_{\xi \in S^{n-1}} \langle \text{Cov}(\mu) \xi, \xi \rangle \\
&= \lambda_1^2 \|\text{Cov}(\mu)\|_{\text{op}}.
\end{aligned} \quad (8.10)$$

Combining (8.9) and (8.10) we obtain the assertion of the lemma.  $\square$

**Remark 8.12.** For every  $\xi \in S^{n-1}$  we have  $\nabla g_\xi = \xi$ , and hence

$$\langle \text{Cov}(\mu)\xi, \xi \rangle = \text{Var}_\mu(g_\xi) \leq \frac{1}{\lambda_1} \int_{\mathbb{R}^n} |\nabla g_\xi|^2 d\mu = \frac{1}{\lambda_1}.$$

This shows that

$$\|\text{Cov}(\mu)\|_{\text{op}} \leq \frac{1}{\lambda_1}. \quad (8.11)$$

*Proof of Theorem 8.10.* By Lemma 8.11, using also the assumption that  $\nabla^2\varphi \geq tI_n$ , we have that

$$t\lambda_\mu = t \int_{\mathbb{R}^n} |\nabla w_1|^2 d\mu \leq \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1 \rangle d\mu \leq \lambda_\mu^3 \|\text{Cov}(\mu)\|_{\text{op}},$$

which implies that

$$\frac{1}{\lambda_\mu^2} \leq \frac{\|\text{Cov}(\mu)\|_{\text{op}}}{t}. \quad (8.12)$$

This is the claim of the theorem.  $\square$

**§ 8.5. Stochastic localization.** Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$  with a regular log-concave density  $\varrho = e^{-\varphi}$ . Set  $\varrho_0 = \varrho$ ,  $\mu_0 = \mu$ ,  $a_0 = \text{bar}(\mu)$ ,  $A_0 = \text{Cov}(\mu) = I_n$  and  $\lambda_0 = \lambda_\mu$ . In particular, since  $\mu$  is isotropic, we have  $\|A_0\|_{\text{op}} = 1$ , and we aim to show that

$$\lambda_0 \geq \frac{c}{\|A_0\|_{\text{op}}} = c$$

for an absolute constant  $c > 0$ . For  $t \geq 0$  and  $\xi \in \mathbb{R}^n$  consider the measure  $\mu_{t,\xi}$  with density

$$\varrho_{t,\xi}(x) = \frac{1}{Z_{t,\xi}} e^{\langle x, \xi \rangle - t|x|^2/2} \varrho_0(x),$$

where  $Z_{t,\xi}$  is a suitable normalizing constant. Note that  $\mu_{t,\xi}$  is  $t$ -uniformly log-concave and if we set  $\lambda_{t,\xi} = \lambda_{\mu_{t,\xi}}$ ,  $A_{t,\xi} = \text{Cov}(\mu_{t,\xi})$  etc., then

$$\lambda_{t,\xi} \geq \sqrt{\frac{t}{\|A_{t,\xi}\|_{\text{op}}}} \geq t$$

by Theorem 8.10. As in Section 7, we choose  $\xi_t$  in a random way, by the stochastic differential equation

$$\xi_0 = 0, \quad d\xi_t = dW_t + a_{t,\xi_t} dt,$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}^n$  with  $W_0 = 0$ . One can check that  $(\xi_t)_{t \geq 0}$  coincides in law with the process  $(tX + W_t)_{t \geq 0}$ , where  $X \sim \mu$  is independent from  $W_t$ . We set

$$\mu_t = \mu_{t,\xi_t}, \varrho_t = \varrho_{t,\xi_t}, a_t = \text{bar}(\mu_t), A_t = \text{Cov}(\mu_t), \lambda_t = \lambda_{\mu_t}.$$

From Itô's formula we see that

$$d\varrho_t(x) = \varrho_t(x) \langle x - a_t, dW_t \rangle, \quad x \in \mathbb{R}^n.$$

The process  $(\varrho_t(x))_{t \geq 0}$  is a martingale with respect to the filtration induced by the Brownian motion, and in particular

$$\mathbb{E}(\varrho_t(x)) = \varrho_0(x) = \varrho(x) \quad (8.13)$$

for all  $t \geq 0$  and  $x \in \mathbb{R}^n$ . Recall that  $A_0 = I_n$  because  $\mu$  is isotropic, therefore  $\|A_0\|_{\text{op}} = 1$ . One of the crucial properties of the covariance process  $(A_t)_{t \geq 0}$  is the following fact.

**Theorem 8.13.** *For every  $0 < t \leq c/(\psi_n^2 \ln n)$  we have*

$$\mathbb{E} \|A_t\|_{\text{op}} \leq C, \quad (8.14)$$

where  $C > 0$  is an absolute constant.

In Section 9, this estimate will be combined with the small-ball reduction from Section 6 to yield the slicing bound. A sketch of the proof of Theorem 8.13 will be presented in the Appendix (Section 11). We also need the next lemma.

**Lemma 8.14.** *Let  $f \in L^2(\mu)$ . For every  $t > 0$  we have that*

$$\mathbb{E}(\text{Var}_{\varrho_t}(f)) \leq \text{Var}_{\varrho_0}(f) \leq \left(2 + \frac{t}{\lambda_0}\right) \mathbb{E}(\text{Var}_{\varrho_t}(f)).$$

*Proof.* For  $f \in L^2(\mu)$  we set

$$M_{t,\xi}(f) = \int_{\mathbb{R}^n} f(x) \varrho_{t,\xi}(x) dx.$$

This is a smooth function of  $\xi$ , and differentiation under the integral sign shows that

$$\nabla_{\xi} M_{t,\xi}(f) = \int_{\mathbb{R}^n} (x - a_{t,\xi}) f(x) \varrho_{t,\xi}(x) dx = \int_{\mathbb{R}^n} (x - a_{t,\xi}) (f(x) - M_{t,\xi}(f)) \varrho_{t,\xi}(x) dx. \quad (8.15)$$

From (8.13) and Fubini's theorem we see that

$$\begin{aligned} \text{Var}_{\varrho_0}(f) &= \int_{\mathbb{R}^n} (f - M_0)^2 \varrho_0 = \mathbb{E} \left( \int_{\mathbb{R}^n} (f - M_0)^2 \varrho_t \right) \\ &= \mathbb{E} \left( \int_{\mathbb{R}^n} (f - M_t)^2 \varrho_t \right) + \mathbb{E}(M_t - M_0)^2. \end{aligned} \quad (8.16)$$

Note that

$$\mathbb{E} \left( \int_{\mathbb{R}^n} (f - M_t)^2 \varrho_t \right) = \mathbb{E}(\text{Var}_{\varrho_t}(f)) \leq \text{Var}_{\varrho_0}(f).$$

Next, we observe that

$$\mathbb{E}(M_t - M_0)^2 = \text{Var}(M_t) = \text{Var}(M_{t,\xi_t}(f)).$$

Recall that  $\xi_t \sim tX + W_t$ , where  $X$  has distribution  $\mu$  and  $W_t$  is a Gaussian random vector in  $\mathbb{R}^n$ , independent from  $X$ , with mean zero and covariance matrix  $tI_n$ . From the subadditivity property of the Poincaré constant we get

$$\vartheta_{\xi_t}^2 = \vartheta_{tX+W_t}^2 \leq \vartheta_{tX}^2 + \vartheta_{W_t}^2 = \frac{t^2}{\lambda_0} + t. \quad (8.17)$$

From (8.15) and the Cauchy-Schwarz inequality we see that

$$\begin{aligned} |\nabla_{\xi} M_{t,\xi}(f)| &= \sup_{u \in S^{n-1}} \int_{\mathbb{R}^n} \langle x - a_{t,\xi}, u \rangle (f(x) - M_{t,\xi}(f)) \varrho_{t,\xi}(x) dx \\ &\leq \sqrt{\|A_{t,\xi}\|_{\text{op}}} \sqrt{\text{Var}_{\varrho_{t,\xi}}(f)} \end{aligned}$$

for any  $\xi \in \mathbb{R}^n$ . Using the Poincaré inequality for the random vector  $\xi_t$  and (8.17), we get

$$\begin{aligned} \text{Var}(M_{t,\xi_t}(f)) &\leq \left(1 + \frac{t}{\lambda_0}\right) t \mathbb{E} |\nabla_{\xi_t} M_{t,\xi_t}(f)|^2 \\ &\leq \left(1 + \frac{t}{\lambda_0}\right) t \mathbb{E} (\|A_t\|_{\text{op}} \text{Var}_{\varrho_t}(f)) \\ &\leq \left(1 + \frac{t}{\lambda_0}\right) \mathbb{E} (\text{Var}_{\varrho_t}(f)), \end{aligned} \tag{8.18}$$

where we have also used the fact that  $\|A_t\|_{\text{op}} \leq 1/t$ . Combining (8.16) and (8.18) we see that

$$\text{Var}_{\varrho_0}(f) \leq \mathbb{E} (\text{Var}_{\varrho_t}(f)) + \left(1 + \frac{t}{\lambda_0}\right) \mathbb{E} (\text{Var}_{\varrho_t}(f)) = \left(2 + \frac{t}{\lambda_0}\right) \mathbb{E} (\text{Var}_{\varrho_t}(f)),$$

as claimed.  $\square$

A consequence of Lemma 8.14 is the next proposition.

**Proposition 8.15.** *For any  $\alpha > 0$  and  $0 < t \leq \alpha \lambda_0$ ,*

$$\frac{1}{\lambda_0} \leq c_1(\alpha + 2) \mathbb{E}(1/\lambda_t) \leq c_1(\alpha + 2) \frac{\mathbb{E}(\sqrt{\|A_t\|_{\text{op}}})}{\sqrt{t}}, \tag{8.19}$$

where  $c_1 > 0$  is an absolute constant.

*Proof.* Recall E. Milman's Theorem 8.2 which states that

$$c\lambda_0^{-1} \leq \sup_{\varphi} \text{Var}_{\mu}(f), \tag{8.20}$$

where the supremum is over all 1-Lipschitz functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $c > 0$  is an absolute constant. Therefore, we may choose a 1-Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\text{Var}_{\mu}(f) \geq c\lambda_0^{-1}/2. \tag{8.21}$$

Lemma 8.14 and the Poincaré inequality imply that

$$\begin{aligned} \text{Var}_{\mu}(f) = \text{Var}_{\varrho_0}(f) &\leq (2 + \alpha) \mathbb{E} (\text{Var}_{\varrho_t}(f)) \leq (2 + \alpha) \mathbb{E} \left( \frac{1}{\lambda_t} \int_{\mathbb{R}^n} |\nabla f|^2 \varrho_t \right) \\ &\leq (2 + \alpha) \mathbb{E}(\lambda_t^{-1}). \end{aligned} \tag{8.22}$$

Now, the left-hand side inequality in (8.19) is a consequence of (8.21) and (8.22), while the right-hand side inequality follows from (8.8).  $\square$

**§ 8.6. Upper bound for the KLS constant.** We are now ready to prove Klartag's upper bound for  $\psi_n$ .

**Theorem 8.16.** *For any  $n \geq 2$  we have that*

$$\psi_n \leq C \sqrt{\ln n},$$

where  $C > 0$  is an absolute constant.

*Proof.* We may choose an isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  so that

$$\psi_\mu \geq \frac{\psi_n}{2}. \quad (8.23)$$

From (8.14) we know that, for  $t = c\psi_n^{-2}/\ln n$ ,

$$\mathbb{E}\left(\sqrt{\|A_t\|_{\text{op}}}\right) \leq \sqrt{\mathbb{E}\|A_t\|_{\text{op}}} \leq C_1, \quad (8.24)$$

where  $C_1 > 0$  is an absolute constant. By the definition of  $t$  and the fact that  $\psi_n \approx \psi_\mu \approx \vartheta_\mu$  and  $\lambda_0 = \vartheta_\mu^{-2}$  we get

$$t \leq c_1\psi_n^{-2} \leq c_2\psi_\mu^{-2} \leq c_3\lambda_0. \quad (8.25)$$

Then, we can apply Proposition 8.15 with  $\alpha = c_3$ . Combining (8.19) with (8.24) and the choice of  $t$  gives

$$\lambda_0^{-1} \leq C_2 \frac{\mathbb{E}(\sqrt{\|A_t\|_{\text{op}}})}{\sqrt{t}} \leq \frac{C_3}{\sqrt{t}} \leq C_4\psi_n \sqrt{\ln n}. \quad (8.26)$$

Then, from (8.23), (8.26) and the equivalence  $\lambda_0^{-1} = \vartheta_\mu^2 \approx \psi_\mu^2$  we get

$$\psi_n^2 \leq 4\psi_\mu^2 \leq C_5\lambda_0^{-1} \leq C_6\psi_n \sqrt{\ln n},$$

which shows that  $\psi_n \leq C \sqrt{\ln n}$  for an absolute constant  $C > 0$ .  $\square$

## 9. THE SLICING THEOREM

Our aim in this section is to describe the affirmative answer to the slicing problem. More precisely, we shall show that the equivalent isotropic constant conjecture is true.

**Theorem 9.1.** *There exists an absolute constant  $C > 0$  such that  $L_n \leq C$  for all  $n \geq 1$ .*

We shall present the proof of P. Bizeul [16] which makes use of the reduction to small-ball estimates that we presented in Section 6. In what follows, we say that a random vector  $X$  in  $\mathbb{R}^n$  is  $b$ -subgaussian if for every  $p \geq 1$  and any  $\xi \in S^{n-1}$

$$(\mathbb{E}|\langle X - \mathbb{E}(X), \xi \rangle|^p)^{1/p} \leq b \sqrt{p}.$$

The small-ball estimate of Theorem 6.8 can be also stated for a not necessarily isotropic log-concave random vector. Then, it takes the following form (see Paouris [118]).

**Theorem 9.2.** *Let  $X$  be a  $b$ -subgaussian log-concave random vector in  $\mathbb{R}^n$  with covariance matrix  $A$ . For any  $0 < \varepsilon \leq c_0$  and any  $y \in \mathbb{R}^n$ ,*

$$\mathbb{P}\left(|X - y|^2 \leq \varepsilon \operatorname{tr}(A)\right) \leq \varepsilon \frac{c_0 \operatorname{tr}(A)}{b^2 \|A\|_{\text{op}} \|A^{-1}\|_{\text{op}}}$$

where  $c_0 > 0$  is an absolute constant.

One can check that the theorem is tight, up to absolute constants, by considering the case where  $X$  is a standard Gaussian random vector. However, for general isotropic log-concave random vectors, it provides a suboptimal exponent of order  $\sqrt{n}$ . Lee and Vempala [94] obtained an improved estimate using stochastic localization.

**Theorem 9.3.** *Let  $X$  be an isotropic log-concave random vector in  $\mathbb{R}^n$ . For any  $y \in \mathbb{R}^n$  and any  $0 < \varepsilon \leq c_0$ ,*

$$\mathbb{P}\left(|X - y|^2 \leq \varepsilon n\right) \leq \varepsilon^{\frac{c_0 n}{\psi_n^2 \ln(n)}}$$

where  $c_0 > 0$  is an absolute constant.

In the previous section we discussed the best known bound for the KLS constant, due to Klartag [78]: we saw that

$$\psi_n^2 \leq C \ln n$$

for some absolute constant  $C > 0$ . Inserting this estimate into Theorem 9.3 we get an exponent which is of the order of  $n$  up to a factor  $(\ln n)^2$ . However, even if we knew that the KLS conjecture has an affirmative answer, the estimate of Lee and Vempala would still include a suboptimal factor  $\ln n$ .

**§ 9.1. The slicing theorem.** Following Bizeul [16] we shall see how one can remove the extra logarithmic factor that appears in Theorem 9.3 and establish an optimal small-ball estimate.

**Theorem 9.4.** *Let  $X$  be an isotropic log-concave random vector in  $\mathbb{R}^n$ . For any  $0 < \varepsilon \leq c_0$  and any  $y \in \mathbb{R}^n$ ,*

$$\mathbb{P}\left(|X - y|^2 \leq \varepsilon n\right) \leq \varepsilon^{c_0 n}$$

where  $c_0 > 0$  is an absolute constant.

Then, we will show that Theorem 9.4 implies the isotropic constant conjecture. In Section 6 we described in detail the work of Dafnis and Paouris and the reduction of the isotropic constant conjecture to the optimal small-ball estimate. Theorem 9.4 is actually equivalent to Theorem 9.1. For the inverse direction note that if  $L_n$  is bounded above by some constant  $C_1 > 0$  then we have seen that the same holds true for  $\tilde{L}_n$ . This means that for any isotropic log-concave random vector  $X$  with density  $f$  and any  $\varepsilon > 0$  and  $y \in \mathbb{R}^n$ , we have that

$$\mathbb{P}\left(|X - y|^2 \leq \varepsilon n\right) = \int_{B(y, \sqrt{\varepsilon n})} f(x) dx \leq C_1^n \text{vol}_n(\sqrt{\varepsilon n} B_2^n) \leq (C_2 \varepsilon)^{n/2},$$

which implies Theorem 9.4.

Before entering the details, we briefly outline the main steps of the proof.

- By the reduction of Section 6, it suffices to establish the small-ball estimate stated in Theorem 9.4.
- To this end, we introduce Eldan's stochastic localization scheme and its associated covariance process, whose behavior is controlled using the results of Appendix 11.
- These estimates are then combined with Guan's results to derive the required small-ball bound.

- Finally, the slicing estimate follows from the small-ball bound via Theorem 6.15.

We start with some preliminary facts that will be used in the proof. The first one is a variant of Theorem 9.2.

**Lemma 9.5.** *Let  $X$  be a  $b$ -subgaussian log-concave random vector with covariance matrix  $A$ . Then, for any  $y \in \mathbb{R}^n$  and any  $0 < \varepsilon \leq c_0$  we have that*

$$\mathbb{P}\left(|X - y|^2 \leq \varepsilon \operatorname{tr}(A)\right) \leq \left(\frac{4n\|A\|_{\text{op}}}{\operatorname{tr}(A)} \varepsilon\right)^{\frac{c_0 \operatorname{tr}(A)^3}{8n^2 b^2 \|A\|_{\text{op}}^2}}$$

where  $c_0 > 0$  is the constant from Theorem 9.2.

*Proof.* Let  $\lambda_1 \geq \dots \geq \lambda_n$  be the eigenvalues of the matrix  $A$ . For any  $2 \leq k \leq n$  we have that

$$n\lambda_k \geq \sum_{i=k}^n \lambda_i = \operatorname{tr}(A) - \sum_{i=1}^{k-1} \lambda_i \geq \operatorname{tr}(A) - k\lambda_1. \quad (9.1)$$

We choose

$$k = \max\left\{\left\lfloor \frac{\operatorname{tr}(A)}{2\lambda_1} \right\rfloor, 1\right\}.$$

If  $k \geq 2$  then (9.1) implies that

$$\lambda_k \geq \frac{\operatorname{tr}(A)}{2n},$$

which remains true when  $k = 1$ . Let  $E$  be the  $k$ -dimensional subspace associated to the eigenvalues  $\lambda_1, \dots, \lambda_k$ , and let  $X_E = P_E X$  be the projection of  $X$  onto  $E$ , and  $A_E = P_E A P_E$  the covariance matrix of  $X_E$ . Then,  $X_E$  is  $b$ -subgaussian and its covariance matrix satisfies :

$$\|A_E\|_{\text{op}} \leq \|A\|_{\text{op}}, \quad \|A_E^{-1}\|_{\text{op}} \leq \frac{2n}{\operatorname{tr}(A)}, \quad \operatorname{tr}(A_E) \geq \frac{k}{n} \operatorname{tr}(A) \geq \frac{\operatorname{tr}(A)^2}{4n\|A\|_{\text{op}}}.$$

Let  $y \in \mathbb{R}^n$  and define  $y_E = P_E y$ . If  $0 < \varepsilon \leq c_0$  then

$$\begin{aligned} \mathbb{P}\left(|X - y|^2 \leq \varepsilon \operatorname{tr}(A)\right) &\leq \mathbb{P}\left(|X_E - y_E|^2 \leq \varepsilon \operatorname{tr}(A)\right) \\ &\leq \left(\frac{\varepsilon \operatorname{tr}(A)}{\operatorname{tr}(A_E)}\right)^{\frac{c_0 \operatorname{tr}(A_E)}{b^2 \|A_E\|_{\text{op}} \|A_E^{-1}\|_{\text{op}}}} \\ &\leq \left(\frac{4n\varepsilon\|A\|_{\text{op}}}{\operatorname{tr}(A)}\right)^{\frac{c_0 \operatorname{tr}(A)^3}{8b^2 n^2 \|A\|_{\text{op}}^2}} \end{aligned}$$

where the second inequality follows by applying Theorem 9.2 to  $X_E$ .  $\square$

We shall also use some classical facts about log-concave vectors. The next lemma is a special case of (4.3).

**Lemma 9.6.** *There exists an absolute constant  $\kappa_0 > 1$  such that for any log-concave real random variable  $Y$  and any  $p \geq 2$*

$$(\mathbb{E}|Y|^p)^{1/p} \leq \kappa_0 p (\mathbb{E}|Y|^2)^{1/2}.$$

Recall also that a random vector, or its density, is called  $t$ -uniformly log-concave for some  $t > 0$  if it is log-concave with respect to a Gaussian of variance  $\frac{1}{t}$ . The next lemma follows from a log-concave/convex correlation inequality of Hargé [68].

**Lemma 9.7.** *Let  $X$  be a  $t$ -uniformly log-concave random vector. Then,  $X$  is  $\frac{1}{\sqrt{t}}$ -subgaussian.*

*Proof.* Without loss of generality we may assume that  $t = 1$  so that  $X$  is log-concave with respect to the standard Gaussian measure denoted by  $\gamma_n$ . We may also assume that  $\mathbb{E}(X) = 0$ . Hargé's result states that if  $f$  is log-concave and  $g$  is convex then

$$\text{Cov}_{\gamma_n}(f, g) \leq 0. \quad (9.2)$$

Let  $f$  be the relative density of  $X$  with respect to  $\gamma_n$ . Applying (9.2) with  $g(x) = |\langle x, \xi \rangle|^p$  for all  $p \geq 1$  and all  $\xi \in S^{n-1}$  we obtain the result.  $\square$

**§ 9.2. Reduction to small diameter.** A first observation, which has been also used in previous approaches to the problem, is that in order to prove Theorem 9.4 we may assume that  $X$  is supported in a ball of radius of the order of  $\sqrt{n}$  and that  $y = 0$ . In this subsection we explain this reduction.

**Lemma 9.8.** *Let  $X$  be an isotropic log-concave random vector in  $\mathbb{R}^n$ . There exists an isotropic log-concave random vector  $Y$  which has support inside the ball  $B(0, \sqrt{c_1 n})$ , where  $c_1 > 0$  is an absolute constant, and satisfies*

$$\mathbb{P}(|X - y| \leq \varepsilon \sqrt{n})^2 \leq \mathbb{P}(|Y| \leq 2\varepsilon \sqrt{n})$$

for every  $0 < \varepsilon < 1$  and any  $y \in \mathbb{R}^n$ .

*Proof.* We consider the symmetrized version  $X_1 = \frac{X - \tilde{X}}{\sqrt{2}}$  of  $X$ , where  $\tilde{X}$  is an independent copy of  $X$ . Then,  $X_1$  is an isotropic symmetric log-concave random vector with the property that, for any  $r > 0$ ,

$$\begin{aligned} \mathbb{P}(|X - y| \leq r) &= \left[ \mathbb{P}(|X - y| \leq r, |\tilde{X} - y| \leq r) \right]^{1/2} \\ &\leq \left[ \mathbb{P}\left( \left| \frac{1}{\sqrt{2}}(X - \tilde{X}) \right| \leq \sqrt{2}r \right) \right]^{1/2} \\ &= \left[ \mathbb{P}(|X_1| \leq \sqrt{2}r) \right]^{1/2}. \end{aligned} \quad (9.3)$$

We set  $K = B(0, 8\kappa_0^2 \sqrt{n}) = \{x \in \mathbb{R}^n : |x| \leq 8\kappa_0^2 \sqrt{n}\}$ , where  $\kappa_0 \geq 1$  is the constant from Lemma 9.6, and define the random vector

$$X_2 = X_1 \cdot \mathbb{1}_{X_1 \in K}.$$

If  $f_1$  is the density of  $X_1$ , then  $X_2$  has density

$$f_2 = \frac{f_1 \mathbb{1}_K}{\int_K f_1} \leq 2f_1 \mathbb{1}_K, \quad (9.4)$$

where the inequality above holds because  $\int_K f_1 \geq \frac{1}{2}$ . To see this, we first use the fact that  $X_1$  is isotropic and Markov's inequality to write

$$\mathbb{P}(X_1 \in K^c) = \mathbb{P}(|X_1|^2 > 64\kappa_0^4 n) \leq \frac{1}{64\kappa_0^4 n} \mathbb{E}|X_1|^2 = \frac{1}{64\kappa_0^4} < \frac{1}{2}, \quad (9.5)$$

and this implies that

$$\int_K f_1 = 1 - \mathbb{P}(X_1 \in K^c) \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

We know that  $X_2$  is symmetric, and hence, from (9.4) we get

$$\text{Cov}(X_2) \leq 2I_n.$$

In fact, we claim that

$$\frac{1}{2}I_n \leq \text{Cov}(X_2) \leq 2I_n. \quad (9.6)$$

To see the left-hand side inequality, for any  $\xi \in S^{n-1}$  we write

$$\begin{aligned} \int_{\mathbb{R}^n} \langle x, \xi \rangle^2 f_2(x) dx &\geq \int_K \langle x, \xi \rangle^2 f_1(x) dx \\ &= 1 - \int_{K^c} \langle x, \xi \rangle^2 f_1(x) dx \\ &\geq 1 - \mathbb{P}(X_1 \in K^c)^{1/2} \left( \int_{\mathbb{R}^n} \langle x, \xi \rangle^4 f_1(x) dx \right)^{1/2} \\ &\geq 1 - \frac{1}{8\kappa_0^2} (2\kappa_0)^2 \int_{\mathbb{R}^n} \langle x, \xi \rangle^2 f_1(x) dx = \frac{1}{2}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality, Lemma 9.6 and (9.5). Finally, we set

$$X_3 = \text{Cov}(X_2)^{-1/2} X_2.$$

Note that  $X_3$  is an isotropic log-concave random vector by construction. Furthermore, using (9.6), we see that it has support inside the ball  $B(0, 8\sqrt{2}\kappa_0^2 \sqrt{n})$ . For any  $0 \leq \varepsilon < 1$ , setting  $r = \varepsilon \sqrt{n}$  we get

$$\begin{aligned} \mathbb{P}(X \in B(y, r))^2 &\leq \mathbb{P}(X_1 \in B(0, \sqrt{2}r)) \\ &\leq \mathbb{P}(X_2 \in B(0, \sqrt{2}r)) \\ &= \mathbb{P}(X_3 \in \text{Cov}(X_2)^{-1/2} B(0, \sqrt{2}r)) \\ &\leq \mathbb{P}(X_3 \in B(0, 2r)) \end{aligned} \quad (9.7)$$

where we used successively (9.3), the fact that  $B(0, \sqrt{2}r) \subset K = B(0, 8\kappa_0^2 \sqrt{n})$  and in the last line the inequality (9.6).  $\square$

It is now clear from (9.7) that in order to prove Theorem 9.4 we may assume  $X$  has support inside a ball of diameter  $c_1 \sqrt{n}$ , where  $c_1 > 0$  is an absolute constant.

**§ 9.3. Bounds for the shrinkage of sets.** The main idea for the proof of Theorem 9.4 is again to use stochastic localization as in the previous section. We recall some basic facts from Section 7. If  $\mu$  is a log-concave probability measure with

density  $\varrho$  with respect to the Lebesgue measure, for  $t \in \mathbb{R}$  and  $\xi \in \mathbb{R}^n$  we define a density

$$\varrho_{t,\xi}(x) = \frac{1}{Z_{t,\xi}} e^{\langle x, \xi \rangle - \frac{t}{2}|x|^2} \varrho(x), \quad x \in \mathbb{R}^n \quad (9.8)$$

where  $Z_{t,\xi}$  is a normalizing factor. Note that  $\varrho_{t,\xi}$  is  $t$ -uniformly log-concave. We also define

$$a_{t,\xi} = \int_{\mathbb{R}^n} x \varrho_{t,\xi}(x) dx$$

and

$$A_{t,\xi} = \int_{\mathbb{R}^n} (x - a_{t,\xi})^{\otimes 2} \varrho_{t,\xi}(x) dx,$$

the barycenter and covariance matrix of  $\varrho_{t,\xi}$  respectively. The tilt process  $(\xi_t)_{t \geq 0}$  is defined as the solution of the stochastic differential equation:

$$d\xi_t = a_{t,\xi_t} dt + dW_t \quad (9.9)$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion. The process  $\varrho_{t,\xi_t}$  is the stochastic localization process starting at  $\varrho$ . We abbreviate  $\varrho_t = \varrho_{t,\xi_t}$  and similarly for the barycenter and covariance,  $a_t$  and  $A_t$ . We also define  $\mu_t$  to be the random measure with density  $\varrho_t$ . For any  $x \in \mathbb{R}^n$ ,  $\varrho_t$  is an Itô process satisfying the stochastic differential equation

$$d\varrho_t(x) = \langle x - a_t, dW_t \rangle \varrho_t(x). \quad (9.10)$$

Integrating (9.10) we see that for any integrable function  $\varphi$ , the process  $(\int_{\mathbb{R}^n} \varphi d\mu_t)_{t \geq 0}$  is a martingale. In particular, for any  $t \geq 0$

$$\mathbb{E}_\mu(\varphi) = \mathbb{E}(\mathbb{E}_{\mu_t}(\varphi)). \quad (9.11)$$

Therefore, for any  $t \geq 0$  we have decomposed the original log-concave probability measure  $\mu$  into a mixture of measures  $\mu_t$  which are  $t$ -uniformly log-concave.

By the log-concave Lichnerowicz inequality (8.7) we have that

$$A_t \leq \frac{1}{t} I_n \quad (9.12)$$

almost surely, in the sense of symmetric matrices. Using Klartag's improved log-concave Lichnerowicz inequality of Theorem 8.10, Guan [65] obtained a lower-bound on the trace of  $A_t$  up to a time of order 1:

**Theorem 9.9.** *Assume that the starting measure  $\mu$  is an isotropic log-concave probability measure. Then,*

$$\mathbb{E}(\text{tr}(A_{\delta_1})) \geq \delta_1 n,$$

where  $\delta_1 > 0$  is an absolute constant.

A sketch of the proof of Theorem 9.9 will be presented in the Appendix (Section 11).

Our goal in this subsection is to obtain estimates for the evolution of the measure of sets along the stochastic localization process under the additional assumption that  $\mu$  has bounded support of diameter  $D$ .

**Proposition 9.10.** *Let  $\mu$  be an isotropic log-concave probability measure with bounded support of diameter  $D$ . For any measurable set  $S$  and any  $\lambda > 1$  we have that*

$$\mu(S) \leq e^{\frac{D^2 t}{2}} \mu_t(S)^{\frac{1}{\lambda}}$$

*with probability at least  $1 - \frac{1}{\lambda}$ , where  $(\mu_t)_{t \geq 0}$  is the stochastic localization process starting at  $\mu$ .*

*Proof.* Let  $S \subset \mathbb{R}^n$  be a set of positive measure and let  $g_t = \mu_t(S)$ . Using (9.10), we can check that

$$dg_t = \left\langle \int_S (x - a_t) d\mu_t(x), dW_t \right\rangle.$$

Then,

$$d[g]_t = \left| \int_S (x - a_t) d\mu_t(x) \right|^2 dt.$$

Note that  $\mu_t$  is supported in the same set as  $\mu$ , which has diameter  $D$ , and hence

$$\left| \int_S (x - a_t) d\mu_t(x) \right|^2 \leq \sup_{x \in \text{supp}(\mu_t)} |x - a_t|^2 \left( \int_S d\mu_t(x) \right)^2 \leq D^2 \mu_t(S)^2 = D^2 g_t^2.$$

It follows that

$$d[g]_t \leq D^2 g_t^2 dt. \tag{9.13}$$

Using Itô's formula (7.1) we compute  $d(\ln(g_t^{-1})) = -\frac{dg_t}{g_t} + \frac{1}{2} \frac{d[g]_t}{g_t^2}$ , which implies that

$d\mathbb{E}(\ln(g_t^{-1})) \leq \frac{D^2 t}{2}$ , and hence

$$\mathbb{E}(\ln(g_t^{-1})) \leq \ln(g_0^{-1}) + \frac{D^2 t}{2}. \tag{9.14}$$

Now, let  $\lambda > 1$ . From (9.14), using Markov's inequality, we see that with probability at least  $1 - \frac{1}{\lambda}$

$$\ln(g_t^{-1}) \leq \lambda \left( \ln(g_0^{-1}) + \frac{D^2 t}{2} \right).$$

Equivalently,

$$\ln g_0 \leq \frac{1}{\lambda} \ln g_t + \frac{D^2 t}{2}$$

and the result follows.  $\square$

**§ 9.4. Proof of the slicing theorem.** We start with the proof of the optimal small-ball estimate of Theorem 9.4.

*Proof of Theorem 9.4.* Lemma 9.8 shows that it is enough to consider an isotropic log-concave random vector  $X$  distributed according to an isotropic log-concave probability measure  $\mu$  with support of diameter  $c_1 \sqrt{n}$ , where  $c_1 > 0$  is an absolute constant. Let  $(\mu_t)_{t \geq 0}$  be the stochastic localization process starting at  $\mu$  and let  $X_t$  be

the random vector with law  $\mu_t$ . Let  $\delta_1$  be the constant in Guan's Theorem 9.9. We consider the event

$$E_0 = \{\text{tr}(A_{\delta_1}) \geq \delta_1 n/2\}.$$

**Lemma 9.11.** *We have  $\mathbb{P}(E_0) \geq \delta_1^2/2$ .*

*Proof.* We set  $p_0 = \mathbb{P}(E_0)$ . From (9.12) we know that  $A_{\delta_1} \leq \frac{1}{\delta_1} I_n$  almost surely, and hence

$$\delta_1 n \leq \mathbb{E}(\text{tr}(A_{\delta_1})) = \mathbb{E}(\text{tr}(A_{\delta_1}) \mathbb{1}_{E_0}) + \mathbb{E}(\text{tr}(A_{\delta_1}) \mathbb{1}_{E_0^c}) \leq \frac{np_0}{\delta_1} + \frac{\delta_1 n}{2},$$

which implies that  $p_0 \geq \delta_1^2/2$ .  $\square$

Let  $\bar{c}_0 > 0$  be the constant from Lemma 9.5. For every  $0 < \varepsilon < \bar{c}_0$  we set  $S_\varepsilon = B(0, \sqrt{\varepsilon n})$ . We also fix  $\lambda = \frac{4}{\delta_1^2}$ . Proposition 9.10 shows that the probability  $p_1$  of the event

$$E_1 = \left\{ \mu(S_\varepsilon) \leq e^{\frac{c_1^2 \delta_1 n}{2}} \mu_{\delta_1}(S_\varepsilon)^{\frac{1}{\lambda}} \right\}$$

satisfies  $p_1 \geq 1 - \frac{\delta_1^2}{4}$ , therefore

$$\mathbb{P}(E_0 \cap E_1) \geq p_0 + p_1 - 1 \geq \frac{\delta_1^2}{4} > 0.$$

On the event  $E_0 \cap E_1$  we have  $\text{tr}(A_{\delta_1}) \geq \delta_1 n/2$ , and also  $\|A_{\delta_1}\|_{\text{op}} \leq 1/\delta_1$ . Moreover,  $X_{\delta_1}$  is  $1/\sqrt{\delta_1}$ -subgaussian by Lemma 9.7. Inserting all these estimates into Lemma 9.5 we get that

$$\mu(S_\varepsilon) \leq e^{\frac{c_1^2 \delta_1 n}{2}} \mu_{\delta_1}(S_\varepsilon)^{\frac{4}{\delta_1^2}} \leq e^{\frac{c_1^2 \delta_1 n}{2}} \left( \frac{16\varepsilon}{\delta_1^3} \right)^{\frac{c_0 \delta_1^8 n}{256}} \leq \varepsilon^{\frac{c_0 \delta_1^8 n}{512}}$$

if we assume that

$$\varepsilon < \varepsilon_0 := \min \left\{ \frac{\bar{c}_0 \delta_1}{2}, \frac{\delta_1^6}{256} \exp \left( -\frac{256 c_1^2}{\bar{c}_0 \delta_1^7} \right) \right\}.$$

This establishes Theorem 9.4 with  $c_0 = \min \left\{ \frac{\bar{c}_0 \delta_1}{2}, \frac{\delta_1^6}{256} \exp \left( -\frac{256 c_1^2}{\bar{c}_0 \delta_1^7} \right), \frac{\bar{c}_0 \delta_1^8}{512} \right\}$ .  $\square$

*Proof of Theorem 9.1.* Basically, we repeat the proof of Theorem 6.15. From Proposition 6.14 we know that there exists an isotropic convex body  $K$  in  $\mathbb{R}^n$  with  $L_n \leq C_1 L_K$  and  $\text{vol}_n(K \cap \sqrt{C_2 n} B_2^n) > c_2^n$  for some absolute constants  $C_1, C_2, c_2 > 0$ .

Theorem 9.4 shows that there exists an absolute constant  $c_0 > 0$  such that for any  $0 < \varepsilon \leq c_0$ ,

$$\text{vol}_n(K \cap \varepsilon \sqrt{n} L_K B_2^n) \leq \varepsilon^{c_0 n}.$$

Choosing  $\varepsilon_1 = \min \{c_0, c_2^{\frac{1}{c_0}}\}$ , and comparing the above inequalities, we get  $L_K \leq \sqrt{C_2}/\varepsilon_1$ , and hence

$$L_n^2 \leq C_1^2 L_K^2 \leq \frac{C_1^2 C_2}{\varepsilon_1^2}.$$

This shows that  $L_n$  is bounded by an absolute constant.  $\square$

**§ 9.5. The thin-shell conjecture.** A third well-known conjecture about isotropic log-concave probability measures is the thin-shell conjecture which has its origin in the central limit problem, the question to identify those high-dimensional distributions that have approximately Gaussian marginals. A typical example is given by the random vector  $X = (X_1, \dots, X_n)$  which is distributed uniformly in the cube  $Q(n) = [-\sqrt{3}, \sqrt{3}]^n$  (the normalization is so that  $\text{Var}(X_j^2) = 1$  for all  $1 \leq j \leq n$ ). It is well-known that, if the  $\xi_j$ 's satisfy e.g. Lindeberg's condition, then the distribution of

$$\langle X, \xi \rangle = \sum_{j=1}^n \xi_j X_j$$

is approximately Gaussian. A second example is given by the ball  $D(n) = \sqrt{n+2}B_2^n$ . Let  $X$  be a random vector which is uniformly distributed in  $D(n)$ . From Maxwell's observation that, if  $n$  is large enough, then

$$\sigma(\{\xi \in S^{n-1} : \xi_j \leq t\}) \sim \sqrt{\frac{n}{2\pi}} \int_{-\infty}^t \exp(-s^2 n/2) ds$$

for all  $t \in [-1, 1]$ , as well as the symmetry of  $D(n)$ , one can check that the distribution of  $\langle X, \xi \rangle$  is close to the standard normal distribution for any  $\xi \in S^{n-1}$ .

Assume now that  $\mu$  is an isotropic Borel probability measure on  $\mathbb{R}^n$ , i.e. normalized so that

$$\mathbb{E}_\mu(x_j) = 0 \quad \text{and} \quad \mathbb{E}_\mu(x_i x_j) = \delta_{ij}, \quad i, j = 1, \dots, n.$$

It has been observed that if  $\mu$  satisfies a "thin-shell bound" then the central limit problem has an affirmative answer for  $\mu$ . More precisely, if

$$\mu(\{x \in \mathbb{R}^n : ||x| - \sqrt{n}| \geq \varepsilon \sqrt{n}\}) \leq \varepsilon$$

for some  $\varepsilon \in (0, 1/2)$ , then, for all directions  $\xi$  in a subset  $A$  of  $S^{n-1}$  of measure  $\sigma(A) \geq 1 - \exp(-c_1 \sqrt{n})$ , we have

$$|\mathbb{P}(\langle X, \xi \rangle \leq t) - \Phi(t)| \leq c_2(\varepsilon + n^{-\alpha}) \quad \text{for all } t \in \mathbb{R},$$

where  $\Phi(t)$  is the standard Gaussian distribution function and  $c_1, c_2, \alpha > 0$  are absolute constants.

Note that in the statement above we have assumed that the dimension is large enough but we have not assumed independence of the coordinate functions  $x \mapsto x_j$  and we have not made any symmetry assumptions about  $\mu$ . Sudakov's work [132] is probably the first place where it is observed that a thin-shell condition implies that most marginals of a high-dimensional distribution are approximately Gaussian. Related early works are the ones by Diaconis and Freedman [43], and by von Weizsäcker [137]. The case where  $\mu = \mu_K$  is the measure with density  $L_K^n \mathbb{1}_{L_K}$  for an isotropic symmetric convex body  $K$  in  $\mathbb{R}^n$  was studied by Anttila, Ball and Perissinaki in [1] who showed that a thin-shell estimate implies an affirmative answer to the central limit problem for  $\mu_K$ . This work made the central limit problem widely known among people working in convex geometry. A clear exposition of both the general and the log-concave case can be found in Bobkov's article [20].

As Klartag notes in [76], it is not hard to construct simple examples of isotropic distributions for which a thin-shell estimate is not possible. If we write  $\sigma_t$  for the uniform probability measure on the sphere  $tS^{n-1}$  then, for  $t_1 = \sqrt{n}/2$  and  $t_2 = \sqrt{7n}/2$ ,

the isotropic probability measure  $\mu = \frac{1}{2}(\sigma_{t_1} + \sigma_{t_2})$  does not have Gaussian marginals, and hence it does not satisfy a thin-shell bound. However, it was conjectured that the assumption of log-concavity guarantees a thin-shell bound. Bobkov and Koldobsky [22] defined the parameter

$$\sigma_\mu^2 = \frac{1}{n} \text{Var}_\mu(|x|^2)$$

and asked if

$$\sigma_n = \sup\{\sigma_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leq C$$

for some absolute constant  $C > 0$ . It is easily checked that this is equivalent to the question if there exists an absolute constant  $C > 0$  such that, for any  $n \geq 1$  and any isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ , one has

$$\mathbb{E}_\mu(|x| - \sqrt{n})^2 \leq C^2.$$

Moreover, applying the Poincaré inequality for the function  $f(x) = |x|^2$  we see that

$$\text{Var}_\mu(|x|^2) \leq 4\vartheta_\mu^2 \int_{\mathbb{R}^n} |x|^2 d\mu(x) = 4\vartheta_\mu^2 n,$$

which shows that  $\sigma_\mu \leq 2\vartheta_\mu$ , and hence

$$\sigma_n \leq C\psi_n$$

for all  $n$ , where  $C > 0$  is an absolute constant. In other words, the KLS conjecture implies the thin-shell conjecture. An important fact is that the thin-shell conjecture is also related to the isotropic constant conjecture. Eldan and Klartag [46] proved that there exists an absolute constant  $C > 0$  such that

$$L_n \leq C\sigma_n$$

for every  $n \geq 1$ .

The first non-trivial upper bound for  $\sigma_n$  was given by Klartag in [73] in his proof of the central limit theorem for convex bodies. He proved that

$$\sigma_n \leq C\sqrt{n}/\ln n.$$

This estimate was then improved to  $\sigma_n \leq Cn^{2/5+o(1)}$  by Klartag [74], to  $\sigma_n \leq Cn^{3/8}$  by Fleury [49], to  $\sigma_n \leq Cn^{1/3}$  by Guédon and E. Milman [66], and to  $\sigma_n \leq Cn^{1/4}$  by Lee and Vempala [93]. A consequence of the more recent developments on the KLS conjecture is that  $\sigma_n \leq C\sqrt{\ln n}$  by the corresponding bound  $\psi_n = O(\sqrt{\ln n})$  of Klartag in [78]. An even more recent breakthrough of Guan [65] showed that

$$\sigma_n \leq C \ln \ln n.$$

Finally, Klartag and Lehec [83] announced an affirmative answer to the thin-shell conjecture.

**Theorem 9.12.** *There exists an absolute constant  $C > 0$  such that*

$$\text{Var}_\mu(|x|^2) \leq Cn$$

*for every  $n \geq 1$  and every isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ . Equivalently,  $\sup_n \sigma_n \leq C$ .*

Using reverse Hölder inequalities for polynomials of a random vector distributed uniformly in a convex body (see [28], [115]) one can deduce from Theorem 9.12 that, for any isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  and for any  $t > 0$ ,

$$\mu(\{x \in \mathbb{R}^n : ||x| - \sqrt{n}| \geq t\}) \leq c_1 \exp(-c_2 \sqrt{t}).$$

This is a thin-shell estimate because it shows that if  $1 \ll t \ll \sqrt{n}$  then most of the mass of  $\mu$  is concentrated on a thin spherical cell  $\{x \in \mathbb{R}^n : \sqrt{n} - t \leq |x| \leq \sqrt{n} + t\}$  whose width  $t$  is much smaller than the central radius  $\sqrt{n}$ .

The proof of Theorem 9.12 starts from ideas that Klartag used in [75], where he established an optimal thin-shell bound for unconditional log-concave measures. This approach was extended by Barthe and Cordero-Erausquin in [12] who adapted Klartag's techniques to provide spectral gap estimates for log-concave measures with many symmetries, and by Barthe and Klartag in [13]. Given a log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ , let  $H^1(\mu)$  be the space of all functions  $f \in L^2(\mu)$  with weak partial derivatives in  $L^2(\mu)$ , equipped with the norm

$$\|f\|_{H^1(\mu)}^2 = \left( \int_{\mathbb{R}^n} |f|^2 d\mu + \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \right)^{1/2}.$$

Barthe and Klartag showed in [13] that  $C_c^\infty(\mathbb{R}^n)$  is dense in  $H^1(\mu)$ . Now, for every  $f \in L^2(\mu)$  with  $\int f d\mu = 0$ , define

$$\begin{aligned} \|f\|_{H^{-1}(\mu)} &= \sup \left\{ \int_{\mathbb{R}^n} fg d\mu : g \in H^1(\mu), \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^n} fg d\mu : g \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \leq 1 \right\}. \end{aligned}$$

In [13] and [75] it is shown, by using Bochner's formula, that

$$\|f\|_{L^2(\mu)}^2 \leq \|\nabla f\|_{H^{-1}(\mu)}^2 := \sum_{i=1}^n \|\partial_i f\|_{H^{-1}(\mu)}^2 \quad (9.15)$$

for every smooth function  $f \in H^1(\mu)$  that satisfies  $\int f d\mu = 0$  and  $\int \nabla f d\mu = 0$ . Applying (9.15) for an isotropic log-concave probability measure  $\mu$  and the function  $f(x) = |x|^2 - n$ , we see that

$$\text{Var}_\mu(|x|^2) \leq 4 \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2.$$

Klartag and Lehec prove the next inequality, which immediately implies Theorem 9.12.

**Theorem 9.13.** *There exists an absolute constant  $C > 0$  such that*

$$\sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2 \leq Cn$$

for every  $n \geq 1$  and every isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ .

The proof is based again on stochastic localization. Klartag and Lehec consider the family of exponential tilts of log-affine perturbations of the original measure  $\mu$  and construct suitable couplings between these tilts. Then, they use these couplings to bound the  $H^{-1}(\mu)$ -norm by the growth of the covariance process  $(A_t)_{t \geq 0}$  using a variant of Guan's technique from [65].

## 10. THE $MM^*$ -ESTIMATE FOR ISOTROPIC CONVEX BODIES

Let  $K$  be a convex body in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ . In this section we discuss the parameters

$$M(K) = \int_{S^{n-1}} p_K(\xi) d\sigma(\xi) \quad \text{and} \quad M^*(K) = \int_{S^{n-1}} h_K(\xi) d\sigma(\xi)$$

where  $p_K$  is the Minkowski functional and  $h_K$  is the support function of  $K$ .

Recall that  $\text{vrad}(K) = (\text{vol}_n(K)/\text{vol}_n(B_2^n))^{1/n}$  denotes the volume radius of  $K$ . It is known that

$$M(K)^{-1} \leq \text{vrad}(K) \leq M^*(K) = M(K^\circ),$$

where  $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$  is the polar body of  $K$ . The right-hand side inequality is a classical inequality of Urysohn, while the left-hand side inequality follows by integration in spherical coordinates and an application of Hölder's inequality. Therefore,

$$M(K)M^*(K) \geq 1.$$

As mentioned in the introduction, a fundamental fact in the other direction, following from results of Figiel–Tomczak-Jaegermann [48], Lewis [95] and Pisier's estimate [121] on the norm of the Rademacher projection, states that for any symmetric convex body  $K$  in  $\mathbb{R}^n$  there exists  $T \in GL_n$  such that

$$M(T(K))M^*(T(K)) \leq c \ln n,$$

where  $c > 0$  is an absolute constant. In the general case, without the symmetry assumption for  $K$ , one considers the parameter

$$E(K) = \inf M(T(K))M^*(T(K))$$

where the infimum is taken over all invertible affine transformations of  $\mathbb{R}^n$  for which  $0 \in \text{int}(T(K))$ . The study of the question to obtain a sharp upper bound for  $\max E(K)$  started with a work of Banaszczyk, Litvak, Pajor and Szarek, who showed in [11] that if  $K$  is a convex body in  $\mathbb{R}^n$  which is in John's position (i.e. the ellipsoid of maximal volume inscribed in  $K$  is the Euclidean unit ball) then

$$M^*(K) \leq c \sqrt{n \ln n}.$$

When  $K$  is in John's position, from the inclusion  $K \supseteq B_2^n$  we also obtain the trivial upper bound  $M(K) \leq M(B_2^n) = 1$ , and this implies that  $E(K) \leq c \sqrt{n \ln n}$ . This estimate was improved by Rudelson in [126]: he showed that if  $K$  is a convex body in  $\mathbb{R}^n$  then

$$E(K) \leq c \sqrt[3]{n} (\ln n)^b$$

where  $c, b > 0$  are absolute constants.

Our goal in this section is to present the known upper bounds for the parameters  $M^*(K)$  and  $M(K)$ , when  $K$  is in the isotropic position. E. Milman proved in [108] that if  $K$  is a symmetric isotropic convex body in  $\mathbb{R}^n$  then

$$M^*(K) \leq c_1 \sqrt{n}(\ln n)^2 L_K \leq c_2 \sqrt{n}(\ln n)^2$$

where the second inequality takes into account the fact that now we know that  $L_n \leq C$ . Urysohn's inequality shows that the dependence on  $n$  is optimal up to the logarithmic term. Some first non-trivial results about the dual problem, to estimate  $M(K)$  in the isotropic position, were obtained in [60] and later in [58], where it was shown that  $M(K) \leq C(n \ln n)^{1/3} / \sqrt{n}$  (see also [128]). Very recently, Bizeul and Klartag [17] obtained the optimal (up to the  $\ln n$ -term) upper bound

$$M(K) \leq \frac{C \ln n}{\sqrt{n}}.$$

In the next subsections we present these developments.

**§ 10.1. A general approach.** A starting point for the question to bound  $M^*(K)$  is Dudley's entropy estimate (see e.g. [123, Theorem 5.5]):

$$\sqrt{n}M^*(K) \leq C_1 \sum_{k=1}^n \frac{1}{\sqrt{k}} e_k(K, B_2^n) \quad (10.1)$$

where  $e_k(A, B)$  is the  $k$ -th entropy number of  $A$  and  $B$ , defined for  $k \geq 1$  as

$$e_k(A, B) = \inf\{t > 0 : N(A, tB) \leq 2^{k-1}\}.$$

We set  $e_k(K) := e_k(K, B_2^n)$ . Although (10.1) is usually stated for symmetric convex bodies, it should be noted that the same estimate holds true for nonsymmetric convex bodies too. This follows from the observation that  $M^*(K) = \frac{1}{2}M^*(K - K)$  and

$$e_{k-1}(K - K, B_2^n) \leq 2e_{k/2}(K, B_2^n) \quad (10.2)$$

for every even integer  $k$ , as a consequence of the simple estimate

$$N(K - K, 2t_0 B_2^n) \leq N(K, t_0 B_2^n) N(-K, t_0 B_2^n) \leq 2^{k-2}$$

if  $t_0 = e_{k/2}(K, B_2^n)$ .

The  $MM^*$ -estimates that were obtained in [108] and [58] depend on the following volumetric parameters that can be defined for any convex body  $K$  in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ :

$$w_k(K) := \sup\{\text{vrad}(K \cap E) : E \in G_{n,k}\}, \quad v_k(K) := \sup\{\text{vrad}(P_E(K)) : E \in G_{n,k}\},$$

and

$$w_k^-(K) := \inf\{\text{vrad}(K \cap E) : E \in G_{n,k}\}, \quad v_k^-(K) := \inf\{\text{vrad}(P_E(K)) : E \in G_{n,k}\}.$$

If  $K$  is symmetric then we have that

$$0 < c \leq w_k^-(K)v_k(K^\circ) \leq 1 \quad \text{and} \quad 0 < c \leq v_k^-(K)w_k(K^\circ) \leq 1$$

where  $c > 0$  is an absolute constant, by the Blaschke-Santaló inequality and the Bourgain-Milman inequality. Note that, for every  $F \in G_{n,k}$ ,

$$\begin{aligned} \text{vol}_k(P_F(K)) &\leq N(P_F(K), e_k(K)P_F(B_2^n)) \text{vol}_k(e_k(K)B_F) \leq N(K, e_k(K)B_2^n) e_k(K)^k \text{vol}_k(B_F) \\ &\leq (2e_k(K))^k \text{vol}_k(B_F), \end{aligned} \quad (10.3)$$

and hence,

$$v_k(K) \leq 2e_k(K). \quad (10.4)$$

In view of (10.4), the next theorem of V. Milman and Pisier [113] is an alternative, and sometimes stronger, version of Dudley's bound.

**Theorem 10.1.** *For every symmetric convex body  $K$  in  $\mathbb{R}^n$  one has*

$$\sqrt{n}M^*(K) \leq c_2 \sum_{k=1}^n \frac{1}{\sqrt{k}} \text{Rad}_k(K) v_k(K), \quad (10.5)$$

where  $\text{Rad}_k(K) := \sup\{\text{Rad}(X_{P_F(K)}) : F \in G_{n,k}\}$ , and  $\text{Rad}(Y) \leq c_3 \ln(d(Y, \ell_2^{\dim(Y)})) + 1$  is the Rademacher constant of  $Y$ .

We refer to [2, Chapter 6] for more information on the Rademacher constant; what we really use here is that  $\text{Rad}_k(K) = O(\ln n)$  for all  $1 \leq k \leq n$ . When  $K$  is assumed isotropic, Theorem 10.1 is the basis for the  $M^*$ -estimate of E. Milman [108], which is essentially optimal and, as we will see, can be easily transferred to non-symmetric convex bodies.

For the  $M$ -estimate we need to introduce the so-called Gelfand numbers. Given any pair of convex bodies  $K, L$  in  $\mathbb{R}^n$  with  $0 \in \text{int}(K) \cap \text{int}(L)$ , the Gelfand numbers  $c_k(K, L)$ ,  $0 \leq k \leq n-1$  are defined as follows:

$$c_k(K, L) = \inf\{\text{diam}_{L \cap F}(K \cap F) : F \in G_{n, n-k}\},$$

where  $\text{diam}_A(B) := \inf\{R > 0 : B \subseteq RA\}$ . We set  $c_k(K) := c_k(K, B_2^n)$ .

In the symmetric case, Carl's theorem [36] relates any reasonable Lorentz norm of the sequence of entropy numbers  $\{e_m(K, L)\}$  with that of the Gelfand numbers  $\{c_m(K, L)\}$ . In particular, for any  $\alpha > 0$ , there exist constants  $c(\alpha), c'(\alpha) > 0$  such that, for any  $n \geq 1$

$$\sup_{k=1, \dots, n} k^\alpha e_k(K, L) \leq c(\alpha) \sup_{k=1, \dots, n} k^\alpha c_k(K, L), \quad (10.6)$$

and

$$\sum_{k=1}^n k^{-1+\alpha} e_k(K, L) \leq c'(\alpha) \sum_{k=1}^n k^{-1+\alpha} c_k(K, L). \quad (10.7)$$

In fact, Pisier deduces the covering estimates of Theorem 3.2 from an application of Carl's theorem, after establishing the following estimates:

$$\max\{c_k(K, \mathcal{E}), c_k(K^\circ, \mathcal{E}^\circ)\} \leq c_1(\alpha) \left(\frac{n}{k}\right)^{1/\alpha} \quad \text{for all } k \in \{1, \dots, n\}, \quad (10.8)$$

where  $c_1(\alpha) = O((2-\alpha)^{-1/2})$  as  $\alpha \rightarrow 2^-$ .

The next theorem from [58] can serve as a basis for the  $M$ -estimate.

**Theorem 10.2.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Then for any  $k = 1, \dots, n/2$ ,*

$$c_{2k}(K) \leq c \frac{n}{k} \ln\left(e + \frac{n}{k}\right) \omega_k(K). \quad (10.9)$$

*In other words, there exists  $F \in G_{n, n-2k}$  such that*

$$K \cap F \subseteq c \frac{n}{k} \ln\left(e + \frac{n}{k}\right) \omega_k(K) B_F, \quad (10.10)$$

*and dually, there exists  $F \in G_{n, n-2k}$  such that*

$$P_F(K) \supseteq \frac{1}{c \frac{n}{k} \ln\left(e + \frac{n}{k}\right)} v_k^-(K) B_F. \quad (10.11)$$

*Proof.* Given  $k = 1, \dots, n/2$ , let  $\mathcal{E}$  be an  $\alpha$ -regular  $M$ -ellipsoid for  $K$ , for some  $\alpha \in [1, 2)$  to be determined. By the second estimate in (10.8), we know that there exists  $E \in G_{n, n-k}$  so that:

$$P_E(K) \supseteq \frac{1}{c_1(\alpha)} \left(\frac{k}{n}\right)^{1/\alpha} P_E(\mathcal{E}). \quad (10.12)$$

Consider the ellipsoid  $\mathcal{E}' = P_E(\mathcal{E})$ , let  $H$  be the subspace spanned by the  $k$  shortest axes of  $\mathcal{E}'$ , and set  $F$  to be its orthogonal complement into  $E$ . Then,  $F \in G_{n, n-2k}$  and

$$P_F(\mathcal{E}') \supseteq \text{vrad}(P_H(\mathcal{E}')) B_F. \quad (10.13)$$

On the other hand, we can give a lower bound for  $\text{vrad}(P_H(K))$ : From Theorem 3.2 we know that  $N_t = N(K, t\mathcal{E}) \leq \exp(c(\alpha)n/t^\alpha)$  for every  $t \geq c(\alpha)^{1/\alpha}$ , where  $c(\alpha) = [c_1(\alpha)]^\alpha$ . Let  $x_1, \dots, x_{N_t}$  be points in  $\mathbb{R}^n$  such that  $K \subseteq \bigcup_{i=1}^{N_t} (x_i + t\mathcal{E})$ . Then,  $P_H(K) \subseteq \bigcup_{i=1}^{N_t} (P_H(x_i) + tP_H(\mathcal{E}))$ , and hence

$$\text{vol}_k(P_H(K)) \leq N_t \text{vol}_k(tP_H(\mathcal{E})) \leq \exp(c(\alpha)n/t^\alpha) t^k \text{vol}_k(P_H(\mathcal{E})).$$

Choosing  $t = c(\alpha)^{1/\alpha} (n/k)^{1/\alpha} = c_1(\alpha) (n/k)^{1/\alpha}$  we see that

$$\text{vrad}(P_H(\mathcal{E})) \geq \frac{c_2}{c_1(\alpha)} \left(\frac{k}{n}\right)^{1/\alpha} \text{vrad}(P_H(K)) \geq \frac{c_2}{c_1(\alpha)} \left(\frac{k}{n}\right)^{1/\alpha} v_k^-(K). \quad (10.14)$$

From (10.13) and (10.14) we get

$$P_F(\mathcal{E}') \supseteq \frac{c_2}{c_1(\alpha)} \left(\frac{k}{n}\right)^{1/\alpha} v_k^-(K) B_F$$

and then (10.12) shows that

$$P_F(K) \supseteq \frac{c_2}{[c_1(\alpha)]^2} \left(\frac{k}{n}\right)^{2/\alpha} v_k^-(K) B_F. \quad (10.15)$$

Choosing  $\alpha = 2 - \frac{1}{\ln(en/k)}$  we obtain (10.11). The estimate (10.10) then follows by duality.  $\square$

**Corollary 10.3.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . For every  $k = 1, \dots, n$  and  $\alpha > 0$ ,*

$$e_k(K, B_2^n) \leq c(\alpha) \sup_{m=1, \dots, k} \left(\frac{m}{k}\right)^\alpha \frac{n}{m} \ln\left(e + \frac{n}{m}\right) w_m(K),$$

where  $c(\alpha) > 0$  is a constant depending only on  $\alpha$ .

*Proof.* The claim follows from Theorem 10.2 and Carl's theorem (see (10.6)). Since  $k \mapsto c_k(K, B_2^n)$  is non-increasing, there is no difference whether we take the supremum on the right-hand-side just on the even integers.  $\square$

**Corollary 10.4.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$  such that  $K \subseteq RB_2^n$ . Then,*

$$\sqrt{n}M^*(K) \leq c \sum_{k=1}^n \frac{1}{\sqrt{k}} \min \left\{ R, \frac{n}{k} \ln \left( e + \frac{n}{k} \right) w_k(K) \right\}.$$

*Dually, for every symmetric convex body  $K$  in  $\mathbb{R}^n$  with  $K \supseteq rB_2^n$  we have that*

$$\sqrt{n}M(K) \leq c \sum_{k=1}^n \frac{1}{\sqrt{k}} \min \left\{ \frac{1}{r}, \frac{n}{k} \ln \left( e + \frac{n}{k} \right) \frac{1}{v_k^-(K)} \right\}.$$

*Proof.* We explain the first claim, and then the second follows by duality. For the proof it is enough to combine Dudley's entropy estimate (10.1) with Carl's theorem (see (10.7)): we have

$$\sqrt{n}M^*(K) \leq c_1 \sum_{k=1}^n \frac{1}{\sqrt{k}} e_k(K) \leq c_2 \sum_{k=1}^n \frac{1}{\sqrt{k}} c_k(K).$$

Since  $c_k(K) \leq R$  for all  $k$ , the assertion follows from the estimate (10.9) of Theorem 10.2.  $\square$

**§ 10.2.  $MM^*$ -estimate for isotropic symmetric convex bodies.** Let  $K$  be an isotropic symmetric convex body in  $\mathbb{R}^n$ . First we describe a simplified version of E. Milman's proof of the next almost sharp estimate for the mean width of  $K$ .

**Theorem 10.5.** *Let  $K$  be an isotropic symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$M^*(K) \leq c \sqrt{n}(\ln n)^2 \quad (10.16)$$

where  $c > 0$  is an absolute constant.

*Proof.* A direct consequence of Theorem 10.1 is the inequality

$$\sqrt{n}M^*(K) \leq c_1(\ln n) \sum_{k=1}^n \frac{1}{\sqrt{k}} v_k(K). \quad (10.17)$$

We shall apply (10.17) for an isotropic convex body  $K$  in  $\mathbb{R}^n$ . Let  $F \in G_{n,k}$ . From Theorem 5.4 we know that  $\text{vol}_k(P_F(K))^{1/k} \leq c_1 \frac{n}{k}$  where  $c_1 > 0$  is an absolute constant, and hence

$$\text{vrad}(P_F(K)) \approx \sqrt{k} \text{vol}_k(P_F(K))^{1/k} \leq c_2 \frac{n}{\sqrt{k}}.$$

Therefore,  $v_k(K) \leq c_2 n / \sqrt{k}$  for all  $1 \leq k \leq n-1$ . Going back to (10.17) we get

$$\sqrt{n}M^*(K) \leq c_3(\ln n) \sum_{k=1}^n \frac{1}{\sqrt{k}} \frac{n}{\sqrt{k}} = c_3 n (\ln n) \sum_{k=1}^n \frac{1}{k} \leq c_4 n (\ln n)^2,$$

and the theorem follows.  $\square$

Next, we prove the upper bound for  $M(K)$ .

**Theorem 10.6.** *Let  $K$  be an isotropic symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$M(K) \leq \frac{c(n \ln n)^{1/3}}{\sqrt{n}} \quad (10.18)$$

where  $c > 0$  is an absolute constant.

*Proof.* From Corollary 10.4 we know that

$$\sqrt{n}M(K) \leq c_1 \sum_{k=1}^n \frac{1}{\sqrt{k}} \min \left\{ \frac{1}{r}, \frac{n}{k} \ln \left( e + \frac{n}{k} \right) \frac{1}{v_k^-(K)} \right\} \quad (10.19)$$

where  $v_k^-(K) := \inf\{\text{vrad}(P_F(K)) : F \in G_{n,k}\}$  and  $r$  is the "inradius" of  $K$ . Since  $K$  is isotropic and symmetric, we know that

$$h_K(\xi) = \|\langle \cdot, \xi \rangle\|_{L_\infty(K)} \geq \|\langle \cdot, \xi \rangle\|_{L_2(K)} = L_K \approx 1,$$

and hence  $K \supseteq c_1 B_2^n$ . Therefore, we may use (10.19) with  $r \approx 1$ . From Theorem 5.4 we also know that  $\text{vol}_k(P_F(K))^{1/k} \geq c_2$  for every  $F \in G_{n,k}$ , and hence  $\text{vrad}(P_F(K)) \geq c_3 \sqrt{k}$ , which gives

$$v_k^-(K) \geq c_3 \sqrt{k}.$$

Set  $k_n = (n \ln n)^{2/3}$ . Inserting the above estimates into (10.19) we get

$$\begin{aligned} \sqrt{n}M(K) &\leq c_4 \sum_{k=1}^n \frac{1}{\sqrt{k}} \min \left\{ 1, \frac{n(\ln n)}{k^{3/2}} \right\} \\ &\leq c_5 \left( \sum_{k=1}^{k_n} \frac{1}{\sqrt{k}} + \sum_{k=k_n}^n \frac{n \ln n}{k^2} \right) \approx (n \ln n)^{1/3}. \end{aligned}$$

This proves the theorem. □

**§ 10.3. The nonsymmetric case.** In the nonsymmetric case the upper bound for  $M^*(K)$  remains the same. In fact, the proof does not present any difficulties. Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$  and consider the difference body  $K - K$  of  $K$ . Since  $0 \in K$  we have that  $K \subseteq K - K$  and hence  $M^*(K) \leq M^*(K - K)$ . In fact,  $M^*(K) = \frac{1}{2}M^*(K - K)$ . We apply Theorem 10.1 for  $K - K$  to get

$$\sqrt{n}M^*(K - K) \leq c_1(\ln n) \sum_{k=1}^n \frac{1}{\sqrt{k}} v_k(K - K). \quad (10.20)$$

Given  $1 \leq k \leq n - 1$ , for every  $F \in G_{n,k}$  we have

$$\text{vol}_k(P_F(K - K))^{1/k} = \text{vol}_k(P_F(K) - P_F(K))^{1/k} \leq 4\text{vol}_k(P_F(K))^{1/k},$$

by the Rogers-Shephard inequality  $\text{vol}_k(C - C) \leq \binom{2k}{k} \text{vol}_k(C)$  where  $C$  is a  $k$ -dimensional convex body. Then, as in the proof of Theorem 10.5, we see that  $v_k(K - K) \leq c_2 n / \sqrt{k}$ , and inserting these bounds into (10.17) we get  $\sqrt{n}M^*(K) \leq c_3 n (\ln n)^2$ .

An upper bound for  $M(K)$ , when  $K$  is isotropic but not necessarily symmetric, was given by Vritsiou in [138]. For every isotropic convex body  $K$  in  $\mathbb{R}^n$  one has that

$$M(K) \leq \frac{cn^{10/22}(\ln n)^{5/22}}{\sqrt{n}} \quad (10.21)$$

where  $c > 0$  is an absolute constant. The proof is based on the existence of  $\beta$ -regular  $M$ -ellipsoids for nonsymmetric convex bodies. It is proved in [138] that for every  $\beta \in (0, \frac{2}{5})$  there exists a constant  $d(\beta) \geq 1$  such that the following holds: For every convex body  $K$  in  $\mathbb{R}^n$  with either  $\text{bar}(K) = 0$  or  $s(K) = 0$  there exists a linear image  $K_\beta$  of  $K$  such that

$$\max\{N(K_\beta, tB_2^n), N(K_\beta^\circ, tB_2^n), N(B_2^n, tK_\beta), N(B_2^n, tK_\beta^\circ)\} \leq \exp(d(\beta) n/t^\beta) \quad (10.22)$$

for every  $t \geq d(\beta)^{1/\beta}$ . Moreover, the constants  $d(\beta)$  satisfy  $d(\beta) \approx \left(c \left(\frac{4\beta}{2-3\beta}\right)\right)^{\frac{2-3\beta}{2}} = O\left((2-5\beta)^{-\beta}\right)$  as  $\beta \rightarrow \frac{2}{5}^-$ , where  $c(\alpha)$  with  $\alpha = \frac{4\beta}{2-3\beta}$  are the constants appearing in Theorem 3.2.

**§ 10.4. The optimal  $M$ -estimate of Bizeul and Klartag.** Very recently, Bizeul and Klartag [17] established an (almost) optimal upper bound for the  $M$ -functional of isotropic convex bodies.

**Theorem 10.7.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then,*

$$M(K) \leq \frac{c \ln n}{\sqrt{n}}$$

where  $c > 0$  is an absolute constant.

The purpose of this subsection is to outline the main ideas underlying this result and to explain its connection with recent advances in comparison inequalities between log-concave measures and Gaussian measures, going back to an inequality due to Eldan and Lehec [47], which quantifies how expectations of norms under isotropic log-concave measures compare to their Gaussian counterparts.

Eldan and Lehec considered the parameter

$$\tau_n^2 = \sup_{\mu} \sup_{\xi \in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_{\mu}(x_i x_j \langle x, \xi \rangle)^2$$

where the first supremum is over all isotropic log-concave probability measures  $\mu$  on  $\mathbb{R}^n$ , and proved the following result.

**Theorem 10.8.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . For any symmetric convex body  $K$  in  $\mathbb{R}^n$  we have that*

$$\int_{\mathbb{R}^n} \|x\|_K d\mu(x) \leq c_1 \sqrt{\ln n} \tau_n \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x)$$

where  $\gamma_n$  is the standard Gaussian measure on  $\mathbb{R}^n$  and  $c_1 > 0$  is an absolute constant.

A fundamental result of Eldan [45] asserts that  $\tau_n$  can be controlled in terms of the thin-shell constant

$$\sigma_n = \sup_{\mu} \sqrt{\text{Var}_{\mu}(|x|)}$$

where the supremum is over all isotropic log-concave probability measures  $\mu$  on  $\mathbb{R}^n$ . Eldan proved that

$$\tau_n^2 \leq c_2 \sum_{k=1}^n \frac{\sigma_k^2}{k}$$

where  $c_2 > 0$  is an absolute constant. Consequently, progress on the thin-shell conjecture has direct implications for comparison inequalities between log-concave measures and Gaussian measures.

Bizeul and Klartag observed in [17] that these comparison results extend naturally from norms to gauge functions, that is, non-negative, convex, 1-homogeneous functions on  $\mathbb{R}^n$ . Moreover, they obtained two-sided estimates in this broader setting.

**Theorem 10.9.** *Let  $X$  be an isotropic, log-concave random vector in  $\mathbb{R}^n$  and let  $G$  be a standard Gaussian random vector in  $\mathbb{R}^n$ . Then for any gauge function  $\|\cdot\|$  on  $\mathbb{R}^n$ ,*

$$\frac{c_1}{\psi_n \sqrt{\ln n}} \mathbb{E}\|G\| \leq \mathbb{E}\|X\| \leq c_2 \psi_n \sqrt{\ln n} \mathbb{E}\|G\|, \quad (10.23)$$

where  $c_1, c_2 > 0$  are absolute constants.

Theorem 10.9 shows that any isotropic log-concave measure is sandwiched, in the gauge order, between two centered Gaussian measures whose covariances differ by logarithmic factors. In particular, such a measure may be compared to Gaussian measures with covariance matrices of order  $(\ln n)^2 I_n$  and  $(\ln n)^{-2} I_n$ . The right-hand inequality in (10.23) is essentially due to Eldan and Lehec, while the left-hand inequality relies on new ideas.

The proof of Theorem 10.9 once again exploits Eldan's stochastic localization scheme. A key step is to truncate the spectrum of the covariance process  $A_t$  by introducing a bounded matrix-valued process  $\Sigma_t = f(A_t)$ , where  $f$  acts on eigenvalues and satisfies

$$\frac{1}{2} I_n \leq \Sigma_t \leq 2 I_n \quad \text{almost surely.}$$

The associated martingale

$$v_t = \int_0^t \Sigma_s dW_s$$

(where  $(W_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}^n$  with  $W_0 = 0$ ) has quadratic variation uniformly comparable to  $t I_n$ , which allows for robust Gaussian comparison estimates.

An observation attributed to Maurey, and made explicit by Eldan and Lehec, asserts that if a continuous martingale  $(M_t)$  satisfies

$$[M]_T \leq r \cdot I_n \quad \text{almost surely,}$$

then its terminal value can be represented as

$$M_T = \sqrt{r} \frac{Z_1 + Z_2}{2},$$

where  $Z_1$  and  $Z_2$  are (possibly dependent) standard Gaussian vectors. As a consequence, for any convex function  $F$ ,

$$\mathbb{E}(F(M_T)) \leq \mathbb{E}(F(\sqrt{r} G)),$$

yielding sharp upper bounds in terms of Gaussian expectations.

Conversely, it is shown in [17] that if a martingale admits a stochastic integral representation

$$M_t = \int_0^t \Sigma_s dW_s \quad \text{with} \quad \Sigma_s \Sigma_s^* \geq r \cdot I_n,$$

then the quadratic variation dominates that of  $\sqrt{r} W_t$ . In this regime, a convexity argument combined with the heat semigroup and Itô calculus shows that for any convex function  $F$  with at most polynomial growth,

$$\mathbb{E}(F(M_T)) \geq \mathbb{E}(F(\sqrt{r} W_T)).$$

Taken together, these two comparison principles allow one to sandwich the localized martingale between Gaussians at appropriate time scales, provided the covariance process is controlled.

A further essential ingredient is a reverse Hölder inequality for norms and, more generally, gauge functions of Gaussian random vectors. A version of Kahane’s inequality asserts that for a standard Gaussian vector  $G \sim \gamma_n$  and any  $p \geq 2$ ,

$$(\mathbb{E}\|G\|^p)^{1/p} \leq c \sqrt{p} \mathbb{E}\|G\|.$$

In the present context, Kahane-type inequalities serve two purposes. First, they allow one to control second moments of gauge functions in terms of their first moments, which is crucial when estimating error terms arising from exceptional events in the stochastic localization process. Second, they ensure that expectations of gauge functions behave regularly under Gaussian scaling, making it possible to compare  $\mathbb{E}\|v_i\|$  and  $\mathbb{E}\|W_i\|$  uniformly in the dimension.

Once Theorem 10.9 is established, the proof of Theorem 10.7 follows readily. Applying the two-sided comparison (10.23) to the gauge function  $x \mapsto p_K(x)$  and to the uniform measure on a convex body  $K$  in isotropic position, we obtain

$$\sqrt{n}M(K) = \sqrt{n} \int_{S^{n-1}} p_K(x) d\sigma(x) \approx \mathbb{E}(p_K(G)) \leq c\psi_n \sqrt{\ln n} \mathbb{E}(p_K(X)) \leq c\psi_n \sqrt{\ln n},$$

since  $p_K(x) \leq 1$  for all  $x \in K$ . Under the KLS conjecture, the logarithmic factors in this estimate would improve to their conjecturally optimal order.

In summary, the optimal  $M$ -estimate of Bizeul and Klartag emerges from a synthesis of Gaussian comparison inequalities, stochastic localization, martingale bounds on quadratic variation, and refined moment inequalities for gauge functions. Beyond yielding sharp bounds for  $M(K)$ , this framework provides a unified perspective on the geometry of isotropic log-concave measures that is likely to find further applications.

**§ 10.5. Diameter of the Minkowski compactum.** The definition of the Banach–Mazur distance can be extended to the setting of not necessarily symmetric convex bodies as follows. If  $K$  and  $D$  are convex bodies in  $\mathbb{R}^n$ , their Banach–Mazur distance is defined by

$$d_{\text{BM}}(K, D) = \min\{t > 0 : T(K) + y \subseteq D + x \subseteq t(T(K) + y)\}$$

where the minimum is taken over all  $x, y \in \mathbb{R}^n$  and all linear isomorphisms  $T \in GL_n$ .

A fundamental open problem for many years was to determine a sharp upper bound for the diameter

$$d_n = \max_{K, D} d_{\text{BM}}(K, D)$$

of the Minkowski compactum with respect to the Banach–Mazur distance. This question was finally resolved in the recent work [17] of Bizeul and Klartag.

The connection between the diameter problem and  $MM^*$ -estimates has been known for a long time. Recall that for any convex body  $K \subset \mathbb{R}^n$  we define

$$E(K) := \min M(TK)M^*(TK),$$

where the minimum is taken over all invertible affine transformations  $T$  of  $\mathbb{R}^n$  such that  $0 \in \text{int}(TK)$ . The following theorem reduces the problem of bounding the diameter of the Minkowski compactum to that of minimizing the product  $MM^*$  in the nonsymmetric setting.

**Theorem 10.10.** *Let  $K$  and  $D$  be two convex bodies in  $\mathbb{R}^n$ . Then,*

$$d_{\text{BM}}(K, D) \leq cn \sqrt{\ln n} (E(K) + E(D)),$$

where  $c > 0$  is an absolute constant.

A proof of Theorem 10.10 may be found in [3, Theorem 6.3.1]. It relies on the method of random orthogonal factorizations, introduced by Benyamini and Gordon [15]; see also the related works of Tomczak–Jaegermann [133] and Davis, V. Milman, and Tomczak–Jaegermann [42].

In view of Theorem 10.10, obtaining sharp upper bounds for  $d_n$  reduces to finding a position of a convex body  $K$  for which the quantity  $M(K)M^*(K)$  is as small as possible. Recall that in the centrally symmetric case one always has

$$E(K) := \min_{T \in GL_n} M(T(K))M^*(T(K)) \leq C \ln n, \quad (10.24)$$

where  $C > 0$  is an absolute constant. A position  $T(K)$  satisfying (10.24) is called an  $\ell$ -position of  $K$ . The almost optimal  $M^*$ - and  $M$ -estimates obtained by E. Milman and by Bizeul–Klartag show that the isotropic position already provides a canonical  $\ell$ -position for arbitrary convex bodies. Indeed, if  $K$  is an isotropic convex body in  $\mathbb{R}^n$ , then

$$M(K)M^*(K) \leq c(\ln n)^3$$

for an absolute constant. Combining this estimate with Theorem 10.10 yields the following essentially optimal bound for the diameter of the Minkowski compactum.

**Theorem 10.11.** *Let  $K$  and  $D$  be two convex bodies in  $\mathbb{R}^n$ . Then,*

$$d_{\text{BM}}(K, D) \leq cn(\ln n)^{7/2},$$

where  $c > 0$  is an absolute constant. In particular,  $d_n = O(n(\ln n)^{7/2})$ .

Theorem 10.11 reveals the striking fact that the diameters of the symmetric and nonsymmetric Banach–Mazur compacta are of essentially the same order.

**§ 10.6. Isotropic position is a regular  $M$ -position.** Another important consequence of the  $MM^*$ -estimate for isotropic convex bodies is an almost optimal non-symmetric analogue of Pisier’s Theorem 3.2 on the existence of  $\alpha$ -regular  $M$ -positions for centrally symmetric convex bodies.

In addition to Theorems 10.5 and 10.7, the proof relies crucially on the Sudakov and dual Sudakov inequalities (see [2, Chapter 4]). These inequalities assert that if  $K$  is a symmetric convex body in  $\mathbb{R}^n$ , then for every  $t > 0$ ,

$$N(K, tB_2^n) \leq \exp\left(cnM^*(K)^2/t^2\right) \quad \text{and} \quad N(B_2^n, tK) \leq \exp\left(cnM(K)^2/t^2\right),$$

where  $c > 0$  is an absolute constant.

In what follows, for every convex body  $K$  in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$  we define the outer and inner regularizations of  $K$  by

$$K_{\text{out}} = \text{conv}(K, -K) \quad \text{and} \quad K_{\text{in}} = K \cap (-K).$$

It is clear that  $K_{\text{in}} \subseteq K \subseteq K_{\text{out}}$ , and both  $K_{\text{out}}$  and  $K_{\text{in}}$  are origin-symmetric. Using the identities  $(\text{conv}(A \cup B))^\circ = A^\circ \cap B^\circ$  and  $(A \cap B)^\circ = \text{conv}(A^\circ \cup B^\circ)$ , valid for convex bodies  $A, B$  with  $0 \in \text{int}(A \cap B)$ , we obtain the duality relations

$$(K_{\text{out}})^\circ = (K^\circ)_{\text{in}} \quad \text{and} \quad (K_{\text{in}})^\circ = (K^\circ)_{\text{out}}, \quad (10.25)$$

The third named author observed in [136] that the  $M^*$ -estimate of Theorem 10.5, together with the  $M$ -estimate of Theorem 10.7, implies that every isotropic convex body  $K \subset \mathbb{R}^n$  is (almost) in a 2-regular  $M$ -position.

In the next theorem we set  $r_n = \omega_n^{-1/n}$ ; recall that  $r_n \simeq \sqrt{n}$ .

**Theorem 10.12.** *Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$ . Then, for every  $t > 0$ ,*

$$\max\{N(K, tr_n B_2^n), N(B_2^n, tr_n K^\circ)\} \leq \exp\left(\frac{\gamma_n^2 n}{t^2}\right), \quad (10.26)$$

$$\max\{N(r_n B_2^n, tK), N(r_n K^\circ, tB_2^n)\} \leq \exp\left(\frac{\delta_n^2 n}{t^2}\right), \quad (10.27)$$

where  $\gamma_n \leq c_1(\ln n)^2$  and  $\delta_n \leq c_2 \ln n$ .

*Proof.* Let  $K_{\text{in}}$  and  $K_{\text{out}}$  denote the inner and outer regularizations of  $K$ . Observe that

$$M(K_{\text{in}}) = M^*((K^\circ)_{\text{out}}) \leq M^*(K^\circ - K^\circ) = 2M^*(K^\circ) = 2M(K) \quad (10.28)$$

and

$$M^*(K_{\text{out}}) \leq M^*(K - K) = 2M^*(K). \quad (10.29)$$

Since  $K_{\text{in}} \subseteq K \subseteq K_{\text{out}}$ , it is clear that  $M(K_{\text{out}}) \leq M(K)$  and  $M^*(K_{\text{in}}) \leq M^*(K)$ .

Combining the Sudakov inequality with (10.29), we get

$$\begin{aligned} N(K, tr_n B_2^n) &\leq N(K_{\text{out}}, tr_n B_2^n) \leq \exp\left(cnM^*(K_{\text{out}})^2/(r_n t)^2\right) \\ &\leq \exp\left(4cnM^*(K)^2/(r_n t)^2\right) \leq \exp\left(\gamma_n^2 n/t^2\right). \end{aligned} \quad (10.30)$$

Similarly, using (10.28) we obtain

$$\begin{aligned} N(r_n K^\circ, tB_2^n) &\leq N(r_n (K_{\text{in}})^\circ, tB_2^n) \leq \exp\left(cnr_n^2 M^*((K_{\text{in}})^\circ)^2/t^2\right) \\ &= \exp\left(cnr_n^2 M(K_{\text{in}})^2/t^2\right) \leq \exp\left(4cnr_n^2 M(K)^2/t^2\right) \leq \exp\left(\delta_n^2 n/t^2\right). \end{aligned} \quad (10.31)$$

Applying the dual Sudakov inequality and (10.28) yields

$$\begin{aligned} N(r_n B_2^n, tK) &\leq N(r_n B_2^n, tK_{\text{in}}) \leq \exp\left(cnr_n^2 M(K_{\text{in}})^2/t^2\right) \\ &\leq \exp\left(4cnr_n^2 M(K)^2/t^2\right) \leq \exp\left(\delta_n^2 n/t^2\right), \end{aligned} \quad (10.32)$$

and, using (10.29), we similarly have

$$\begin{aligned} N(B_2^n, tr_n K^\circ) &\leq N(B_2^n, tr_n (K_{\text{out}})^\circ) \leq \exp\left(cnM((K_{\text{out}})^\circ)^2/(r_n t)^2\right) \\ &\leq \exp\left(cnM^*(K_{\text{out}})^2/(r_n t)^2\right) = \exp\left(4cnM^*(K)^2/(r_n t)^2\right) \leq \exp\left(\gamma_n^2 n/t^2\right). \end{aligned} \quad (10.33)$$

Combining (10.30)–(10.33) completes the proof.  $\square$

Some first geometric applications of Theorem 10.12 were obtained in [136], including new inequalities comparing the volumes of sections and projections of a convex body whose barycenter or Santaló point lies at the origin with those of its inner and outer regularizations. These results highlight the role of isotropic normalization as a unifying framework bridging the geometry of symmetric and general convex bodies.

**§ 10.7. A multi-integral norm.** Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . For any  $s$ -tuple  $C = (C_1, \dots, C_s)$  of symmetric convex bodies  $C_j$  of volume 1 in  $\mathbb{R}^n$ , consider the norm on  $\mathbb{R}^s$ , defined by

$$\|\mathbf{t}\|_{C,K} = \int_{C_1} \cdots \int_{C_s} \left\| \sum_{j=1}^s t_j x_j \right\|_K dx_s \cdots dx_1$$

where  $\mathbf{t} = (t_1, \dots, t_s)$ . If  $C = (C, \dots, C)$  then we write  $\|\mathbf{t}\|_{C^s, K}$  instead of  $\|\mathbf{t}\|_{C,K}$ . A question posed by V. Milman is to determine if, in the case  $C = K$ , one has that  $\|\cdot\|_{K^s, K}$  is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension, and in particular, if under some cotype condition on the norm induced by  $K$  to  $\mathbb{R}^n$  one has equivalence between  $\|\cdot\|_{K^s, K}$  and the Euclidean norm.

This question was studied by Bourgain, Meyer, V. Milman and Pajor. For simplicity let us assume that  $\text{vol}_n(K) = 1$  (this is only a matter of normalization). It was proved in [31] that

$$\|\mathbf{t}\|_{C,K} \geq c \sqrt{s} \left( \prod_{j=1}^s |t_j| \right)^{1/s}$$

where  $c > 0$  is an absolute constant. Later, Gluskin and V. Milman obtained a better lower bound in [61], in fact working in a more general context: Let  $\mathcal{A} = (A_1, \dots, A_s)$  be an  $s$ -tuple of measurable sets of volume 1 in  $\mathbb{R}^n$  and let  $K$  be a star body of volume 1 in  $\mathbb{R}^n$  with  $0 \in \text{int}(K)$ . Then,

$$\|\mathbf{t}\|_{\mathcal{A},K} \geq c |\mathbf{t}| \tag{10.34}$$

for all  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ . Their argument was based on the Brascamp-Lieb-Luttinger inequality (see also [3, Chapter 4]).

In this subsection we shall see that the original question of V. Milman has an affirmative answer. Since  $\|\mathbf{t}\|_{K^s, K} = \|\mathbf{t}\|_{(TK)^s, TK}$  for any  $T \in SL_n$ , we may restrict our attention to the case where  $K$  is isotropic.

For any centered log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  and any symmetric convex body  $K$  in  $\mathbb{R}^n$ , consider the parameter

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K d\mu(x). \tag{10.35}$$

Let  $C$  be an isotropic symmetric convex body in  $\mathbb{R}^n$  and let  $X_1, \dots, X_s$  be independent random vectors, uniformly distributed in  $C$ . For any  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$  we write  $\nu_{\mathbf{t}}$  for the distribution of the random vector  $t_1 X_1 + \cdots + t_s X_s$ . Since  $\|\mathbf{t}\|_{C^s, K}$  is a norm, we may always assume that  $|\mathbf{t}| = 1$ . Note that  $\nu_{\mathbf{t}}$  is an even log-concave probability measure on  $\mathbb{R}^n$  (this is a consequence of the Prékopa-Leindler inequality; see [2]). We write  $g_{\mathbf{t}}$  for the density of  $\nu_{\mathbf{t}}$ . Our starting point is the next observation from [38].

**Lemma 10.13.** *For any  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ , we write  $\nu_{\mathbf{t}}$  for the distribution of the random vector  $t_1 X_1 + \cdots + t_s X_s$ . Then,*

$$\|\mathbf{t}\|_{C^s, K} = \int_{\mathbb{R}^n} \|x\|_K d\nu_{\mathbf{t}}(x).$$

It is easily verified that the covariance matrix  $\text{Cov}(v_t)$  of  $v_t$  is a multiple of the identity: more precisely,

$$\text{Cov}(v_t) = L_C^2 I_n.$$

It follows that the function  $f_t(x) = L_C^n g_t(L_C x)$  is the density of an isotropic log-concave probability measure  $\mu_t$  on  $\mathbb{R}^n$ . Indeed, we have

$$\int_{\mathbb{R}^n} f_t(x) x_i x_j dx = L_C^n \int_{\mathbb{R}^n} g_t(L_C x) x_i x_j dx = L_C^{-2} \int_{\mathbb{R}^n} g_t(y) y_i y_j dy = \delta_{i,j}$$

for all  $1 \leq i, j \leq n$ .

One may easily check that if  $\mu$  is an isotropic log-concave probability measure on  $\mathbb{R}^n$  and  $K$  is a symmetric convex body in  $\mathbb{R}^n$  then

$$\begin{aligned} \int_{O(n)} I_1(\mu, U(K)) dv(U) &= \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)} dv(U) d\mu(x) \\ &= M(K) \int_{\mathbb{R}^n} |x| d\mu(x) \approx \sqrt{n} M(K) \end{aligned}$$

where  $\nu, \sigma$  denote the Haar probability measures on  $O(n)$  and  $S^{n-1}$  respectively. It follows that

$$\int_{O(n)} \|\mathbf{t}\|_{C^s, U(K)} dv(U) \approx (L_C \sqrt{n} M(K)) |\mathbf{t}|. \quad (10.36)$$

Therefore, one might hope to obtain a quantity of the order of  $L_C \sqrt{n} M(K) |\mathbf{t}|$  as an upper estimate for  $\|\mathbf{t}\|_{C^s, K}$ . The next theorem of Skarmogiannis [128] provides a logarithmic in  $n$ , and independent from  $s$ , upper bound for  $\|\mathbf{t}\|_{C^s, K}$ .

**Theorem 10.14.** *Let  $C$  be an isotropic symmetric convex body in  $\mathbb{R}^n$  and  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$\|\mathbf{t}\|_{C^s, K} \leq c_1 \sqrt{n} (\ln n) M(K) |\mathbf{t}|$$

for every  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ , where  $c_1 > 0$  is an absolute constant.

In the case  $C = K$ , Theorem 10.14 and (10.34) show that, for any isotropic symmetric convex body  $K$  of volume 1 in  $\mathbb{R}^n$ ,

$$c_1 |\mathbf{t}| \leq \|\mathbf{t}\|_{K^s, K} \leq c_2 \sqrt{n} (\ln n) M(K) L_K |\mathbf{t}|$$

for every  $s \geq 1$  and every  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ , where  $c_i > 0$  are absolute constants and. Thus, we have a reduction of V. Milman's question to the problem of estimating the parameter  $M(K)$  for an isotropic symmetric convex body  $K$  in  $\mathbb{R}^n$ . Now, combining the boundedness of  $L_n$  with Theorem 10.7, and taking (10.34) into account, we obtain an affirmative answer.

**Theorem 10.15.** *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

$$c_1 |\mathbf{t}| \leq \|\mathbf{t}\|_{K^s, K} \leq c_2 (\ln n)^2 |\mathbf{t}|$$

for every  $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ , where  $c_1, c_2 > 0$  are absolute constants.

The estimate of Theorem 10.14 is based on the next result.

**Theorem 10.16.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . For any symmetric convex body  $K$  in  $\mathbb{R}^n$  we have that*

$$I_1(\mu, K) \leq c_2 \sqrt{n}(\ln n) M(K)$$

where  $c_2 > 0$  is an absolute constant.

*Proof.* Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . We shall use Theorem 10.8. Recall that

$$\tau_n^2 \leq c_1 \sum_{k=1}^n \frac{\sigma_k^2}{k}$$

where  $c_1 > 0$  is an absolute constant. Since  $\sigma_n \leq C$  where  $C > 0$  is an absolute constant, one gets

$$\tau_n^2 \leq c_2 \sum_{k=1}^n \frac{\sigma_k^2}{k} \leq c_3 \ln n.$$

Therefore, Theorem 10.8 immediately implies that

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K d\mu(x) \leq c_4 \ln n \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x)$$

where  $c_4 > 0$  is an absolute constant. Finally, integration in spherical coordinates shows that

$$\int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x) \approx \sqrt{n} \int_{S^{n-1}} \|\xi\|_K d\sigma(\xi) \approx \sqrt{n} M(K)$$

and hence the proof of Theorem 10.16 is complete.  $\square$

We can now complete the proof of Theorem 10.14, and hence of Theorem 10.15 too. We combine the approach of [38] with Theorem 10.16.

*Proof of Theorem 10.14.* We compute

$$\|\mathbf{t}\|_{C^s, K} = \int_{\mathbb{R}^n} \|x\|_K dv_{\mathbf{t}}(x) = L_C^{-n} \int_{\mathbb{R}^n} \|x\|_K f_{\mathbf{t}}(x/L_C) dx = L_C \int_{\mathbb{R}^n} \|y\|_K d\mu_{\mathbf{t}}(y)$$

and hence we get

$$\|\mathbf{t}\|_{C^s, K} = L_C I_1(\mu_{\mathbf{t}}, K) \tag{10.37}$$

for all  $\mathbf{t} \in \mathbb{R}^s$  with  $|\mathbf{t}| = 1$ . Now, we use Theorem 10.16 to estimate  $I_1(\mu_{\mathbf{t}}, K)$ . As a result, we obtain the upper bound

$$\|\mathbf{t}\|_{C^s, K} \leq c_1 L_C \sqrt{n}(\ln n) M(K) \approx \sqrt{n}(\ln n) M(K),$$

which is the assertion of Theorem 10.14.  $\square$

## 11. APPENDIX: COVARIANCE ESTIMATES

In this appendix we study the covariance process  $(A_t)_{t \geq 0}$  associated with the stochastic localization of an isotropic log-concave measure and sketch the proofs of the two main technical covariance estimates that were used for the bound  $\psi_n = O(\sqrt{\ln n})$  and the bound  $L_n = O(1)$ . We first derive stochastic differential equations for the barycenter and covariance (Lemma 11.2), then use a matrix-valued Itô argument

combined with the improved log-concave Lichnerowicz inequality to control the operator norm of  $A_t$  (Theorem 11.7). Finally, we adapt Guan's argument to obtain a dimension-independent bound on  $\mathbb{E}(\text{tr}(A_t^2))$  (Theorem 11.14).

Our first main goal is to prove Theorem 8.13 which we state again below for the convenience of the reader.

**Theorem 11.1.** *For every  $0 < t \leq c/(\psi_n^2 \ln n)$  we have*

$$\mathbb{E} \|A_t\|_{\text{op}} \leq C, \quad (11.1)$$

where  $C > 0$  is an absolute constant.

In fact, using the improved log-concave Lichnerowicz inequality (Theorem 8.10) we can show that the statement of Theorem 11.1 is true for all  $0 < t \leq c/(\ln n)^2$ .

As in Section 9, if  $\mu$  is a log-concave probability measure with density  $\varrho$  with respect to the Lebesgue measure, we consider the stochastic localization process starting at  $\varrho$ . The tilt process  $(\xi_t)_{t \geq 0}$  is defined as the solution of the stochastic differential equation  $d\xi_t = a_{t,\xi_t} dt + dW_t$  where  $(W_t)_{t \geq 0}$  is a standard Brownian motion. For  $t \geq 0$  we denote by  $\mu_t := \mu_{t,\xi_t}$  the measure with density  $\varrho_{t,\xi_t}(x) = \frac{1}{Z_{t,\xi_t}} e^{\langle x, \xi_t \rangle - \frac{t}{2}|x|^2} \varrho(x)$ , and we write  $a_t := a_{t,\xi_t}$  and  $A_t := A_{t,\xi_t} = \text{Cov}(\mu_{t,\xi_t})$  for the barycenter and covariance matrix of  $\mu_{t,\xi_t}$  respectively.

**§ 11.1. Preliminary observations.** In Section 7 we saw that, for any test function  $f$ , the martingale  $M_t = \int_{\mathbb{R}^n} f d\mu_t$  satisfies

$$dM_t = \left\langle \int_{\mathbb{R}^n} f(x)(x - a_t) d\mu_t, dW_t \right\rangle,$$

where  $(W_t)_{t \geq 0}$  is some standard Brownian motion. This extends to vector valued functions as follows. If  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is a vector valued function that grows mildly at infinity, then the process  $(M_t)_{t \geq 0}$  given by

$$M_t = \int_{\mathbb{R}^n} F d\mu_t$$

is a martingale, and

$$dM_t = \left\langle \int_{\mathbb{R}^n} F(x) \otimes (x - a_t) d\mu_t, dW_t \right\rangle.$$

To see this, writing  $x_i$  for the  $i$ -th coordinate of a vector  $x \in \mathbb{R}^n$  we get

$$dM_t = \sum_{i=1}^n \left( \int_{\mathbb{R}^n} F(x)(x - a_t)_i d\mu_t \right) dW_{t,i}. \quad (11.2)$$

Applying (11.2) to the tensors  $F(x) = x$  and  $F(x) = x \otimes x$  and then rearranging the terms appropriately, we obtain the next lemma.

**Lemma 11.2.** *If  $a_t = \int x d\mu_t(x)$  and  $A_t = \int (x - a_t)^{\otimes 2} d\mu_t(x)$  are the barycenter and covariance matrix of  $\mu_t$ , introduced in Section 7, then  $da_t = A_t dW_t$  and*

$$dA_t = \sum_{i=1}^n \left( \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} (x - a_t)_i d\mu_t \right) dW_{t,i} - A_t^2 dt.$$

As Lemma 11.2 shows, the derivative of the barycenter is expressed in terms of the covariance, and the derivative of the covariance depends on 3-tensors.

**§ 11.2. Auxiliary results.** We start with an inequality about the Hessian of the function  $\phi(A) = \text{tr}(e^A)$  where  $A$  is a symmetric matrix.

**Lemma 11.3.** *Let  $\phi$  be the function defined on the space  $S_n(\mathbb{R})$  of symmetric  $n \times n$  matrices by  $\phi(A) = \text{tr}(e^A)$ . For every pair of symmetric matrices  $A, H$  we have that*

$$\nabla^2 \phi(A)(H, H) \leq \langle \nabla \phi(A), H^2 \rangle = \text{tr}(e^A H^2),$$

where  $\nabla^2 \phi(A)$  is the Hessian matrix of  $\phi$  at  $A$ , viewed as a bilinear form on  $S_n(\mathbb{R})$ .

*Proof.* Assume first that the matrix  $A$  is positive. We use the following fact (see [81] for a simple proof): If  $K$  and  $H$  are symmetric matrices, and  $K$  is positive semi-definite, then for any  $s, m \in \mathbb{N}$  we have

$$\text{tr}(K^s H K^m H) \leq \text{tr}(K^{s+m} H^2).$$

Applying this inequality we get

$$\begin{aligned} \nabla^2 \phi(A)(H, H) &= \sum_{k \geq 1} \frac{1}{k!} \sum_{s=0}^{k-1} \text{tr}(A^s H A^{k-s-1} H) \\ &\leq \sum_{k \geq 1} \frac{1}{k!} k \text{tr}(A^{k-1} H^2) = \text{tr}(e^A H^2). \end{aligned}$$

For the case where  $A$  has some negative eigenvalues, we observe that  $\phi$  satisfies the identity

$$\phi(A + tI_n) = e^t \phi(A)$$

and differentiating this equality with respect to  $A$  we see that  $\nabla \phi$  and  $\nabla^2 \phi$  also satisfy the same equation, which means that adding a multiple of the identity to  $A$  leads to an equivalent inequality. This reduces the proof of the lemma to the case of positive  $A$ .  $\square$

Recall the equation for  $A_t$

$$dA_t = \sum_{i=1}^n H_{t,i} dW_{t,i} - A_t^2 dt,$$

where

$$H_{t,i} = \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} (x - a_t)_i d\mu_t$$

and  $a_t$  is the barycenter of  $\mu_t$ . Therefore, the matrix  $H_{t,i}$  is of the form  $\mathbb{E}(X_i X^{\otimes 2})$  for some random vector with mean 0. In what follows, for any  $u \in S^{n-1}$  we define  $H_u = \mathbb{E}(\langle X, u \rangle X^{\otimes 2})$ .

**Lemma 11.4.** *Let  $X$  be a centered log-concave random vector. Then,*

$$\sup_{u \in S^{n-1}} \|H_u\|_{\text{op}} \leq c_1 \|\text{Cov}(X)\|_{\text{op}}^{3/2}$$

where  $c_1 > 0$  is an absolute constant.

*Proof.* Let  $u, v \in S^{n-1}$ . Using the Cauchy-Schwarz inequality we get

$$\langle H_u(v), v \rangle = \mathbb{E}(\langle X, u \rangle \langle X, v \rangle^2) \leq \left( \mathbb{E} \langle X, u \rangle^2 \right)^{1/2} \left( \mathbb{E} \langle X, v \rangle^4 \right)^{1/2}.$$

Since the random variable  $\langle X, v \rangle$  is centered and log-concave, its fourth moment and the square of its second moment are of the same order. It follows that

$$\langle H_u(v), v \rangle \leq c_1 \left( \mathbb{E} \langle X, u \rangle^2 \right)^{1/2} \left( \mathbb{E} \langle X, v \rangle^2 \right) \leq c_1 \|\text{Cov}(X)\|_{\text{op}}^{3/2}$$

for some absolute constant  $c_1 > 0$ . Taking the supremum over all  $u$  and  $v$  in  $S^{n-1}$  we conclude the proof.  $\square$

**Lemma 11.5.** *Let  $X$  be a centered random vector with finite Poincaré constant  $\vartheta(X)$ . Then,*

$$\left\| \sum_{i=1}^n \left( \mathbb{E} (X_i X^{\otimes 2}) \right)^2 \right\|_{\text{op}} \leq 4\vartheta^2(X) \|\text{Cov}(X)\|_{\text{op}}^2.$$

*Proof.* For every coordinate vector  $e_i$  we set  $H_i := H_{e_i}$ . In this notation the lemma asserts that

$$\sum_{i=1}^n \langle H_i^2 u, u \rangle \leq 4\vartheta^2(X) \|\text{Cov}(X)\|_{\text{op}}^2$$

for every  $u \in S^{n-1}$ . We check that  $\sum_{i=1}^n \langle H_i^2 u, u \rangle = \text{tr}(H_u^2)$  and, using the assumption that  $X$  is centered, from the Cauchy-Schwarz inequality and the Poincaré inequality we get

$$\begin{aligned} \text{tr}(H_u^2) &= \mathbb{E}(\langle X, u \rangle \langle H_u X, X \rangle) \\ &\leq \left( \mathbb{E} \langle X, u \rangle^2 \right)^{1/2} \left( \text{Var}(\langle H_u X, X \rangle) \right)^{1/2} \\ &\leq \left( \mathbb{E} \langle X, u \rangle^2 \right)^{1/2} \left( 4\vartheta^2(X) \mathbb{E} |H_u X|^2 \right)^{1/2} \\ &= \langle \text{Cov}(X) u, u \rangle^{1/2} \left( 4\vartheta^2(X) \text{tr}(H_u^2 \text{Cov}(X)) \right)^{1/2} \\ &\leq \|\text{Cov}(X)\|_{\text{op}} \left( 4\vartheta^2(X) \text{tr}(H_u^2) \right)^{1/2}. \end{aligned}$$

This shows that  $\text{tr}(H_u^2) \leq 4\vartheta^2(X) \|\text{Cov}(X)\|_{\text{op}}^2$ , which is exactly the assertion of the lemma.  $\square$

The last result that we need is a deviation inequality for martingales (see Freedman [51]).

**Lemma 11.6.** *Let  $(M_t)_{t \geq 0}$  be a continuous local martingale with  $M_0 = 0$ . For every  $u > 0$  and  $\sigma \neq 0$  we have*

$$\mathbb{P}(\exists t > 0 : M_t \geq u \text{ and } [M]_t \leq \sigma^2) \leq e^{-u^2/(2\sigma^2)}.$$

The argument for Lemma 11.6 follows the usual proof of Freedman's inequality for martingales with bounded quadratic variation. We first check that if  $(Z_t)$  is a square integrable martingale such that  $[Z]_t \leq \sigma^2$  for all  $t > 0$  and almost surely, then  $Z_\infty = \lim_{t \rightarrow +\infty} Z_t$  exists and satisfies

$$\mathbb{P}(Z_\infty \geq u) \leq e^{-u^2/(2\sigma^2)}$$

for all  $u > 0$ . Then, we consider the stopping time

$$\tau = \inf\{t > 0 : [M]_t > \sigma^2\}$$

and apply the previous claim to the martingale  $(M_t)$  stopped at time  $\tau$ .

**§ 11.3. Proof of Theorem 11.1.** The basic step is to prove the next theorem.

**Theorem 11.7.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$  and let  $(A_t)_{t \geq 0}$  be the covariance process of the stochastic localization starting from  $\mu$ . Then,*

$$\mathbb{P}\left(\exists s \leq t : \|A_s\|_{\text{op}} \geq 2\right) \leq \exp\left(-\frac{1}{c_0 t}\right)$$

for all  $0 \leq t \leq \frac{1}{c_0(\ln n)^2}$ , where  $c_0 > 0$  is an absolute constant.

*Proof.* We consider the function  $h_\beta(M) := \frac{1}{\beta} \ln(\text{tr}(e^{\beta M}))$  on  $S_n(\mathbb{R})$ . Then,  $h_\beta$  is a smooth function and

$$\lambda_{\max}(M) \leq \frac{1}{\beta} \ln(\text{tr}(e^{\beta M})) \leq \lambda_{\max}(M) + \frac{\ln n}{\beta}.$$

We choose  $\beta$  of the order of  $\ln n$  so that  $h_\beta(A_t)$  tracks the largest eigenvalue  $\lambda_{\max}(A_t)$  of  $A_t$  uniformly in  $n$ .

From Itô's formula we see that

$$dh_\beta(A) = \left\langle \nabla h_\beta(A), \sum_{i=1}^n H_i dB_i \right\rangle - \left\langle \nabla h_\beta(A), A^2 dt \right\rangle + \frac{1}{2} \sum_{i=1}^n \nabla^2 h_\beta(A)(H_i, H_i) dt.$$

The matrix

$$M = \nabla h_\beta(A) = \frac{e^{\beta A}}{\text{tr}(e^{\beta A})}$$

is positive semi-definite and has trace 1. Using Lemma 11.3 we see that the second derivative of  $h_\beta$  satisfies

$$\nabla^2 h_\beta(A)(H_i, H_i) \leq \beta \text{tr}(MH_i^2).$$

This implies that

$$dh_\beta(A) \leq \sum_{i=1}^n \text{tr}(MH_i) dB_i + \frac{\beta}{2} \text{tr}\left(M \sum_{i=1}^n H_i^2\right) dt.$$

We concentrate on the absolutely continuous part. Since  $M$  is positive and has trace 1, from Lemma 11.5 we see that

$$\text{tr}\left(M \sum_{i=1}^n H_i^2\right) \leq \left\| \sum_{i=1}^n H_i^2 \right\|_{\text{op}} \leq 4\vartheta_{\mu_t}^2 \|A_t\|_{\text{op}}^2.$$

Since  $\mu_t$  is  $t$ -uniformly log-concave almost surely, from the improved log-concave Lichnerowicz inequality (Theorem 8.10) we get

$$\vartheta_{\mu_t}^2 \leq \left(\frac{\|A_t\|_{\text{op}}}{t}\right)^{1/2},$$

and hence

$$dh_\beta(A) \leq \sum_{i=1}^n \text{tr}(MH_i) dB_i + \frac{2\beta}{\sqrt{t}} \|A_t\|_{\text{op}}^{5/2} dt.$$

Next, we give an upper bound for the quadratic variation of the martingale part. For any  $u \in S^{n-1}$  we set  $H_u = \sum H_i u_i$ . Then, Lemma 11.4 shows that

$$\sum_{i=1}^n \operatorname{tr}(MH_i) u_i = \operatorname{tr}(MH_u) \leq \|H_u\|_{\text{op}} \leq c_1 \|A_t\|_{\text{op}}^{3/2}.$$

It follows that

$$\sum_{i=1}^n \operatorname{tr}(MH_i)^2 \leq c_1^2 \|A_t\|_{\text{op}}^3.$$

The above calculations show that

$$\begin{aligned} \|A_t\|_{\text{op}} &\leq h_\beta(A_t) \leq h_\beta(A_0) + Z_t + 2\beta \int_0^t r^{-1/2} \|A_r\|_{\text{op}}^{5/2} dr \\ &= 1 + \frac{\ln n}{\beta} + Z_t + 2\beta \int_0^t r^{-1/2} \|A_r\|_{\text{op}}^{5/2} dr \end{aligned} \quad (11.3)$$

where  $(Z_t)$  is a continuous martingale starting from 0 with quadratic variation that satisfies

$$[Z]_t \leq c_2 \int_0^t \|A_r\|_{\text{op}}^3 dr. \quad (11.4)$$

We choose  $\beta = 2 \ln n$ , and assume that there exists  $s \leq t$  such that  $\|A_s\|_{\text{op}} \geq 2$ . If  $s$  is the smallest such time, then before time  $s$  the operator norm of  $A$  is less than 2, and then (11.3) shows that

$$2 = \|A_s\|_{\text{op}} \leq \frac{3}{2} + Z_s + c_3 s^{1/2} \ln n \leq \frac{3}{2} + Z_s + c_3 t^{1/2} \ln n,$$

where  $c_3 > 0$  is an absolute constant. If  $t$  is a sufficiently small multiple of  $(\ln n)^{-2}$  then this last inequality implies that  $Z_s \geq \frac{1}{4}$ . Moreover, (11.4) shows that  $[Z]_s \leq c_4 s \leq c_4 t$ . Therefore,

$$\mathbb{P}(\exists s \leq t : \|A_s\|_{\text{op}} \geq 2) \leq \mathbb{P}\left(\exists s > 0 : Z_s \geq \frac{1}{4} \text{ and } [Z]_s \leq c_4 t\right).$$

Applying Lemma 11.6 we conclude the proof.  $\square$

Theorem 11.7 implies the upper bound for the expectation of  $A_t$ .

*Proof of Theorem 11.1.* Since  $\mu_t$  is  $t$ -uniformly log-concave, we have  $A_t \leq \frac{1}{t} I_n$ , and in particular  $\|A_t\|_{\text{op}} \leq 1/t$ , almost surely. Therefore,

$$\mathbb{E}\|A_t\|_{\text{op}} \leq 2 + \frac{1}{t} \mathbb{P}(\|A_t\|_{\text{op}} > 2).$$

Since  $x \exp\left(-\frac{1}{c_0} x\right)$  is a bounded function of  $x$ , Theorem 11.7 shows that

$$\mathbb{E}\|A_t\|_{\text{op}} \leq C$$

on the time interval  $\left[0, \frac{1}{c_0(\ln n)^2}\right]$ .  $\square$

*Note.* Instead of the improved log-concave Lichnerowicz inequality, we could have bounded  $\mathfrak{V}_{\mu_t}$  by the KLS constant  $\psi_n$ . It is not hard to see that for any log-concave random vector  $X$

$$\mathfrak{V}^2(X) \leq \psi_n^2 \|\text{Cov}(X)\|_{\text{op}},$$

so we can use the inequality

$$\mathfrak{V}_{\mu_t}^2 \leq \psi_n^2 \|A_t\|_{\text{op}}.$$

Using this estimate instead of Theorem 8.10 we would get Theorem 11.1 in the (weaker) form that we have stated it.

The second main estimate, which played a key role in the proof of the isotropic constant conjecture, is Guan's theorem [65], which we state again below for the convenience of the reader.

**Theorem 11.8.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . Then,*

$$\mathbb{E}(\text{tr}(A_{\delta_1})) \geq \delta_1 n,$$

where  $\delta_1 > 0$  is an absolute constant.

**§ 11.4. Auxiliary results.** For  $i, j, k = 1, \dots, n$  we denote

$$R_{ij}(t) := R_{ij}(t, \xi_t) = \int_{\mathbb{R}^n} (x - a_t)(x - a_t)_i (x - a_t)_j \varrho_{t, \xi_t}(x) dx$$

and

$$R_{ijk}(t) := R_{ijk}(t, \xi_t) = \int_{\mathbb{R}^n} (x - a_t)_i (x - a_t)_j (x - a_t)_k \varrho_{t, \xi_t}(x) dx$$

where  $x_i$  are the coordinates of  $x$  with respect to an orthonormal basis  $\{u_i(t, \xi_t)\}_{i=1}^n$  of eigenvectors corresponding to the eigenvalues  $0 < \lambda_1(t, \xi_t) \leq \dots \leq \lambda_n(t, \xi_t) \leq 1/t$  of  $A_{t, \xi_t}$ . We set  $\lambda_i(t) := \lambda_i(t, \xi_t)$  for  $i = 1, \dots, n$ .

For any  $r > 0$  let

$$d(r; t) = \max\{i \geq 1 : \lambda_i(t) \leq r\}.$$

**Lemma 11.9.** *Let  $r > 0$  and  $t > 0$ . For any  $1 \leq k \leq n$  we have*

$$\sum_{i, j=1}^{d(r; t)} R_{ijk}^2(t) \leq 4t^{-1/2} r^{3/2} \lambda_k(t).$$

*Proof.* Fix  $t$  and let  $E$  be the subspace of  $\mathbb{R}^n$  spanned by the first  $d(r; t)$  eigenvectors of  $A_{t, \xi_t}$ . The measure  $\nu_E = (P_E)_* \mu_{t, \xi_t}$  is  $t$ -uniformly log-concave and  $\text{Cov}(\nu_E) \leq r P_E$ . We set

$$g_k(x) := g_k(x; t) = \sum_{i, j=1}^{d(r; t)} R_{ijk}(t) (x - a_t)_i (x - a_t)_j.$$

Using the Cauchy-Schwarz inequality and Theorem 8.10 we write

$$\begin{aligned}
\sum_{i,j=1}^{d(r;t)} R_{ijk}^2(t) &= \int_E g_k(x)(x - a_t)_k d\nu_E(x) \leq \sqrt{\text{Var}_{\nu_E}(g_k)} \left( \int_E (x - a_t)_k^2 d\mu_{t,\xi_t}(x) \right)^{1/2} \\
&= \sqrt{\lambda_k(t)} \sqrt{\text{Var}_{\nu_E}(g_k)} \leq \sqrt{\lambda_k(t)} \left( \frac{\|\text{Cov}(\nu_E)\|_{\text{op}}}{t} \right)^{1/4} \left( \int_E |\nabla g_k|^2 d\nu_E \right)^{1/2} \\
&\leq 2 \sqrt{\lambda_k(t)} \left( \frac{r}{t} \right)^{1/4} \left( \sum_{i,j=1}^{d(r;t)} \lambda_j(t) R_{ijk}^2(t) \right)^{1/2} \leq 2 \sqrt{\lambda_k(t)} t^{-1/4} r^{3/4} \left( \sum_{i,j=1}^{d(r;t)} R_{ijk}^2(t) \right)^{1/2}
\end{aligned}$$

and the lemma follows.  $\square$

The starting point of Guan's work is the next lemma.

**Lemma 11.10.** *For any smooth function  $f : [0, \infty) \rightarrow \mathbb{R}$  with bounded second derivative we have*

$$\frac{d}{dt} \mathbb{E}(\text{tr}(f(A_t))) = \mathbb{E} \left( \frac{1}{2} \sum_{i,j=1}^n |R_{i,j}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \right) - \mathbb{E} \left( \sum_{i=1}^n \lambda_i^2(t) f'(\lambda_i(t)) \right),$$

where the quotient in the above formula is interpreted as  $f''(\lambda_i(t))$  if  $\lambda_i(t) = \lambda_j(t)$ .

Lemma 11.10 is applied for a function  $f(\lambda)$  which is quadratic when  $\lambda$  is relatively large and exponential when  $\lambda$  is small. In this way, the analysis of  $A_t$  is restricted on the large eigenvalues. The precise construction of such an  $f$  is given in [65, Lemma 2.2].

**Lemma 11.11.** *Let  $D_0 > 4$  and  $\frac{7}{3} \leq r_0 \leq \frac{8}{3}$ . There exists  $b \in [\frac{1}{20}, \frac{1}{5}]$  and an increasing twice differentiable positive function  $f$  on  $[0, \infty)$  such that  $f(r) = e^{D_0(r-r_0)}$  if  $0 \leq r \leq r_0 - 1/D_0$  and  $f(r) = br^2$  if  $r \geq r_0$ , which satisfies  $|f''(r)| \leq D_0^2 f(r)$  for all  $r > 0$ .*

A direct consequence of Lemma 11.9 and of the fact that  $\lambda_i(t) \leq 1/t$  is that if  $i \geq d(r_0; t) + 1$  then

$$b \sum_{j,k=1}^i R_{ijk}^2(t) \leq 4bt^{-1/2} \lambda_i(t)^{3/2} \lambda_i(t) \leq 4bt^{-1} \lambda_i(t)^2 = 4t^{-1} f(\lambda_i(t)). \quad (11.5)$$

Instead of estimating  $\mathbb{E}(\text{tr}(A_t))$  directly, Guan is using the above in order to estimate  $\mathbb{E}(F_t)$ , where

$$F_t = \sum_{i=1}^n f(\lambda_i(t)) = \text{tr}(f(A_t)).$$

To this end, we apply the formula of Lemma 11.10. The first step is the next lemma (see [79]) which exploits (11.5) and the fact that, for any  $x$ ,

$$\sum_{\substack{i,j,k=1 \\ \lambda_j(t) > x}}^n R_{ijk}^2(t) \leq 3 \sum_{i=d(x;t)}^n \sum_{j,k=1}^i R_{ijk}^2(t).$$

This last inequality holds true because  $R_{ijk}^2(t)$  is symmetric in  $i, j, k$ , therefore we may sum over those triples that satisfy  $i \geq \max\{j, k\}$  and then multiply by 3.

**Lemma 11.12.** For any  $t > 0$ ,

$$\frac{d}{dt} \mathbb{E}(F_t) \leq \left( \frac{300}{t} + \frac{400D_0^2}{\sqrt{t}} \right) \mathbb{E}(F_t).$$

*Proof.* Since the second summand in Lemma 11.10 is negative, we have

$$\frac{d}{dt} \mathbb{E}(\text{tr}(f(A_t))) \leq \mathbb{E} \left( \frac{1}{2} \sum_{i,j=1}^n |R_{i,j}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \right). \quad (11.6)$$

Consider the sum

$$\frac{1}{2} \sum_{i,j=1}^n |R_{i,j}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)}. \quad (11.7)$$

We split this quantity into four sums, according to whether  $i, j \leq d(r_0; t)$  or not. From (11.5) we see that

$$b \sum_{i,j=d(r_0;t)+1}^n |R_{i,j}(t)|^2 = b \sum_{i,j=d(r_0;t)+1}^n \sum_{k=1}^n R_{ijk}^2(t) \leq 3b \sum_{i=d(r_0;t)+1}^n \sum_{j,k=1}^i R_{ijk}^2(t) \leq 12t^{-1} \sum_{i=d(r_0;t)+1}^n f(\lambda_i(t)). \quad (11.8)$$

Recall that  $r_0 \leq \frac{8}{3}$ . Using again (11.5) and the fact that  $x/(x-y) \leq 3(r_0 + \frac{1}{3})$  when  $x \geq r_0 + \frac{1}{3}$  and  $y \leq r_0$ , we see that the indices with  $i \leq d(r_0; t)$  and  $j > d(r_0 + 1/3; t)$  or  $i > d(r_0 + 1/3; t)$  and  $j \leq d(r_0; t)$  contribute to the sum (11.7) at most

$$\sum_{i=1}^{d(r_0;t)} \sum_{j=d(r_0+1/3;t)+1}^n \sum_{k=1}^n R_{ijk}^2(t) \frac{2b\lambda_j(t)}{\lambda_j(t) - \lambda_i(t)} \leq 2b \cdot 3(r_0 + 1/3; t) \sum_{i=1}^{d(r_0;t)} \sum_{j=d(r_0+1/3;t)+1}^n \sum_{k=1}^n R_{ijk}^2(t) \quad (11.9)$$

$$\leq 54b \sum_{i=d(r_0+1/3;t)+1}^n \sum_{j,k=1}^i R_{ijk}^2(t) \leq 216t^{-1} \sum_{i=d(r_0;t)+1}^n f(\lambda_i(t)).$$

Since  $|f''(r)| \leq D_0^2 f(r)$  for all  $r > 0$ , the indices with  $i, j \leq d(r_0 + 1/3; t)$  contribute to the sum (11.7) at most

$$\frac{1}{2} \sum_{i,j=1}^{d(r_0+1/3;t)} \sum_{k=1}^n R_{ijk}^2(t) \frac{|f'(\lambda_j(t)) - f'(\lambda_i(t))|}{|\lambda_j(t) - \lambda_i(t)|} \leq \frac{D_0^2}{2} \sum_{i,j=1}^{d(r_0+1/3;t)} \sum_{k=1}^n R_{ijk}^2(t) f(\max\{\lambda_i(t), \lambda_j(t)\}). \quad (11.10)$$

We split the last sum into two parts. From Lemma 11.9 we see that the terms with  $k \leq d(r_0 + 1/3; t)$  contribute at most

$$\begin{aligned} & \frac{3D_0^2}{2} \sum_{i=1}^{d(r_0+1/3;t)} f(\lambda_i(t)) \sum_{j,k=1}^{d(\lambda_i(t))} R_{ijk}^2(t) \leq 6D_0^2 t^{-1/2} \sum_{i=1}^{d(r_0+1/3;t)} \lambda_i(t)^{3/2} \lambda_i(t) f(\lambda_i(t)) \\ & \leq 6D_0^2 t^{-1/2} (r_0 + 1/3)^{5/2} \sum_{i=1}^{d(r_0+1/3;t)} f(\lambda_i(t)) \leq 6 \cdot 3^{5/2} D_0^2 t^{-1/2} \sum_{i=1}^{d(r_0+1/3;t)} f(\lambda_i(t)). \end{aligned} \quad (11.11)$$

Using Lemma 11.9 again, and taking into account that  $b \in [\frac{1}{20}, \frac{1}{5}]$ ,  $f(3) \leq 2$  and  $\frac{7}{60}x \leq bx^2 = f(x)$  for  $x \geq \frac{8}{3}$ , we see that the terms with  $k > d(r_0 + 1/3; t)$  contribute at

most

$$\begin{aligned}
D_0^2 \sum_{k=d(r_0+1/3;t)+1}^n \sum_{i=1}^{d(r_0+1/3;t)} f(\lambda_i(t)) \sum_{j=1}^{d(r_0+1/3;t)} R_{ijk}^2(t) &\leq 4D_0^2 f(3) t^{-1/2} (r_0 + 1/3)^{3/2} \sum_{k=d(r_0+1/3;t)+1}^n \lambda_k(t) \\
&\leq 8 \cdot 3^{3/2} \cdot \frac{60}{7} D_0^2 t^{-1/2} \sum_{k=d(r_0+1/3;t)+1}^n f(\lambda_k(t)).
\end{aligned} \tag{11.12}$$

From (11.8), (11.9), (11.10) and (11.11) we finally get

$$\begin{aligned}
\frac{d}{dt} \mathbb{E}(F_t) &\leq \left( \frac{12 + 216}{t} + \frac{D_0^2}{\sqrt{t}} \max \left\{ 6 \cdot 3^{5/2}, 8 \cdot 3^{3/2} \cdot \frac{60}{7} \right\} \right) \mathbb{E} \left( \sum_{i=1}^n f(\lambda_i(t)) \right) \\
&\leq \left( \frac{300}{t} + \frac{400D_0^2}{\sqrt{t}} \right) \mathbb{E}(F_t),
\end{aligned}$$

and the proof is complete.  $\square$

A consequence of Lemma 11.12 is that

$$\mathbb{E}(F_s) \leq \left( \frac{s}{t} \right)^{700} \mathbb{E}(F_t) \tag{11.13}$$

for all  $0 < t \leq s \leq D_0^{-4}$ . It is known (see [80, Lemma 5.2]) that this implies the bound

$$\mathbb{P}(\|A_t\|_{\text{op}} \geq 2) \leq c_1 \exp(-c_2/t) \tag{11.14}$$

for all  $0 < t \leq c_3/(\ln n)^2$ . In particular, if  $0 \leq t \leq t_1 = \bar{c}/(\ln n)^2$ , where  $\bar{c} > 0$  is an absolute constant, then

$$\mathbb{P}(\|A_t\|_{\text{op}} \geq 2) \leq \frac{1}{n}. \tag{11.15}$$

**§ 11.5. Proof of Theorem 11.8.** We set  $s_1 = 7/3$  and consider the function  $f_1 = f_{D_0, r_0}$  from Lemma 11.11 with  $r_0 = s_1$  and  $D_0 = (\ln t_1)^4$ . If we assume that  $n \geq n_0$ , where  $n_0$  is a large enough absolute constant, then we know that  $D_0 \geq 3$ . We define  $F_{1,t} = \text{tr}(f_1(A_t))$  and using (11.15) and the fact that  $A_t \leq \frac{1}{t} I_n$  we see that

$$\mathbb{E}(F_{1,t_1}) \leq \left( 1 - \frac{1}{n} \right) n f_1(2) + \frac{1}{n} n f_1(1/t_1) = (n-1)e^{-|\ln t_1|^{4/3}} + b t_1^{-2}. \tag{11.16}$$

For  $k \geq 2$  we define

$$t_k = |\ln t_{k-1}|^{-16}, \quad s_k = \frac{7}{3} + \sum_{i=2}^k |\ln t_i|^{-1/2}, \quad f_k = f_{(\ln t_k)^4, s_k}, \quad F_{k,t} = \text{tr}(f_k(A_t)).$$

We denote by  $b_k$  the constant in  $\left[ \frac{1}{20}, \frac{1}{5} \right]$  from the construction of the function  $f_k = f_{(\ln t_k)^4, s_k}$ . We set  $k_0 = \max\{k \geq 1 : t_k \leq e^{-100}\}$ . Since the summands in the sum of the definition of  $s_k$  decay very fast, if we assume that  $n \geq n_0$  for a large enough absolute constant  $n_0 > 0$  then we see that  $s_k \in \left[ \frac{7}{3}, \frac{8}{3} \right]$  and  $0 < t_1 < t_2 \leq \dots \leq t_{k_0} \leq e^{-100}$ . Also, for every  $2 \leq k \leq k_0$  we have

$$s_{k-1} = s_k - \frac{1}{\sqrt{|\ln t_k|}} \leq s_k - \frac{1}{(\ln t_k)^4}. \tag{11.17}$$

**Lemma 11.13.** *Let  $n \geq n_0$ , where  $n_0$  is a large enough absolute constant. For all  $1 \leq k \leq k_0$  and  $t \in [t_k, t_{k+1}]$ ,*

$$\mathbb{E}(F_{k,t}) \leq n \exp(-(\ln t_k)^2). \quad (11.18)$$

*Proof.* We shall show that for every  $k \geq 2$  and  $r \geq s_{k-1}$ ,

$$f_k(r) \leq 5f_{k-1}(r). \quad (11.19)$$

If  $r \geq s_k$  then

$$\frac{f_k(r)}{f_{k-1}(r)} = \frac{b_k}{b_{k-1}} \leq 4.$$

If  $s_{k-1} \leq r \leq s_k$  then

$$f_k(r) \leq f_k(s_k) = b_k s_k^2 \leq 4b_{k-1} s_k^2 \leq 5b_{k-1} r^2 = 5f_{k-1}(r)$$

So, (11.19) is true. We shall prove (11.18) by induction on  $k$ . For the case  $k = 1$ , recall that  $t_1 = \bar{c}/(\ln n)^2$ . Using (11.13), (11.16) and the fact that  $b \leq 1/5$  we see that

$$\mathbb{E}(F_{1,t}) \leq \left(\frac{t_2}{t_1}\right)^{700} \mathbb{E}(F_{1,t_1}) \leq t_1^{-700} \left( (n-1)e^{-|\ln t_1|^{4/3}} + t_1^{-2}/5 \right) \leq n e^{-(\ln t_1)^2}$$

for every  $t \in [t_1, t_2]$ , provided that  $n \geq n_0$ . Now, let  $k \geq 2$  and assume that (11.18) holds true for  $k-1$ . From (11.17) and (11.19) we have

$$\begin{aligned} \mathbb{E}(F_{k,t_k}) &= \mathbb{E} \left( \sum_{i=1}^n f_k(\lambda_i(t_k)) \right) = \mathbb{E} \left( \sum_{i=1}^{d(s_{k-1}; t_k)} f_{(\ln t_k)^4, s_k}(\lambda_i(t_k)) + \sum_{i=d(s_{k-1}; t_k)+1}^n f_k(\lambda_i(t_k)) \right) \\ &\leq \mathbb{E} \left( d(s_{k-1}; t_k) e^{(\ln t_k)^4 (s_{k-1} - s_k)} + 5 \sum_{i=d(s_{k-1}; t_k)+1}^n f_{k-1}(\lambda_i(t_k)) \right) \\ &\leq n e^{-|\ln t_k|^{7/2}} + 5\mathbb{E}(F_{k-1, t_k}). \end{aligned}$$

From (11.13) and the induction hypothesis we see that if  $t \in [t_k, t_{k+1}]$  then

$$\begin{aligned} \mathbb{E}(F_{k,t}) &\leq \left(\frac{t_{k+1}}{t_k}\right)^{700} \mathbb{E}(F_{k,t_k}) \leq \left(\frac{t_{k+1}}{t_k}\right)^{700} \left[ n e^{-|\ln t_k|^{7/2}} + 5n e^{-(\ln t_{k-1})^2} \right] \\ &\leq n \left[ t_k^{-700} e^{-|\ln t_k|^{7/2}} + 5t_k^{-700} e^{-t_k^{-1/8}} \right] \leq n \left[ \frac{1}{2} e^{-|\ln t_k|^2} + \frac{1}{2} e^{-|\ln t_k|^2} \right], \end{aligned} \quad (11.20)$$

using also the fact that  $t_k \leq \exp(-100)$  and  $t_k = |\ln t_{k-1}|^{-16}$ . Now, (11.20) gives (11.18) and the proof is complete.  $\square$

We can proceed now with Guan's main estimate in [65].

**Theorem 11.14.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . If  $(A_t)_{t \geq 0}$  is the covariance process associated with the stochastic localization starting from  $\mu$ , then*

$$\mathbb{E}(\text{tr}(A_t^2)) \leq cn$$

for all  $t > 0$ , where  $c > 0$  is an absolute constant.

*Proof.* We first assume that  $n \geq n_0$  where  $n_0$  is a large enough absolute constant. Let  $c_1 > 0$  be an absolute constant such that  $c_1 < t_{k_0+1}$ . If  $t \in [t_1, c_1]$  then there exists  $1 \leq k \leq k_0$  such that  $t_k \leq t \leq t_{k+1}$ . From Lemma 11.13 we know that

$$\begin{aligned} \mathbb{E}(\text{tr}(A_t^2)) &= \mathbb{E}\left(\sum_{i=1}^n \lambda_i^2(t)\right) \leq \left(\frac{8}{3}\right)^2 n + \mathbb{E}\left(\sum_{\{i:\lambda_i(t) \geq 8/3\}} \lambda_i^2(t)\right) \\ &\leq \frac{64}{9}n + 20\mathbb{E}(F_{k,t}) \leq n\left(\frac{64}{9} + e^{-(\ln t_k)^2}\right) \leq 8n. \end{aligned}$$

On the other hand, if  $0 < t \leq t_1$  then (11.14) and the fact that  $A_t \leq \frac{1}{t}I_n$  show that

$$\mathbb{E}(\text{tr}(A_t^2)) \leq 4n \mathbb{P}(\|A_t\|_{\text{op}} \leq 2) + \frac{n}{t^2} \mathbb{P}(\|A_t\|_{\text{op}} \geq 2) \leq 4n + \frac{c_2 n}{t^2} \exp(-c_3/t) \leq 8n,$$

where we have taken into account that  $t \leq \bar{c}/(\ln n)^2$  and  $n \geq n_0$ . Note also that if  $t > c_1$  then

$$\mathbb{E}(\text{tr}(A_t^2)) \leq \frac{1}{c_1^2}n$$

because  $A_t \leq \frac{1}{c_1}I_n$ . Finally, the case  $n < n_0$  is covered e.g. by [80, Corollary 5.4] which provides a bound for  $\mathbb{E}\|A_t\|_{\text{op}}^2$  and hence for  $\mathbb{E}(\text{tr}(A_t^2))$ .  $\square$

It is not hard now to prove Theorem 11.8.

*Proof of Theorem 11.8.* Using the formula

$$dA_t = \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} \langle x - a_t, dW_t \rangle d\mu_t(x) - A_t^2 dt$$

and taking into account that  $A_0 = I_n$ , from the estimate  $\mathbb{E}(\text{tr}(A_t^2)) \leq cn$  for every  $t \geq 0$  we see that

$$\mathbb{E}(\text{tr}(A_t)) \geq \text{tr}(A_0) - ctn = (1 - ct)n$$

and hence

$$\mathbb{E}(\text{tr}(A_t)) \geq \delta_1 n$$

for all  $0 \leq t \leq \delta_1$ , where  $\delta_1$  is a suitably small positive absolute constant (one may choose  $\delta_1 = \frac{1}{c+1}$ ).  $\square$

Theorem 11.1 and Theorem 11.8 play an important role in the derivation of the main results  $\psi_n = O(\sqrt{\ln n})$  and  $L_n = O(1)$  that we discussed in Sections 8 and 9.

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