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INFINITE CYCLIC COVERS OF THE PROJECTIVE LINE

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Abstract. In this article, Schreier's Lemma is employed in order to compute the fundamental group of the open Riemann surface obtained as a cyclic cover of the projective line. Then an action of the Galois group of the corresponding cover on abelianized fundamental group of the projective line with branch points removed is computed and it is shown to be independent of the choice of the Schreier transversal.

Introduction

The study of fundamental groups of punctured complex projective lines is a classical topic in algebraic topology and combinatorial group theory. A projective line is an affine and projective variety and a Riemann surface [6, Ch. 1]. Despite of this we will study the line as a topological space. Let

$$X = \mathbb{CP}^1 \setminus \{p_1, \ldots, p_{s+1}\}$$

denote the complex projective line punctured in s distinct points. Its fundamental group has the well-known presentation

$$\pi_1(X) = \langle x_1, \ldots, x_{s+1} \mid x_1 x_2 \cdots x_{s+1} = 1 \rangle,$$

which is generated by loops around the punctures, subject to the single relation that the product of all loops is trivial.

Consider a homomorphism

$$\alpha : \pi_1(X) \longrightarrow b\mathbb{Z}, \quad x_i \mapsto d_i, \tag{1}$$

where $b = \gcd(d_1, \ldots, d_s)$ and the image of the last generator x_{s+1} is determined by the relation among the generators. Such a homomorphism naturally arises in the study of cyclic coverings of punctured spheres and in the investigation of branched covers of Riemann surfaces.

The kernel of α, denoted $\ker(\alpha)$, is naturally isomorphic to the fundamental group of the corresponding infinite cyclic covering of X [10, Ch. 14, Sec. b]. More precisely, there is a Galois covering map [5, Ch. 1, Sec. 3]

$$p : \widetilde{X} \longrightarrow X$$

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whose fundamental group is exactly the kernel of α:

$$\pi_1(\widetilde{X}) \cong \ker(\alpha) \subset \pi_1(X).$$

By the correspondence between subgroups of the fundamental group and covering spaces,we have the natural isomorphism Φ:
For each homotopy class of loops $[\tilde{\gamma}] \in \pi_1(\widetilde{X}, \tilde{x}_0)$, consider its projection $\pi \circ \tilde{\gamma}$ to X. Then $\pi \circ \tilde{\gamma}$ is a loop in X based at $x_0 = \pi(\tilde{x}_0)$, and since $\tilde{\gamma}$ is closed in $\widetilde{X}$, its image under α is trivial. Hence,

$$\Phi([\tilde{\gamma}]) = [\, \pi \circ \tilde{\gamma}\,] \in \ker(\alpha) \subset \pi_1(X, x_0).$$

The deck trasformation group of the covering p is cyclic and isomorphic to the image of α:

$$\mathrm{Deck}(\widetilde{X}/X) \cong \mathrm{Im}(a) = b\mathbb{Z}.$$

It is generated by a single transformation corresponding to a loop in $\pi_1(X)$ whose image under a generates $\mathrm{Im}(\alpha)$. Let γ be a loop in X based at $x \in X$ that $\alpha([\gamma]) = b$. The deck transformation associated to $[\gamma] \in \pi_1(X, x)$ can be defined as:

$$T_{\alpha[\gamma]} : \widetilde{X} \longrightarrow \widetilde{X}, \quad \tilde{x} \longmapsto \tilde{\gamma}(1) \quad \text{where } \tilde{\gamma}(0) = \tilde{x}.$$

Here, $\tilde{\gamma}$ denotes the unique lift of γ starting at $\tilde{x} \in \pi^{-1}(x)$.

The previous construction induces a natural $\mathbb{Z}$-action on $\widetilde{X}$, which permutes the sheets of the covering. The covering is infinite, thus the $\mathbb{Z}$-action is free and transitive on the fibers:

$$\mathbb{Z} \times \widetilde{X} \longrightarrow \widetilde{X}, \qquad (n, \tilde{x}) \longmapsto T_b^n(\tilde{x}),$$

where T_b denotes the deck transformation corresponding to $b \in b \cdot \mathbb{Z}$. The action satisfies

$$T_b^0 = \mathrm{id}_{\widetilde{X}}, \qquad T_b^{m+n} = T_b^m \circ T_b^n \quad \text{for all } m, n \in \mathbb{Z}.$$

The number of sheets of the covering coincides with the order of the deck transformation group, which is countable, infinite and equal to the cardinal number of $\mathrm{Im}(a)$ [5, Ch. 3, Sec. 1].

Aim of this article is the computation of the kernel of this morphism using Schreier's lemma [1, Ch. 2, Sec. 8], [3, Sec. 2.3, Thm. 2.7]. A major difficulty in this computation is the selection of a suitable Schreier transversal, which is handled in Section 1 by introducing the notion of pruning and we provide an algorithm for the study of this problem. Using this algorithm a set of free generators is determined.

Our initial motivation was the computation of the covering group and homology for Riemann surfaces given by an equation of the form

$$y^n = \prod_{i=1}^{s} (x - \rho_i)^{b_i}, \quad \sum_{i=1}^{s} b_i \equiv 0 \bmod n,$$

extending the results of [2].

We have developed python code for the algorithms explained in this article, which is available at `https://bulletin.math.uoc.gr/vol/69/paper3-code/`. The first program uses the pruning technique explained in the article, while the second one solves a diophantine equation and pruning is not required.

A geometrical visual example showing how a lift of a loop in X via deck transformations moves through sheets is shown in Fig. 1.

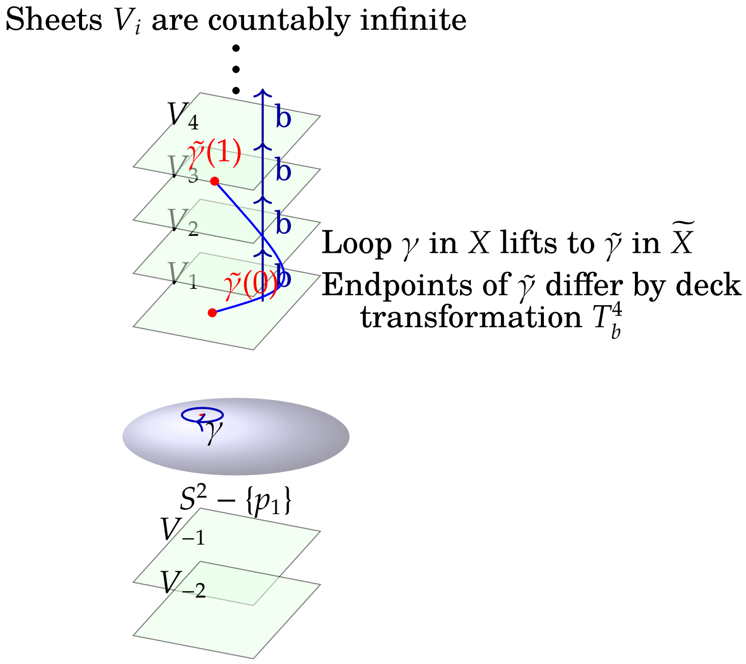

FIGURE 1. The specific example maps $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ via the deck transformation T_b^4

A classical result in combinatorial group theory, the Nielsen–Schreier theorem, guarantees that the kernel of a nontrivial homomorphism from a finitely generated free group (or a free group modulo a single relation) to $\mathbb{Z}$ is itself a free group of infinite rank. Consequently, $\pi_1(\widetilde{X}) \cong \ker(a)$ is a free group of infinite rank, and one can construct an explicit infinite basis for it.

In summary, analyzing the kernel and image of a provides a complete picture of the associated cyclic covering: the kernel describes the fundamental group of the covering space, the image describes the cyclic deck transformation group, its generator, the $\mathbb{Z}$-action, describes how the covering space fibers are permuted, and the number of sheets corresponds to the order of the deck group. This construction not only elucidates the algebraic structure of $\pi_1(X)$ but also provides deep topological insight into the structure of punctured projective lines and their infinite cyclic coverings.

1. COMPUTATIONS

Lemma 1. *Consider the word* $w = x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r} \in \mathscr{F}_s$, *where* $l_i \neq 0$ *for all* $1 \leq i \leq r$. *We denote by* w_{i,s_i} *the word*

$$w_{i,s_i} = x_1^{l_1} x_2^{l_2} \cdots x_{i-1}^{l_i-1} x_i^{s_i} \text{ with } |s_i| \leq |l_i| \text{ and } s_i l_i > 0. \tag{2}$$

If

$$\alpha(w_{k_2,s_{k_2}}) = \alpha(w_{k_1,s_{k_1}}) \text{ for some } 1 \leq k_2 < k_1 \leq r,$$

then

(1) $\alpha(x_{k_2}^{l_{k_2}-s_{k_2}} x_{k_2+1}^{l_{k_2+1}} \cdots x_{k_1-1}^{l_{k_1-1}} x_{k_1}^{s_{k_1}}) = 0$
(2) $\alpha(w_*) = \alpha(w)$, *where*

$$w_* = x_1^{l_1} x_2^{l_2} \cdots x_{k_2-1}^{l_{k_2-1}} \cdot x_{k_2}^{s_{k_2}} \cdot x_{k_1}^{l_{k_1}-s_{k_1}} \cdot x_{k_1+1}^{l_{k_1}+1} \cdots x_r^{l_r}.$$

Proof. The first statement is clear by observing that

$$w_{k_2,s_{k_2}} \cdot x_{k_2}^{l_{k_2}-s_{k_2}} x_{k_2+1}^{l_{k_2+1}} \cdots x_{k_1-1}^{l_{k_1-1}} x_{k_1}^{s_{k_1}} = w_{k_1,s_{k_1}}.$$

For the second statement observe that

$$\begin{aligned}
\alpha(w) &= \alpha(x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r}) \\
&= \alpha\left(x_1^{l_1} x_2^{l_2} \cdots x_{k_2-1}^{l_{k_2-1}} x_{k_2}^{s_{k_2}}\right) + \alpha\left(x_{k_2}^{l_{k_2}-s_{k_2}} x_{k_2+1}^{l_{k_2+1}} \cdots x_{k_1}^{s_{k_1}}\right) \\
&\qquad + \alpha\left(x_{k_1}^{l_{k_1}-s_{k_1}} x_{k_1+1}^{l_{k_1+1}} \cdots x_r^{l_r}\right) \\
&= \alpha(w^*).
\end{aligned}$$

□

Definition 2. If the word $w = x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r} \in \mathscr{F}_s$ has a segment

$$x_{k_2}^{l_{k_2}-s_{k_2}} x_{k_2+1}^{l_{k_2+1}} \cdots x_{k_1-1}^{l_{k_1-1}} x_{k_1}^{s_{k_1}} \tag{3}$$

so that

$$\alpha(x_{k_2}^{l_{k_2}-s_{k_2}} x_{k_2+1}^{l_{k_2+1}} \cdots x_{k_1-1}^{l_{k_1-1}} x_{k_1}^{s_{k_1}}) = 0 \tag{4}$$

then we define

$$w^* = w^*(k_2, s_2, k_1, s_1) = x_1^{l_1} x_2^{l_2} \cdots x_{k_2-1}^{l_{k_2-1}} \cdot x_{k_2}^{s_{k_2}} \cdot x_{k_1}^{l_{k_1}-s_{k_1}} \cdot x_{k_1+1}^{l_{k_1}+1} \cdots x_r^{l_r}.$$

and we call w^* a *pruned word* of w. We repeat this procedure to the pruned word until there is no segment of the form of (3) with the property (4). The resulting word will be called *fully pruned*.

Proposition 3. *For a fully pruned word all initial segments obtain different integers as images of the function* α *defined in (1).*

Proof. Indeed, if two initial segments $w_{k_1,s_{k_1}}$, $w_{k_2,s_{k_2}}$ have the same α-image then Lemma 1 implies the existence of a segment of the form of (3) and the same lemma implies that the fully pruned word can be further pruned, a contradiction.

□

Lemma 4. *Extended Bézout's identity (or Bézout's Lemma) : If* $\gcd(b_1, \dots, b_s) = b$*, then there exist integers* $k_1, \dots, k_s$ *such that:*

$$k_1 b_1 + k_2 b_2 + \dots + k_s b_s = b.$$

Proof. See [4].

□

Consider now the Diophantine equation:

$$y_1 \cdot b_1 + y_2 \cdot b_2 + \dots + y_s \cdot b_s = b, \quad b = \text{g.c.d.}(b_1, \dots, b_s). \tag{5}$$

which has a solution $(a_1, a_2 \dots, a_s)$ from Lemma 4. Without loss of generality we can assume that the first r components are non-zero. We denote by w_b the word

$$w_b = x_1^{a_1} x_2^{a_2} \dots x_r^{a_r}.$$

We assume that this word w_b has been fully pruned, meaning that there are no segments of this word with image 0.

Definition 5 (Initial segment condition)**.** A set T_0^m satisfies the initial segments condition (i.s.c.) if for every $w \in T_0^m$ holds: $\{w_1, \dots, w_n = w_b\} \subseteq T_0^m$, where $\{w_1, \dots, w_n = w_b\}$ are the initial segments of w.

Let $w_1, \dots, w_n = w_b$ be the initial segments of w_b. As we already observed $\alpha(w_j) \neq \alpha(w_i)$ for $1 \leq i \neq j \leq n$. We define the set

$$T_0^1 = \{w_1, \dots, w_n\} \subseteq \mathscr{F}_s.$$

This set T_0^1 satisfies the conditions:

1. It is a finite subset of $\mathscr{F}_s$.
2. $\alpha(T_0^1) \subseteq [-M, M]_b$, where
$$[-M, M]_b := [-M, M] \cap b\mathbb{Z}$$
$$M := \max_{w \in T_0^1} |\alpha(w)| .$$
3. The map $\alpha|_{T_0^1} : T_0^1 \longrightarrow b\mathbb{Z}$ is 1-1 (because of pruning)
4. The set T_0^1 satisfies the initial segments condition (i.s.c.).

We will construct a family of sets of the form:

$$T_0^1 \subseteq T_0^{m_1} \subseteq T_0^{m_2} \subseteq \dots \subseteq T_0^{m_l} = T_0,$$

where:

- The sets $T_0^{m_i}$ for every $i = 1, 2, \dots, l$ will satisfy the conditions (1),(2),(3) and (4), by putting $T_0^{m_i}$ in place of T_0^1.
- The set T_0 will satisfy, apart from conditions (1),(2),(3),(4), the condition $\alpha(T_0) = [-M, M]_b$

With T_0 we will denote the "period" of the transversal. The "period" of the transversal is a finite subset of it that allows us, through an iterative procedure developed in Theorem 15, to describe the entire infinite transversal.

If a $T_0^{m_i}$ satisfy conditions (1),(2),(3),(4) by putting $T_0^{m_i}$ in place of T_0^1, then we will write that it satisfies condition (*) and if it satisfies additionally condition $\alpha(T_0^{m_i}) = [-M, M]_b$ we will write that it satisfies condition (**).

The transversal construction

We set $M := \max_{w \in T_0^1} |\alpha(w)|$. We will define the finite sets $T_0^{m_i} \subset \mathscr{F}_s$ such that $\emptyset \neq T_0^1 \subseteq T_0^{m_i}$.

Definition 6. A set T_0^m has a "hole" in $hb \in b\mathbb{Z}$ if

(1) $hb \in [-M, M]_b$, and
(2) for every $w \in T_0^m$ we have $\alpha(w) \neq hb$.

From now on we will denote by $(a_1, \dots, a_s)$ is a solution of the Diophantine equation (5) and T_0^m satisfies (*) condition and $T_0^1 \subseteq T_0^m$.

Lemma 7. *If hb is a hole of T_0^m, then for every $w \in T_0^m$ we have:*

$$hb = \alpha(w) + k_1 \cdot b_1 + \cdots + k_s \cdot b_s,$$

for some integers $k_i = k_i(w) \in \mathbb{Z}$ depending on w, with the additional property $(k_1, \dots k_s) \neq (0, \dots 0)$.

Proof. Let $w \in T_0^m$. Since, for all $w \in T_0^m$ $\alpha(w) \neq h \cdot b$, we have that

$$hb - \alpha(w) = n \cdot b \text{ with } n \neq 0 \in \mathbb{Z}.$$

Then,

$$\begin{aligned} hb &= \alpha(w) + n(a_1 \cdot b_1 + \cdots + a_s \cdot b_s) \\ &= \alpha(w) + k_1 \cdot b_1 + \cdots + k_s \cdot b_s, \end{aligned}$$

where $k_i = k_i(w) = n \cdot a_i$, and $a_1, \dots, a_s$ are the solutions of eq. (5). Since $n \neq 0$ we have $(k_1, \cdots, k_s) \neq (0, 0, \cdots, 0)$.

□

Theorem 8. *If the set $T_0^m \supseteq T_o^1$ has a hole of the form $hb = \alpha(w_0) + k_i \cdot b_i$ for some $w_0 \in T_0^m$, T_0^m satisfies $(*)$ and $k_i \in \mathbb{Z}_{>0}$, then there exists a set T_0^n, with $T_0^m \subseteq T_0^n$, that satisfies the $(*)$ condition and it does not contain a hole at $hb \in b\mathbb{Z}$.*

Proof. We distinguish the following cases:

- If $\alpha(w_0) + b_i \notin \alpha(T_0^m)$, then:

$$-M < -M + b_i \leq \alpha(w_0) + b_i \leq \alpha(w_0) + k_i \cdot b_i \leq M.$$

The last inequality holds, since the T_0^m has a hole in $\alpha(w_0) + k_i \cdot b_i$.

We now form the word $w_{l+1} = w_0 \cdot x_i$. Then, $\alpha(w_{l+1}) = \alpha(w_0) + b_i$. The initial segments of w_{l+1} is w_0 and the initial segments of w_0. Because $w_0 \in T_0^m$ and T_0^m satisfy (*) condition it holds that the set $T_0^m \cup \{w_{l+1}\}$ satisfies i.s.c. We define $T_0^{m_2} := T_0^m \cup \{w_{l+1}\}$.

The set $T_0^{m_2}$ does not have a hole in $\alpha(w_0) + b_i$. Since $\alpha(w_0) + b_i \notin \alpha(T_0^m)$ we have $\alpha(w) \neq \alpha(w_0) + b_i$ for every $w \in T_0^m$. Thus $\alpha|_{T_0^{m_2}}$ is 1-1.

- If $\alpha(w_0) + b_i \in \alpha(T_0^m)$, then we set $T_0^{m_2} := T_0^m$.
- If $k_i \geq 2$ and $\alpha(w_0) + 2 \cdot b_i \notin \alpha(T_0^{m_2})$, then we argue as before: $\alpha(w_0) + 2 \cdot b_i \in [-M, M]_b$.

Let $\alpha(u) = \alpha(w_0) + b_i$ for some $u \in T_0^{m_2}$. We define $w_{l+2} = u \cdot x_i$ and then $\alpha(w_{l+2}) = \alpha(w_0) + 2 \cdot b_i$. Let $T_0^{m_3} := T_0^{m_2} \cup \{w_{l+2}\}$. If $\alpha(w_0) + 2 \cdot b_i \in \alpha(T_0^{m_2})$, then we set $T_0^{m_3} := T_0^{m_2}$.

Similarly the set $T_0^{m_3}$ satisfies (*) condition and it does not have a hole in $\alpha(w_0) + 2 \cdot b_i$.

In this way we construct the following set family:

$$T_0^1 = T_0^{m_1} \subseteq T_0^{m_2} \subseteq \cdots \subseteq T_0^{m_p}$$

for which we have

1. $\alpha(T_0^{m_j}) \subseteq [-M, M]_b, 1 \leq j \leq p$
2. $\alpha(w_0) + j \cdot b_i \in \alpha(T_0^{m_{j+1}}), 1 \leq j \leq k_i - 1$
3. the sets $\alpha(T_0^{m_j})$ satisfy i.s.c. and $\alpha|T_0^{m_j}$ is injective for every $1 \leq j \leq p$.

From the construction of $T_0^{m_j}$ and because $\alpha(w_0) + k_i \cdot b_i \notin \alpha(T_0^m)$ it is clear that $\alpha(w_0) + k_i \cdot b_i \notin \alpha(T_0^{m_p})$. Let $v \in T_0^{m_p}$ with $\alpha(v) = \alpha(w_0) + (k_i - 1) \cdot b_i$. If $z = v \cdot x_i$ then $\alpha(z) = \alpha(w_0) + k_i \cdot b_i$. It is now clear that the set $T_0^n := T_0^{m_{p+1}} = T_0^{m_p} \cup \{z\}$ satisfies (*) and it has no hole in hb. □

Theorem 9. *If the set $T_0^m \supseteq T_0^1$ contains a hole of the form $hb = \alpha(w_0) - k_i \cdot b_i$, $k_i \in \mathbb{Z}_{>0}$ for some $w_0 \in T_0^m$ and T_0^m obeys condition (*) , then it exists a set $T_0^n \supseteq T_0^m$ that satisfies condition (*) and it does not have a hole in hb.*

Proof. (Similar to the proof of Theorem 8.)

If $\alpha(w_0) - b_i \notin \alpha(T_0^m)$ then

$$-M \leq \alpha(w_0) - k_i \cdot b_i \leq \alpha(w_0) - b_i \leq M - b_i < M$$

because the set T_0^m has a hole in $\alpha(w_0) - k_i \cdot b_i$. Let $u_{t+1} = w_0 \cdot x_i^{-1}$. Then $\alpha(u_{t+1}) = \alpha(w_0) - b_i$ and we set $T_0^{m_2} := T_0^m \cup \{u_{t+1}\}$ otherwise it holds $\alpha(w_0) - b_i \in \alpha(T_0^m)$ and we define $T_0^{m_2} := T_0^m$. Like Theorem 8 the set $T_0^{m_2}$ satisfies condition (*) and it does not have any hole in $\alpha(w_0) - b_i$.

If $k_i \geq 2$ and $\alpha(w_0) - 2 \cdot k_i \notin \alpha(T_0^{m_2})$. Similarly $\alpha(w_0) - 2 \cdot b_i \in [-M, M]_b$ and there exists a $u \in T_0^{m_2}$ such that $\alpha(u) = \alpha(w_0) - 2 \cdot b_i$. If $u_{t+2} = u \cdot x_i^{-1}$ we define $\alpha(u_{t+2}) = \alpha(w_0) - 2 \cdot b_i$ and $T_0^{m_3} := T_0^{m_2} \cup \{w_{t+2}\}$ otherwise we have $\alpha(w_0) - 2 \cdot b_i \in \alpha(T_0^{m_2})$ and we set $T_0^{m_3} := T_0^{m_2}$.

Similarly the set $T_0^{m_3}$ satisfies (*) condition and it has no hole in $\alpha(w_0) - 2 \cdot b_i$.

In this way we construct the following family of sets

$$T_0^1 = T_0^{m_1} \subseteq T_0^{m_2} \subseteq \cdots \subseteq T_0^{m_p}$$

for which we have

1. $\alpha(T_0^{m_j}) \subseteq [-M, M]_b, 1 \leq j \leq p$,
2. $\alpha(w_0) - j \cdot b_i \in \alpha(T_0^{m_{j+1}}), 1 \leq j \leq k_i - 1$,
3. the sets $\alpha(T_0^{m_j})$ satisfy i.s.c. and $\alpha|T_0^{m_j}$ is injective for every $1 \leq j \leq p$.

From the construction of $T_0^{m_j}$ and since $\alpha(w_0) - k_i \cdot b_i \notin \alpha(T_0^m)$ it is clear that $\alpha(w_0) - k_i \cdot b_i \notin \alpha(T_0^{m_p})$.

As in Theorem 8 there exists a $z \in T_0^{m_p}$ with $\alpha(z) = \alpha(w_0) - (k_i - 1) \cdot b_i$. We set $w_p = z \cdot x_i^{-1}$ and we have $\alpha(w_p) = \alpha(w_0) - k_i \cdot b_i$. We take the set $T_0^n := T_0^{m_{p+1}} = T_0^{m_p} \cup \{w_p\}$ and it is clear that it satisfies the conditions of the theorem.

□

Corollary 10. *If the set* $T_0^l \supseteq T_0^1$ *has a hole of the form* $hb = \alpha(w_0) + k_i \cdot b_i$ *with* $0 \neq k_i \in \mathbb{Z}$ *for a* $w_0 \in T_0^l$*, then there exists a set* $T_0^m \supseteq T_0^l$ *that satisfies the* $(*)$ *condition and has no hole in* hb*.*

Proof. Immediate from Theorems 8 and 9.

□

Theorem 11. *If* $\alpha(w_0) + k_i \cdot b_i \in [-M, M]_b$ *with* $w_0 \in T_0^p \supseteq T_0^1$ *so that* $k_i \in \mathbb{Z}^*$*, then there exists a set* $T_0^m \supseteq T_0^p$ *that satisfies the* $(*)$ *condition and has no hole in* $\alpha(w_0) + k_i \cdot b_i$*.*

Proof. If $k_i = 0$ we set $T_0^m := T_0^p$.
If $k_i > 0$ it is immediate from Theorem 8.
If $k_i < 0$ it is immediate from Theorem 9.

□

Lemma 12. *We have*

$$[-M, M]_b := \{k_1 \cdot b_1 + k_2 \cdot b_2 + \cdots + k_s \cdot b_s \,|\, k_i \in \mathbb{Z},\ |k_i| \leq r_1 \cdot a_i,\ i = 1, \ldots, r\},$$

where

$$b = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_s \cdot b_s, \quad M = r_1 \cdot b.$$

Proof. The assumption right after Lemma 4 and Bézout's Lemma impose that $b = a_1 \cdot b_1 + \cdots + a_r \cdot b_r$ for some $a_i \in \mathbb{Z}$. We have supposed without loss of generality that the first r components of a_i are not zero. If $lb \in [-M, M]_b$ then $l \in [-r_1, r_1]_1$ and $lb = (l \cdot a_1) \cdot b_1 + \cdots + (l \cdot a_r) \cdot b_r = k_1 \cdot b_1 + \cdots + k_r \cdot b_r$ with $k_i = l \cdot a_i$ for $i = 1, 2, \ldots, r$.

□

Theorem 13. *There is a set* $T_0 \supseteq T_0^1$ *that satisfies condition (**).*

Proof. We regard the integers $k_r \in \mathbb{Z}$ where $a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r \in [-M, M]_b$. There exists at least one integer that satisfies the above condition. Indeed, $k_r = 0$ satisfies the condition since

$$\begin{aligned} a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} &= a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r \\ &= \alpha(x_1^{a_1} \ldots x_{r-1}^{a_{r-1}}) \in \alpha(T_0^1) \subseteq [-M, M]_b, \end{aligned}$$

because the word $x_1^{a_1} \ldots x_{r-1}^{a_{r-1}}$ is an initial segment of the word w_b.

We have

$$\begin{aligned} a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r &= a_1 \cdot b_1 + \cdots + a_r \cdot b_r - (a_r - k_r) \cdot b_r \\ &= \alpha(w_b) - (a_r - k_r) \cdot b_r, \end{aligned}$$

where $w_b \in T_0^1$, $w_b = x_1^{a_1} \dots x_r^{a_r}$.

From Theorem 11 there exists a set $T_0^{m_{k_r}} \supseteq T_0^1$ such that

$$\alpha(w_{k_r}) = a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r$$

for some $w_{k_r} \in T_0^{m_{k_r}}$. We can repeat Theorem 11 several times (the precise number is $2 \cdot r_r \cdot a_r$) in order to construct the set $T_0^{p_{k_r}}$ which contains all integers of the above form. The set $T_0^{p_{k_r}}$ satisfies the (*) condition and it has no hole in $a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r$.

We regard the integers $a_1 \cdot b_1 + \cdots + a_{r-2} \cdot b_{r-2} + k_{r-1} \cdot b_{r-1} + k_r \cdot b_r \in [-M, M]_b$. Similarly we have:

$$\begin{aligned} a_1 \cdot b_1 + \cdots + a_{r-2} \cdot b_{r-2} + k_{r-1} \cdot b_{r-1} + k_r \cdot b_r & \\ = a_1 \cdot b_1 + \cdots + a_{r-1} \cdot b_{r-1} + k_r \cdot b_r - (a_{r-1} - k_{r-1}) \cdot b_{r-1} & \\ = \alpha(w_{k_r}) - (a_{r-1} - k_{r-1}) \cdot b_{r-1}, & \end{aligned}$$

where $w_{k_r} \in T_0^{p_{k_r}}$. From Theorem 11 there is a set $T_0^{m_{k_{r-1}}} \supseteq T_0^{p_{k_r}}$ that satisfies condition (*) and has no hole in $a_1 \cdot b_1 + \cdots + a_{r-2} \cdot b_{r-2} + k_{r-1} \cdot b_{r-1} + k_r \cdot b_r$ such that

$$\alpha(w_{k_{r-1}}) = a_1 \cdot b_1 + \cdots + a_{r-2} \cdot b_{r-2} + k_{r-1} \cdot b_{r-1} + k_r \cdot b_r$$

for some $w_{k_{r-1}} \in T_0^{m_{k_{r-1}}}$. We can repeat Theorem 11 finitely many times (the number is $2 \cdot r_{r-1} \cdot a_{r-1}$) in order to construct the set $T_0^{p_{k_{r-1}}} \supseteq T_0^{m_{k_{r-1}}}$ such that it satisfies condition (*) and has no holes in integers of the form $a_1 \cdot b_1 + \cdots + a_{r-2} \cdot b_{r-2} + k_{r-1} \cdot b_{r-1} + k_r \cdot b_r$. Let us suppose that we have constructed the sets

$$T_0^1 = T_0^{p_{k_r}} \subseteq T_0^{p_{k_{r-1}}} \subseteq \cdots \subseteq T_0^{p_{k_{r-i}}},$$

for $i = 0, 1 \dots r-1$, such that $w_{k_{r-j}} \in T_0^{p_{k_{r-j}}}$ and the sets $T_0^{p_{k_{r-j}}}$ satisfy the well known properties for $j = 0, 1, \dots, i$,

$$\alpha(w_{k_{r-j}}) = a_1 \cdot b_1 + \cdots + a_{r-(j+1)} \cdot b_{r-(j+1)} + k_{r-j} \cdot b_{r-j} + \cdots + k_r \cdot b_r.$$

Then,

$$\begin{aligned} a_1 \cdot b_1 + \cdots + a_{r-(i+1)} \cdot b_{r-(i+1)} + k_{r-i} \cdot b_{r-i} + \cdots + k_r \cdot b_r - (a_{r-(i+1)} - k_{r-(i+1)}) \cdot b_{r-(i+1)} & \\ = \alpha(w_{k_{r-i}}) - (a_{r-(i+1)} - k_{r-(i+1)}) \cdot b_{r-(i+1)}. & \end{aligned}$$

From Theorem 11 there is a set $T_0^{m_{k_{r-(i+1)}}} \supseteq T_0^{p_{k_{r-i}}}$ such that

$$\alpha(w_{k_{r-(i+1)}}) = a_1 \cdot b_1 + \cdots + k_{r-(i+1)} \cdot b_{r-(i+1)} + k_{r-i} \cdot b_{r-i} + \cdots + k_r \cdot b_r$$

for some $w_{k_{r-(i+1)}} \in T_0^{k_{r-(i+1)}}$. We can repeat Theorem 11 finitely many times (the number is $2 \cdot k_{r-(i+1)} \cdot a_{r-(i+1)}$) in order to construct a set $T_0^{p_{k_{r-(i+1)}}} \supseteq T_0^{m_{k_{r-(i+1)}}}$ that will contain the integers of the above form.

By repeating the procedure for every $i = 1, 2, \dots, r-1$ we construct a set $T_0 = T_0^{p_{k_0}}$. It is clear that the set T_0 satisfies condition (**).

$\square$

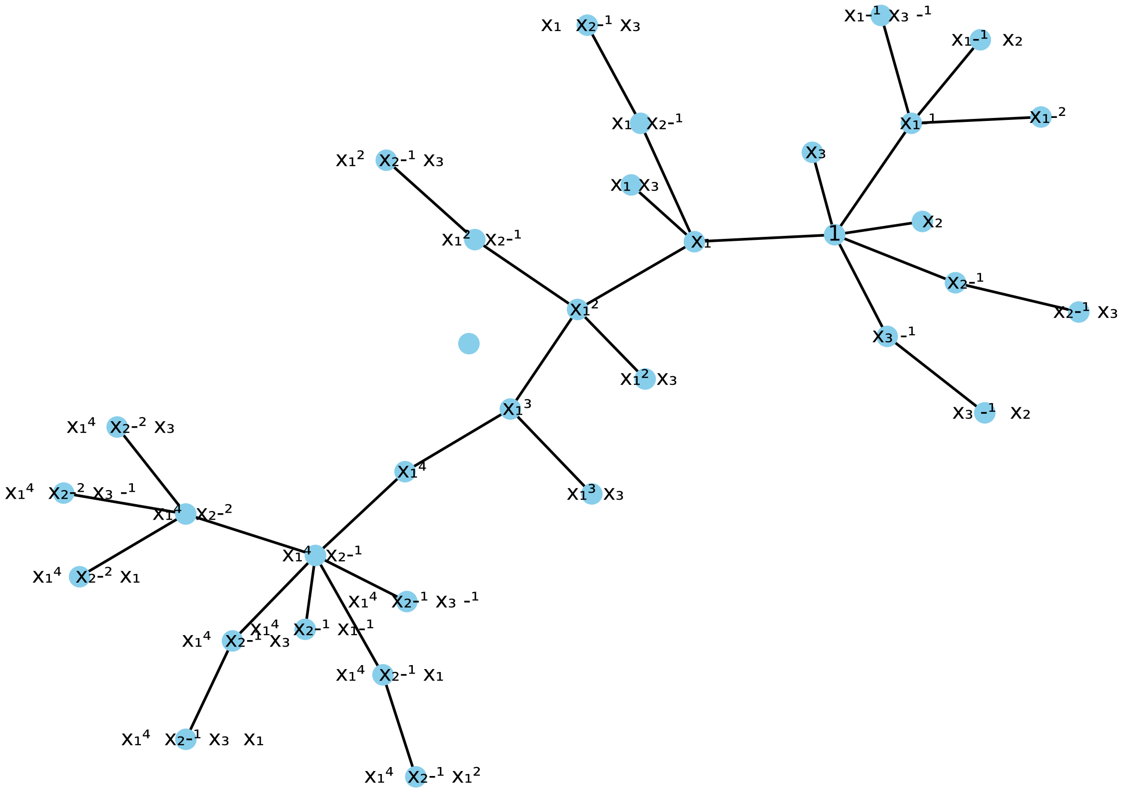

Figure 2. The case $b_1, b_2, b_3 = 4, 15, 2$. The vertices are words of the period of the transversal and the edges connect the initial segments.

Definition 14. A (right) Schreier Transversal for a group H in $\mathscr{F}_{s-1}$ is a set $T = \{t_1 = 1, \ldots, t_n\}$ of reduced words, such that each right coset of H in $\mathscr{F}_{s-1}$ contains a unique word of T, called a representative of this class, and all initial segments of the words $t_1, \ldots, t_n$ also lie in T. In particular $1 \in T$ is a representative of the class H and $Ht_i \neq Ht_j$ for all $i \neq j$.

For any $g \in \mathscr{F}_{s-1}$ denote by $\bar{g}$ the element of T with the property $Hg = H\bar{g}$. If $t_i \in T$ has a decomposition as a reduced word $t_i = x_{i_1}^{e_1} \cdots x_{i_k}^{e_k}$ (with $i_j = 1, \ldots, s-1, e_j = \pm 1$ and $e_j = e_{j+1}$ if $x_{i_j} = x_{i_j+1}$), then for every word $t_i \in T$ we have that

$$t_i = x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} \Rightarrow 1, x_{i_1}^{e_1}, x_{i_1}^{e_1} x_{i_2}^{e_2}, \ldots, x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k} \in T.$$

Theorem 15. *Let $T_0 = \{v_p, \ldots, v_2, v_1, v_0 = u_0 = 1, u_1, u_2, \ldots, u_p\}$ where $\alpha(u_i) = i \cdot b, i \in \mathbb{Z}, i \geq 0$, $\alpha(v_j) = -j \cdot b, j \in \mathbb{Z}, j \geq 0$, such that $T_0 \supseteq T_0^1$ and $M = pb$ satisfies condition*

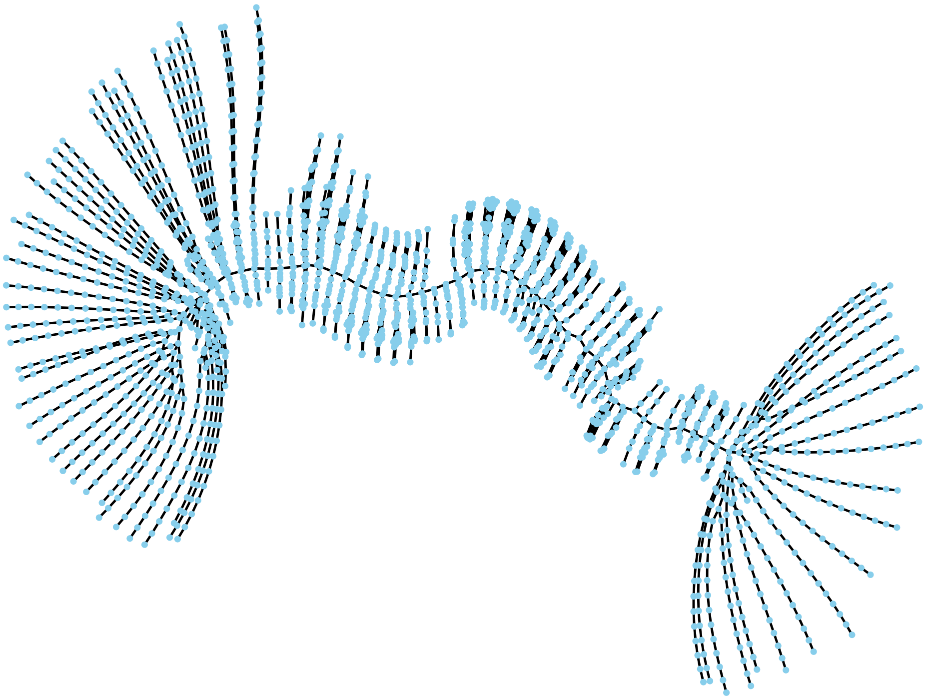

FIGURE 3. The case $b_1, b_2, b_3 = 64, 31, 45$. The vertices are words of the period of the transversal and the edges connect the initial segments. The words do not appear because of the large period of the transversal.

(∗∗). *Then the set*

$$T = \left\{ \begin{array}{l} \dots, v_p^2 \cdot v_1, v_p^2, v_p \cdot v_{p-1}, \dots, v_p \cdot v_2, v_p \cdot v_1, v_p, v_{p-1}, \dots v_2, v_1, v_0 = u_0 = 1, \\ u_1, u_2, \dots u_{p-1}, u_p, u_p \cdot u_1, u_p \cdot u_2, \dots, u_p \cdot u_{p-1}, u_p^2, u_p^2 \cdot u_1, \dots \end{array} \right\}$$

is a Schreier transversal of $\ker\alpha \leq \mathscr{F}_s$. *This means that the transversal element* t_κ *corresponding to* $\kappa \cdot b \in b \cdot \mathbb{Z}$ *is given by*

$$t_\kappa = \begin{cases} u_p^{\lfloor \kappa/p \rfloor} u_{\kappa - \lfloor \kappa/p \rfloor \cdot p} & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ v_p^{\lfloor -\kappa/p \rfloor} v_{-\kappa - \lfloor -\kappa/p \rfloor \cdot p} & \text{if } \kappa < 0. \end{cases} \tag{6}$$

Proof. Clearly $\alpha(T) = b\mathbb{Z}$. Additionally, from the construction of T the homomorphism α is injective. Also T satisfies the i.s.c. because the same holds for the set T_0 and because of the construction of T. We have $u_0 = 1 \in T$ because 1 is an initial segment of w_b and $w_b \in T_0^1 \subseteq T$. If $t_i, t_j \in \ker\alpha$, $i \neq j$, we also have $\ker\alpha \cdot t_i \neq \ker\alpha \cdot t_j$.

Indeed if $i \neq j$ then $\alpha(t_i) \neq \alpha(t_j)$ which gives $\alpha(t_i \cdot t_j^{-1}) \neq 0$ and we conclude $t_i \cdot t_j^{-1} \notin \ker\alpha$. This implies $\ker\alpha \cdot t_i \neq \ker\alpha \cdot t_j$ because $\alpha(t_i \cdot t_j^{-1}) = i - j \neq 0$.

Finally let $\ker\alpha \cdot w \in \mathscr{F}_s/\ker\alpha$. Then $\alpha(w) \in bZ = \alpha(T)$. This gives $\alpha(w) = \alpha(t_i)$ for a unique $t_i \in T$ (because α is injective). From this we conclude that $w^{-1} \cdot t_i \in \ker\alpha$ and $t_i \in \ker\alpha \cdot w$. $\square$

In Fig. 2 and 3 we present as graphs two Schreier transversal sets for different choices of b_1, b_2, b_3, $s = 3$.

Lemma 16 (Schreier's lemma). *Let T be a right Schreier transversal for H in $\mathscr{F}_{s-1}$ and set $\gamma(t,x) = tx\overline{tx}^{-1}$, $t \in T, x \in \{x_1, \dots, x_s\}$ and $tx \notin T$. Then H is freely generated by the set*

$$\{\gamma(t,x) : \gamma(t,x) \neq 1\}.$$

Using the above lemma we arrive at

Theorem 17. *We can assume without loss of generality that $b_i > 0$ for all $i = 1, \dots, s$. A set of free generators for the free groups* $\ker\alpha$ *is given by the set*

$$\left\{ \begin{array}{lll} u_p^{\left\lfloor \frac{\kappa}{p} \right\rfloor} u_{\kappa - \left\lfloor \frac{\kappa}{p} \right\rfloor \cdot p} \cdot x_i \cdot u^{-1}_{\kappa + \frac{b_i}{b} - \left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor \cdot p} u_p^{-\left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor}, & \kappa \geq 0 & \\ v_p^{\left\lfloor -\frac{\kappa}{p} \right\rfloor} v_{-\kappa - \left\lfloor -\frac{\kappa}{p} \right\rfloor \cdot p} \cdot x_i \cdot u^{-1}_{\kappa + \frac{b_i}{b} - \left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor \cdot p} u_p^{-\left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor}, & -\frac{b_i}{b} < \kappa < 0, & \begin{array}{l} \kappa \in \mathbb{Z} \\ 1 \leq i \leq s \end{array} \\ v_p^{\left\lfloor -\frac{\kappa}{p} \right\rfloor} v_{-\kappa - \left\lfloor -\frac{\kappa}{p} \right\rfloor \cdot p} \cdot x_i \cdot v^{-1}_{-\kappa - \frac{b_i}{b} - \left\lfloor -\frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor \cdot p} v_p^{-\left\lfloor -\frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor}, & \kappa \leq -\frac{b_i}{b} & \end{array} \right\}$$

Proof. For an integer $\kappa \cdot b \in b \cdot \mathbb{Z}$ and $1 \leq i \leq s$ we compute

$$\overline{t_\kappa x_i} = \begin{cases} u_p^{\left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor} u_{\kappa + \frac{b_i}{b} - \left\lfloor \frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor \cdot p} & \text{if } \kappa \cdot b + b_i > 0, \\ 1 & \text{if } \kappa \cdot b + b_i = 0, \\ v_p^{\left\lfloor -\frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor} v_{-\kappa - \frac{b_i}{b} - \left\lfloor -\frac{\kappa \cdot b + b_i}{p \cdot b} \right\rfloor \cdot p} & \text{if } \kappa \cdot b + b_i < 0. \end{cases}$$

But since $b_i > 0$, for $\kappa \geq 0$, we have $\kappa \cdot b + b_i > 0$.

$\square$

Notation. We denote by $R_{o,b}$ the kernel of the homomorphism

$$\alpha : \mathscr{F}_s = \langle x_1, x_2, \dots, x_s \rangle \longrightarrow b \cdot \mathbb{Z}.$$

Theorem 18. *The map*

$$* : \mathbb{Z}^b \times R_{o,b}/R'_{o,b} \longrightarrow R_{o,b}/R'_{o,b} : (t^{k \cdot b}, R'_{o,b} r) \longmapsto R'_{o,b} \cdot t_k \cdot r \cdot t_k^{-1}$$

is an action which is independent of the choice of the transversal.

Proof. Indeed, if T' is another transversal such that $\alpha(t_k) = k \cdot b$, $k \in \mathbb{Z}$, and $r \in R_{o,b}$, then $t_k \cdot (t'_k)^{-1} \in R_{o,b}$, thus $t_k = s_k \cdot t'_k$ for some $s_k \in R_{o,b}$. Then:

$$\begin{aligned} R_{o,b} \cdot t_k \cdot r \cdot t_k^{-1} &= R'_{o,b} \cdot s_k \cdot t'_k \cdot r \cdot (s_k \cdot t'_k)^{-1} \\ &= R'_{o,b} \cdot s_k \cdot t'_k \cdot r \cdot (t'_k)^{-1} \cdot s_k^{-1} \\ &= R'_{o,b} \cdot s_k \cdot R'_{o,b} \cdot s_k^{-1} \cdot R'_{o,b} \cdot t'_k \cdot r \cdot (t'_k)^{-1} \\ &= R'_{o,b} \cdot t'_k \cdot r \cdot (t'_k)^{-1}. \end{aligned}$$

□

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