

THE FROBENIUS NUMBER FOR THE TRIPLE OF THE 2-STEP STAR NUMBERS

TAKAO KOMATSU, RITIKA GOEL, AND NEHA GUPTA

ABSTRACT. In this paper, we give closed-form expressions of the Frobenius number for the triple of the 2-step star numbers $an(n-2) + 1$ for an integer $a \geq 4$. These numbers have been studied from different aspects for some a 's. These numbers can also be considered as variations of the well-known star numbers of the form $6n(n-1) + 1$. We also give closed-form expressions of the Sylvester number (genus) for the triple of the 2-step star numbers.

1. INTRODUCTION

The renowned Frobenius problem, also known as the coin exchange problem, is a well-known problem in number theory with applications in algebra, combinatorics, cryptography, and many other areas as well.

Given a set of positive integers $A = \{a_1, a_2, \dots, a_k\}$ with $k \geq 2$ and $\gcd(A) = 1$, the denominator [32], denoted by $d(n) = d(n; a_1, \dots, a_k)$, counts the number of nonnegative integer solutions (x_1, x_2, \dots, x_k) to the equation

$$a_1x_1 + a_2x_2 + \dots + a_kx_k = n. \quad (1.1)$$

For a fixed nonnegative integer p , the set $S_p(A)$ consists of all integers n for which this equation has more than p nonnegative integer solutions, i.e., $d(n) > p$ [20]. This set provides valuable insight into the distribution of representable integers. For more background on the number of representations, refer to sources such as [5, 7, 12, 37]. It is known that $\mathbb{N}_0 \setminus S_p(A)$ is finite if and only if $\gcd(A) = 1$. The largest integer in this finite complement is called the p -Frobenius number, denoted by $g_p(A)$, and this set's cardinality is the p -genus (or p -Sylvester number), denoted by $n_p(A)$.

When $p = 0$, the classical Frobenius number $g_0(A)$ and the Sylvester number $n_0(A)$ is recovered. The publication [26] provides a detailed account of the Frobenius problem. The problem is commonly attributed to Frobenius, though its historical origins trace back to earlier mathematicians. James Joseph Sylvester [33] first posed the problem in 1884 for $k = 2$, asking for the largest integer that cannot be

Date: Submitted 18 September 2024; accepted 7 March 2025.

2020 Mathematics Subject Classification. Primary 11D07; Secondary 05A15, 05A17, 05A19, 11B68, 11D04, 11P81.

Key words and phrases. Frobenius problem, Frobenius number, Sylvester number, Apéry set, 2-step star numbers.

¹N. Gupta is the corresponding author.

expressed as a nonnegative integer combination of two relatively prime numbers. Later that same year, Curran Sharp solved the problem, proving the now well-known formula:

$$g_0(a_1, a_2) = a_1 a_2 - a_1 - a_2, \quad (1.2)$$

Also, it is known that

$$n_0(a_1, a_2) = \frac{(a_1 - 1)(a_2 - 1)}{2}. \quad (1.3)$$

Frobenius' contribution was in generalizing the problem to arbitrary k , though an explicit formula for $k \geq 3$ remains elusive. In addition, for $k = 3$, Curtis [8] demonstrated that no finite set of polynomial formulas can compute $g_0(A)$ in general. Moreover, Ramírez Alfonsín [25] proved that determining $g_0(A)$ is NP-hard for $k \geq 3$. However, Tripathi [35] provided an exact (non-polynomial) formula for $g_0(A)$ when $k = 3$. An alternative method has also been studied for calculating Frobenius number by Tripathi and Vijay [34].

Recent advancements have focused on two primary directions: (i) efficient algorithms for computing Frobenius numbers, such as those developed by Kannan [11] and later refined through computational algebra methods (e.g., [9]), and (ii) explicit closed formulas for specific families of integers. In particular, explicit formulas have been determined for special cases, including arithmetic [4, 36, 39], geometric-like [23], Fibonacci [22], Mersenne [27], and triangular sequences [28], among others. Furthermore, when $p > 0$, the calculation of the p -Frobenius number becomes even more challenging. However, some progress has been made in this direction (see, e.g., [14, 19, 17, 18]). Notably, in [38], the p -Frobenius number of consecutive triples of star numbers is analysed.

A star number, denoted by S_n , is given by the formula

$$S_n = 6n(n - 1) + 1. \quad (1.4)$$

These numbers are also known as centered dodecagonal numbers and represent a centered hexagram, similar to the board used in Chinese checkers. A generalization of these numbers, called 2-step star numbers, is defined as

$$\mathfrak{S}_{a,n} = an(n - 2) + 1, \quad (n \geq 2), \quad (1.5)$$

where $a \geq 4$. The 2-step star numbers have been studied from different aspects. When $a = 4$ in (1.5), the first several values are given by

$$\{\mathfrak{S}_{4,n}\}_{n \geq 2} = 1, 13, 33, 61, 97, 141, 193, 253, 321, 397, 481, 573, \dots$$

[31, A082109]. When $a = 5$ in (1.5), $\mathfrak{S}_{5,n}$ denotes the number of entries required to describe the options and constraints in Don Knuth's formulation of the n nonattacking queens on an $n \times n$ board problem [31, A134593]. Similarly, $\mathfrak{S}_{7,n}$ appears in [31, A131878].

In this paper, we derive a closed-form expression for the Frobenius number of the three consecutive 2-step star numbers $\{\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}\}$ for $a \geq 5$ and $n \geq 3$. We also provide an explicit formula for their Sylvester number.

2. PRILIMINARIES

We introduce the p -Apéry set [2] to obtain the formulas in this paper.

Let p be a nonnegative integer. For a set of positive integers $A = \{a_1, a_2, \dots, a_k\}$ with $\gcd(A) = 1$ and $a_1 = \min(A)$ we denote by

$$\text{Ap}_p(A) = \text{Ap}_p(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\},$$

the p -Apéry set of A , where each positive integer $m_i^{(p)}$ ($0 \leq i \leq a_1 - 1$) satisfies the conditions:

$$(i) m_i^{(p)} \equiv i \pmod{a_1}, \quad (ii) m_i^{(p)} \in S_p(A), \quad (iii) m_i^{(p)} - a_1 \notin S_p(A).$$

Note that $m_0^{(0)}$ is defined to be 0.

It follows that for each p ,

$$\text{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

One of the convenient formulas to obtain the p -Frobenius number is via the elements in the corresponding p -Apéry set [16].

Lemma 1. *Let $\gcd(a_1, \dots, a_k) = 1$ with $a_1 = \min\{a_1, \dots, a_k\}$. Then we have*

$$g_p(a_1, \dots, a_k) = \max_{0 \leq j \leq a_1-1} m_j^{(p)} - a_1, \quad (2.1)$$

$$n_p(a_1, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j^{(p)} - \frac{a_1 - 1}{2}. \quad (2.2)$$

Remark. When $p = 0$, the formulas (2.1) and (2.2) are essentially due to Brauer and Shockley [6], and Selmer [30], respectively. More general formulas, including the p -power sum and the p -weighted sum, can be seen in [15, 16].

Lemma 2. *Let a and n be integers with $a \geq 4$, $n \geq 2$, then we have*

$$\gcd(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) = 1.$$

Proof. From the properties of \gcd , we know that

$$\begin{aligned} \gcd(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \gcd(\mathfrak{S}_{a,n} - \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+1}) \\ &= \gcd(a(2n-1), a(2n+1), a(n-1)(n+1)+1) \\ &= \gcd(a, a(n-1)(n+1)+1) \\ &= 1. \end{aligned}$$

□

It is important to note that the elements of this triple are not necessarily pairwise relatively prime. A significant result established by Johnson in [10] demonstrates that the Frobenius problem for an arbitrary set of integers can always be

reduced to the Frobenius problem for sets where all elements are pairwise relatively prime. Specifically, Johnson's formula provides a method for computing the Frobenius number in such cases:

$$g_0(a_1, a_2, a_3) = d g_0\left(\frac{a_1}{d}, \frac{a_2}{d}, a_3\right) + (d-1)a_3, \quad (2.3)$$

where $d = \gcd(a_1, a_2)$.

3. MAIN RESULT

The following cases are exceptional and these cases will not be considered in this paper:

$$\begin{aligned} a = 5 & \quad \text{and} \quad n \equiv 0, 3 \pmod{4}, n = 6; \\ a = 6 & \quad \text{and} \quad n \equiv 1, 3 \pmod{4}, n = 4; \\ a = 7 & \quad \text{and} \quad n \equiv 0 \pmod{4}, n = 5, 9; \\ a = 9 & \quad \text{and} \quad n = 4, 8, 12; \\ a = 10 & \quad \text{and} \quad n = 3, 5, 7; \\ a = 11 & \quad \text{and} \quad n = 4; \\ a = 13 & \quad \text{and} \quad n = 4; \\ a = 14 & \quad \text{and} \quad n = 3; \\ a = 18 & \quad \text{and} \quad n = 3. \end{aligned} \quad (3.1)$$

Note that, at least the last four exceptional cases in (3.1) can be easily addressed using available algorithms. For example, for $a = 13$ and $n = 4$ we can use Johnson's formula (2.3) to get:

$$g_0(105, 196, 313) = 7g_0(15, 28, 313) + 1878 = 7g_0(15, 28) + 1878,$$

where $g_0(15, 28)$ can be calculated by using (1.2).

We now establish the Frobenius number of the three consecutive 2-step star numbers which is as follows.

Theorem 1. *Let a and n be integers with $a \geq 5$, $n \geq 3$ and not in (3.1) above. When $a \equiv 0 \pmod{4}$, we have*

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (2n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 4(n-1)}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $a \equiv 1 \pmod{4}$, we have

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) = \begin{cases} (4n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)-2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 0 \pmod{4}; \\ (4n-1)\mathfrak{S}_{a,n+1} + \frac{a(n-2)-2n-1}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 1 \pmod{4}; \\ (2n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)+4n-4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 2 \pmod{4}; \\ (3n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)+2n-3}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

When $a \equiv 2 \pmod{4}$, we have

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) = \begin{cases} (2n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)+4n-4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 0 \pmod{2}; \\ (4n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)-2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

When $a \equiv 3 \pmod{4}$, we have

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) = \begin{cases} (4n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)-2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 0 \pmod{4} \\ (3n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)+2n-3}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 1 \pmod{4}; \\ (2n-2)\mathfrak{S}_{a,n+1} + \frac{a(n-2)+4n-4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 2 \pmod{4}; \\ (4n-1)\mathfrak{S}_{a,n+1} + \frac{a(n-2)-2n-1}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Remark. By a result of Brauer and Shockley [6], the Frobenius number $g(a_1, a_2, a_3)$ for $a_1 < a_2 < a_3$ is always of the form

$$g_0(a_1, a_2, a_3) = a_2x_2 + a_3x_3 - a_1$$

for some $x_2, x_3 \in \mathbb{N}$. Furthermore, for $a = 1$, the sequence of 2-step star numbers reduces to the sequence of perfect squares. This leads to an interesting connection with [21], where exact formulas for the Frobenius number of triples of consecutive perfect squares have been established.

3.1. The construction of the Apéry set. We shall use the following structure of the 0-Apéry set frequently, but it is applicable to many other triplets as well. Hence, we establish a general setting. Consider the triple $A := \{a_1, a_2, a_3\}$, where $3 \leq a_1 < a_2 < a_3$ and $\gcd(a_1, a_2, a_3) = 1$. For simplicity, put $t_{y,z} := ya_2 + za_3$ ($y, z \geq 0$). In Table 1, denote (y, z) by its position.

TABLE 1. $\text{Ap}_0(a_1, a_2, a_3)$

$(0, 0)$...	$(y_1 - 1, 0)$	$(y_1, 0)$	$(y_0 - 1, 0)$
\vdots		\vdots	\vdots			\vdots
$(0, z_0 - 1)$...	$(y_1 - 1, z_0 - 1)$	$(y_1, z_0 - 1)$	$(y_0 - 1, z_0 - 1)$
$(0, z_0)$...	$(y_1 - 1, z_0)$				
\vdots		\vdots				
\vdots		\vdots				
$(0, z_1 - 1)$...	$(y_1 - 1, z_1 - 1)$				

Here, (y_0, z_0) and (y_1, z_1) satisfy the following conditions.

- (i) $0 \leq y_1 \leq y_0$ and $0 \leq z_0 \leq z_1$,
- (ii) $a_1 = y_0z_0 + y_1(z_1 - z_0)$,
- (iii) $y_0a_2 - (z_1 - z_0)a_3 \equiv 0 \pmod{a_1}$ and $y_0a_2 > (z_1 - z_0)a_3$,
- (iv) $y_1a_2 + z_0a_3 \equiv 0 \pmod{a_1}$.

Then, every element of the 0-Apéry set appears exactly once in the frame of Table 1. This fact is proved by considering the sequence $\{\ell a_2 \pmod{a_1}\}_{\ell=0}^{a_1-1}$ as follows.

Case 1. Assume that $\gcd(z_0, z_1) = 1$ and $\gcd(a_1, a_2) = 1$. The sequence

$$\{\ell a_2 \pmod{a_1}\}_{\ell=0}^{a_1-1}$$

begins from the position $(0, 0)$ and takes all elements in that row, and moves to another row by increasing $(z_1 - z_0)$ columns beginning from $(0, z_1 - z_0)$ by using the rule (iii) above. If it is still the longer row of length y_0 , the sequence takes all elements in that row, and moves to another row by increasing $(z_1 - z_0)$ columns again. If it is the shorter row of length y_1 , the sequence takes all elements in that row, and moves to another row by decreasing z_0 columns by the rule (iv). Namely,

$$\begin{aligned} \{\ell a_2 \pmod{a_1}\}_{\ell \geq 0} = & \\ & (0, 0), (1, 0), \dots, (y_0 - 1, 0), \\ & (0, z_1 - z_0), (1, z_1 - z_0), \dots, (y_0 - 1, z_1 - z_0), \quad \text{by (iii)} \\ & (0, 2(z_1 - z_0)), (1, 2(z_1 - z_0)), \dots, (y_0 - 1, 2(z_1 - z_0)), \quad \text{by (iii)} \\ & \dots \\ & (0, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil - 1)(z_1 - z_0), (1, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil - 1)(z_1 - z_0), \\ & \dots, (y_0 - 1, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil - 1)(z_1 - z_0), \quad \text{by (iii)} \\ & (0, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil (z_1 - z_0)), (1, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil (z_1 - z_0)), \\ & \dots, (y_1 - 1, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil (z_1 - z_0)), \quad \text{by (iii)} \\ & (0, \left\lceil \frac{z_0}{z_1 - z_0} \right\rceil (z_1 - z_0) - z_0), \dots \quad \text{by (iv)}. \end{aligned}$$

Here, for simplicity, we used the position (y, z) instead of $t_{y,z}$. Thereafter, the sequence repeatedly runs over all the elements of a longer row, then moves to another row by increasing by $(z_1 - z_0)$ columns, or runs over all the elements of a shorter row then moves to another row by decreasing by z_0 columns. Since $\gcd(z_0, z_1) = 1$, all elements of the Apéry set are traced only once without overlap, and finally the end returns to the position $(0, 0)$.

TABLE 2. increasing by condition (iii)

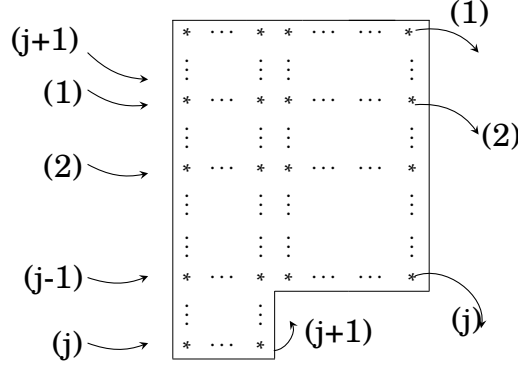
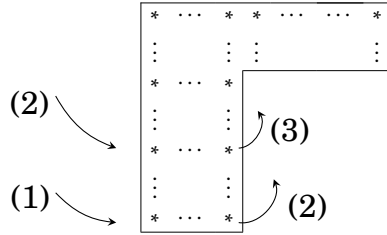


TABLE 3. decreasing by condition (iv)



By $\gcd(a_1, a_2) = 1$, we have $\{\ell a_2 \pmod{a_1}\}_{\ell=0}^{a_1-1} = \{\ell \pmod{a_1}\}_{\ell=0}^{a_1-1}$. Thus, every element of the 0-Apéry set appears exactly once in the frame of Table 1.

Case 2. Assume that $\gcd(z_0, z_1) \neq 1$ or $\gcd(a_1, a_2) \neq 1$ (including the case where $z_0 = z_1$, so $y_0 = y_1$). In fact, if $\gcd(a_1, a_2) = d > 1$, then by (iii) and (iv) with $\gcd(a_3, d) = 1$ (note that otherwise, $\gcd(a_1, a_2, a_3) \neq 1$), we have $\gcd(z_1 - z_0, d) = d$ and $\gcd(z_0, d) = d$, respectively. Hence, $\gcd(z_0, z_1) = d$. Similarly, if $\gcd(z_0, z_1) = d > 1$, then $\gcd(a_1, a_2) = d$.

In this case, the sequence $\{\ell a_2 \pmod{a_1}\}_{\ell=0}^{a_1-1}$ is divided into d non-intersecting subsequences $\{\ell a_2 + j a_3 \pmod{a_1}\}_{\ell=0}^{a_1/d-1}$ ($j = 0, 1, \dots, d-1$) with the length of period a_1/d . Notice that by $d \mid a_2$ and $\gcd(a_3, d) = 1$, $\ell a_2 + j a_3 \equiv j \pmod{d}$ ($j = 0, 1, \dots, d-1$). We also have

$$\frac{a_1}{d} a_2 + j a_3 = \frac{a_2}{d} a_1 + j a_3 \equiv j a_3 \pmod{a_1}.$$

Each subsequence moves over the elements of the longer row, and the shorter row by increasing and decreasing by the columns, respectively, as shown in Case 1. Thus, in this case too, every element of the 0-Apéry set appears exactly once in the frame of Table 1. Namely, for $j = 0, 1, \dots, z_1 - z_0$,

$$(0, j), (1, j), \dots, (y_0 - 1, j),$$

$$\begin{aligned}
& (0, z_1 - z_0 + j), (1, z_1 - z_0 + j), \dots, (y_0 - 1, z_1 - z_0 + j), \quad \text{by (iii)} \\
& (0, 2(z_1 - z_0) + j), (1, 2(z_1 - z_0) + j), \dots, (y_0 - 1, 2(z_1 - z_0) + j), \quad \text{by (iii)} \\
& \dots \\
& (0, \left(\left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor - 1\right)(z_1 - z_0) + j), (1, \left(\left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor - 1\right)(z_1 - z_0) + j), \\
& \quad \dots, (y_0 - 1, (y_0 - 1, \left(\left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor - 1\right)(z_1 - z_0) + j), \quad \text{by (iii)} \\
& (0, \left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor (z_1 - z_0) + j), (1, \left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor (z_1 - z_0) + j), \\
& \quad \dots, (y_1 - 1, \left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor (z_1 - z_0) + j), \quad \text{by (iii)} \\
& (0, \left\lfloor \frac{z_0}{z_1 - z_0} \right\rfloor (z_1 - z_0) - z_0 + j), \dots \quad \text{by (iv)}.
\end{aligned}$$

From the structure of Table 1, the largest element of the 0-Apéry set is either at $(y_0 - 1, z_0 - 1)$ or at $(y_1 - 1, z_1 - 1)$. By comparing both values directly, if $(y_0 - y_1)a_2 - (z_1 - z_0)a_3 > 0$, then the value at $(y_0 - 1, z_0 - 1)$ is the largest and

$$g_0(a_1, a_2, a_3) = (y_0 - 1)a_2 + (z_0 - 1)a_3 - a_1.$$

Otherwise, $(y_1 - 1, z_1 - 1)$ is the largest and

$$g_0(a_1, a_2, a_3) = (y_1 - 1)a_2 + (z_1 - 1)a_3 - a_1.$$

3.2. Proof of Theorem 1. We discuss each case for the possible values of a and calculate g_0 accordingly.

3.2.1. The case $a \equiv 0 \pmod{4}$. For $a = 4\kappa$, in order to satisfy all the conditions (i), (ii), (iii) and (iv), we can determine

$$(y_0 - 1, z_0 - 1) = (4n - 1, (\kappa - 1)n - 2\kappa) = \left(4n - 1, \frac{a(n - 2) - 4n}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = (2n - 2, (\kappa + 1)n - 2\kappa - 1) = \left(2n - 2, \frac{a(n - 2) + 4n}{4} - 1\right).$$

Comparing the two candidates, $t_{2n-2, (\kappa+1)n-2\kappa-1} - t_{4n-1, (\kappa-1)n-2\kappa} = 2an^2 + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{2n-2, (\kappa+1)n-2\kappa-1}$ is the largest element of the Apéry set. So,

$$\begin{aligned}
g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\
&= (2n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n - 2) + 4(n - 1)}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}.
\end{aligned}$$

3.2.2. *The case $a \equiv 1 \pmod{4}$.* For $a = 4\kappa + 1$, when $n \equiv 0 \pmod{4}$, in order to satisfy all the conditions, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 7)n - 8\kappa}{4}\right) = \left(4n - 1, \frac{a(n - 2) - 2(4n - 1)}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(4n - 2, \frac{(4\kappa + 1)n - 8\kappa - 4}{4}\right) = \left(4n - 2, \frac{a(n - 2) - 2}{4}\right).$$

Comparing the two candidates, $t_{4n-2, \frac{(4\kappa+1)n-8\kappa-4}{4}} - t_{4n-1, \frac{(4\kappa-7)n-8\kappa}{4}} = 2an^3 + 2an^2 - 2(a-1)n + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{4n-2, \frac{a(n-2)-2}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (4n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n - 2) - 2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $n \equiv 1 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 1)n - 8\kappa - 3}{4}\right) = \left(4n - 1, \frac{a(n - 2) - 2n - 1}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(n - 2, \frac{(4\kappa + 7)n - 8\kappa - 7}{4}\right) = \left(n - 2, \frac{a(n - 2) + 6n - 5}{4}\right).$$

Comparing the two candidates, $t_{4n-1, \frac{(4\kappa-1)n-8\kappa-3}{4}} - t_{n-2, \frac{(4\kappa+7)n-8\kappa-7}{4}} = an^3 - 2an^2 - (a-1)n - a + 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{4n-1, \frac{a(n-2)-2n-1}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (4n - 1)\mathfrak{S}_{a,n+1} + \frac{a(n - 2) - 2n - 1}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $n \equiv 2 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 3)n - 8\kappa - 2}{4}\right) = \left(4n - 1, \frac{a(n - 2) - 4n}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(2n - 2, \frac{(4\kappa + 5)n - 8\kappa - 6}{4}\right) = \left(2n - 2, \frac{a(n - 2) + 4n - 4}{4}\right).$$

Comparing the two candidates, $t_{2n-2, \frac{(4\kappa+5)n-8\kappa-6}{4}} - t_{4n-1, \frac{(4\kappa-3)n-8\kappa-2}{4}} = 2an^2 + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{2n-2, \frac{a(n-2)+4n-4}{4}}$ is the largest element of the Apéry set. So,

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) =$$

$$= (2n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 4n - 4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}.$$

When $n \equiv 3 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 5)n - 8\kappa - 1}{4}\right) = \left(4n - 1, \frac{a(n-2) - 6n + 1}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(3n - 2, \frac{(4\kappa + 3)n - 8\kappa - 5}{4}\right) = \left(3n - 2, \frac{a(n-2) + 2n - 3}{4}\right).$$

Comparing the two candidates, $t_{3n-2, \frac{(4\kappa+3)n-8\kappa-5}{4}} - t_{4n-1, \frac{(4\kappa-5)n-8\kappa-1}{4}} = an^3 + 2an^2 - (a-1)n + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{3n-2, \frac{a(n-2)+2n-3}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (3n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 2n - 3}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

3.2.3. The case $a \equiv 2 \pmod{4}$. For $a = 4\kappa + 2$, when $n \equiv 0 \pmod{2}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(2\kappa - 1)n - 4\kappa - 2}{2}\right) = \left(4n - 1, \frac{a(n-2) - 4n}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(2n - 2, \frac{(2\kappa + 3)n - 4\kappa - 4}{2}\right) = \left(2n - 2, \frac{a(n-2) + 4n - 4}{4}\right).$$

Comparing the two candidates, $t_{2n-2, \frac{(4\kappa+3)n-8\kappa-6}{4}} - t_{4n-1, \frac{(4\kappa-3)n-8\kappa-2}{4}} = 2an^2 + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{2n-2, \frac{a(n-2)+4n-4}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (2n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 4n - 4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $n \equiv 1 \pmod{2}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(2\kappa - 3)n - 4\kappa - 1}{2}\right) = \left(4n - 1, \frac{a(n-2) - 8n + 2}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(4n - 2, \frac{(2\kappa + 1)n - 4\kappa - 3}{2}\right) = \left(4n - 2, \frac{a(n-2) - 2}{4}\right).$$

Comparing the two candidates, $t_{4n-2, \frac{(4\kappa+1)n-8\kappa-4}{4}} - t_{4n-1, \frac{(4\kappa-7)n-8\kappa}{4}} = 2an^3 + 2an^2 - 2(a-1)n + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{4n-2, \frac{a(n-2)-2}{4}}$ is the largest element of the Apéry set. So,

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) =$$

$$= (4n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) - 2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}.$$

3.2.4. *The case $a \equiv 3 \pmod{4}$.* For $a = 4\kappa + 3$, when $n \equiv 0 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 5)n - 8\kappa - 4}{4}\right) = \left(4n - 1, \frac{a(n-2) - 8n + 2}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(4n - 2, \frac{(4\kappa + 3)n - 8\kappa - 8}{4}\right) = \left(4n - 2, \frac{a(n-2) - 2}{4}\right).$$

Comparing the two candidates, $t_{4n-2, \frac{(4\kappa+3)n-8\kappa-8}{4}} - t_{4n-1, \frac{(4\kappa-5)n-8\kappa-4}{4}} = 2an^3 + 2an^2 + 2n + a - 2an - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{4n-2, \frac{a(n-2)-2}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (4n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) - 2}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $n \equiv 1 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 3)n - 8\kappa - 5}{4}\right) = \left(4n - 1, \frac{a(n-2) - 6n + 1}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(3n - 2, \frac{(4\kappa + 5)n - 8\kappa - 9}{4}\right) = \left(3n - 2, \frac{a(n-2) + 2n - 3}{4}\right).$$

Comparing the two candidates, $t_{3n-2, \frac{(4\kappa+5)n-8\kappa-9}{4}} - t_{4n-1, \frac{(4\kappa-3)n-8\kappa-5}{4}} = an^3 + 2an^2 - (a-1)n + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{3n-2, \frac{a(n-2)+2n-3}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (3n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 2n - 3}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

When $n \equiv 2 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa - 1)n - 8\kappa - 6}{4}\right) = \left(4n - 1, \frac{a(n-2) - 4n}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(2n - 2, \frac{(4\kappa + 7)n - 8\kappa - 10}{4}\right) = \left(2n - 2, \frac{a(n-2) + 4n - 4}{4}\right).$$

Comparing the two candidates, $t_{2n-2, \frac{(4\kappa+7)n-8\kappa-10}{4}} - t_{4n-1, \frac{(4\kappa-1)n-8\kappa-6}{4}} = 2an^2 + a - 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{2n-2, \frac{a(n-2)+4n-4}{4}}$ is the largest element of the Apéry set. So,

$$g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) =$$

$$= (2n - 2)\mathfrak{S}_{a,n+1} + \frac{a(n-2) + 4n - 4}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}.$$

When $n \equiv 3 \pmod{4}$, we can determine

$$(y_0 - 1, z_0 - 1) = \left(4n - 1, \frac{(4\kappa + 1)n - 8\kappa - 7}{4}\right) = \left(4n - 1, \frac{a(n-2) - 2n - 1}{4}\right)$$

and

$$(y_1 - 1, z_1 - 1) = \left(n - 2, \frac{(4\kappa + 9)n - 8\kappa - 11}{4}\right) = \left(n - 2, \frac{a(n-2) + 6n - 5}{4}\right).$$

Comparing the two candidates, $t_{4n-1, \frac{(4\kappa-1)n-8\kappa-3}{4}} - t_{n-2, \frac{(4\kappa+7)n-8\kappa-7}{4}} = an^3 - 2an^2 - (a-1)n - a + 2$. Since $n \geq 3$ and $a \geq 5$, the above value is positive. Hence, $t_{4n-1, \frac{a(n-2)-2n-1}{4}}$ is the largest element of the Apéry set. So,

$$\begin{aligned} g_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= (4n - 1)\mathfrak{S}_{a,n+1} + \frac{a(n-2) - 2n - 1}{4}\mathfrak{S}_{a,n+2} - \mathfrak{S}_{a,n}. \end{aligned}$$

4. SYLVESTER NUMBER (GENUS)

Observing Table 1, the sum of the elements of the p -Apéry set is made by

$$\begin{aligned} \sum_{j=0}^{a_1-1} m_j^{(0)} &= \sum_{z=0}^{z_0-1} \sum_{y=0}^{y_0-1} t_{y,z} + \sum_{z=z_0}^{z_1-1} \sum_{y=0}^{y_1-1} t_{y,z} \\ &= \frac{1}{2} \left((y_0 z_0 (a_2 (y_0 - 1) + a_3 (z_0 - 1)) - y_1 (z_0 - z_1) (a_2 (y_1 - 1) + a_3 (z_0 + z_1 - 1))) \right) \\ &= \frac{a_1}{2} (l a_1 - a_2 + (z_1 - 1) a_3 + (k - l) y_0 z_0). \end{aligned}$$

Here, we used the conditions (ii), (iii) and (iv) as $y_0 a_2 - (z_1 - z_0) a_3 = k a_1$ and $y_1 a_2 + z_0 a_3 = l a_1$ for some integers k and l . By Lemma 1 (2.2), we have

$$\begin{aligned} n_0(a_1, a_2, a_3) &= \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j^{(0)} - \frac{a_1 - 1}{2} \\ &= \frac{1}{2} \left((l - 1) a_1 - a_2 + (z_1 - 1) a_3 + (k - l) y_0 z_0 + 1 \right). \end{aligned} \quad (4.1)$$

In the case of $(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2})$, we apply the condition (iii) to get $y_0 = 4n$ and $z_1 - z_0 = 2n - 1$. So, $k = 2n + 1$. In addition, from the condition (iv), we get

$$l = \begin{cases} \frac{a(n+2)+2n-1}{4} & \text{if } (a, n) \equiv (1, 1), (3, 3) \pmod{4}; \\ \frac{a(n+2)+4n}{4} & \text{if } a \equiv 0, (a, n) \equiv (1, 2), (2, 0), (2, 2), (3, 2) \pmod{4}; \\ \frac{a(n+2)+6n+1}{4} & \text{if } (a, n) \equiv (1, 3), (3, 1) \pmod{4}; \\ \frac{a(n+2)+8n+2}{4} & \text{if } (a, n) \equiv (1, 0), (2, 1), (2, 3), (3, 0) \pmod{4}. \end{cases}$$

Notice that

$$l = \frac{a(n+2) + \beta n + \gamma}{4} \iff z_0 - 1 = \frac{a(n-2) - \beta n + \gamma}{4}$$

for some integers β and γ . Hence,

$$\begin{aligned} n_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) &= \\ &= \frac{1}{2} \left((l-1)\mathfrak{S}_{a,n} - \mathfrak{S}_{a,n+1} + (z_0 + 2n - 2)\mathfrak{S}_{a,n+2} + 4n(2n-l+1)z_0 + 1 \right). \end{aligned}$$

Theorem 2. *Let a and n be integers with $a \geq 5$, $n \geq 3$ and not in (3.1). Then we have*

$$n_0(\mathfrak{S}_{a,n}, \mathfrak{S}_{a,n+1}, \mathfrak{S}_{a,n+2}) = \begin{cases} \frac{1}{8} \left((a^2 + 16a - 12)n^3 - (12a - 8)n^2 - (4a^2 + 14a - 23)n + 4a - 10 \right) \\ \quad \text{if } (a, n) \equiv (1, 1), (3, 3) \pmod{4}; \\ \frac{1}{8} \left((a^2 + 16a - 16)n^3 - 12an^2 - (4a^2 + 14a - 24)n + 4a - 8 \right) \\ \quad \text{if } a \equiv 0, \text{ or } (a, n) \equiv (1, 2), (2, 0), (2, 2), (3, 2) \pmod{4}; \\ \frac{1}{8} \left((a^2 + 16a - 12)n^3 - (12a + 8)n^2 - (4a^2 + 14a - 23)n + 4a - 6 \right) \\ \quad \text{if } (a, n) \equiv (1, 3), (3, 1) \pmod{4}; \\ \frac{1}{8} \left((a^2 + 16a)n^3 - (12a + 16)n^2 - (4a^2 + 14a - 20)n + 4a - 4 \right) \\ \quad \text{if } (a, n) \equiv (1, 0), (2, 1), (2, 3), (3, 0) \pmod{4}. \end{cases}$$

5. p -NUMERICAL SEMIGROUP

As indicated in [13, 14, 16, 17, 18, 19], the p -Frobenius numbers and the p -genus can be obtained by using the p -Apéry set, and each element of the p -Apéry set can be determined uniquely from the corresponding element of the $(p-1)$ -Apéry set. However, the situation becomes more complicated when p becomes larger. In most cases, detailed discussion is required, depending on the particular case.

REFERENCES

- [1] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, Fifth edition (Including illustrations by Karl H. Hofmann), Springer-Verlag, Berlin, 2014, ISBN: 978-3-662-44204-3; 978-3-662-44205-0
- [2] R. Apéry, Sur les branches superlinéaires des courbes algébriques, C. R. Acad. Sci. Paris **222** (1946), 1198–1200.
- [3] A. Assi, M. D’Anna and P. A. Garcia-Sanchez, *Numerical semigroups and applications*, 2nd extended and revised edition, RSME Springer Series 3, Springer (2020).
- [4] P.T. Bateman, Remark on a recent note on linear forms, Amer. Math. Monthly **65** (1958) 517518.
- [5] D. S. Binner, Generalization of a result of Sylvester related to the Frobenius coin problem, J. Integer Seq. **24** (2021), no. 8, Art. 21.8.4, 14 pp.
- [6] A. Brauer and B. M. Shockley, On a problem of Frobenius, J. Reine Angew. Math. **211** (1962), 215–220.
- [7] A. Cayley, On a problem of double partitions, Philos. Mag. **XX** (1860), 337–341.

- [8] F. Curtis, On formulas for the Frobenius number of a numerical semigroup, *Math. Scand.* **67** (1990), 190–192.
- [9] D. Einstein, D. Lichtblau, A. Strzebonski, and S. Wagon, Frobenius numbers by lattice point enumeration, *Integers* 7.1 (2007): A15.
- [10] S.M. Johnson, A linear diophantine problem, *Canad. J. Math.* **12** (1960) 390398.
- [11] R. Kannan, Lattice translates of a polytope and the Frobenius problem, *Combinatorica* **12.2** (1992): 161–177.
- [12] T. Komatsu, On the number of solutions of the Diophantine equation of Frobenius—General case, *Math. Commun.* **8** (2003), 195–206.
- [13] T. Komatsu, The Frobenius number for sequences of triangular numbers associated with number of solutions, *Ann. Comb.* **26** (2022) 757–779.
- [14] T. Komatsu, The Frobenius number associated with the number of representations for sequences of repunits, *C. R. Math., Acad. Sci. Paris* **361** (2023), 73–89. <https://doi.org/10.5802/crmath.394>
- [15] T. Komatsu, Sylvester power and weighted sums on the Frobenius set in arithmetic progression, *Discrete Appl. Math.* **315** (2022), 110–126.
- [16] T. Komatsu, On the determination of p -Frobenius and related numbers using the p -Apéry set, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **118** (2024), Article 58, 17 pages. DOI: 10.1007/s13398-024-01556-5
- [17] T. Komatsu and J. Mu, p -numerical semigroups of Pell triples, *J. Ramanujan Math. Soc.* (in press).
- [18] T. Komatsu and C. Pita-Ruiz, The Frobenius number for Jacobsthal triples associated with number of solutions, *Axioms* **12** (2023), no. 2, Article 98, 18 pp.
- [19] T. Komatsu and H. Ying, The p -Frobenius and p -Sylvester numbers for Fibonacci and Lucas triplets, *Math. Biosci. Eng.* **20** (2023), No.2, 3455–3481.
- [20] T. Komatsu and H. Ying, p -numerical semigroups with p -symmetric properties, *J. Algebra Appl.* 2024.
- [21] M. Lepilov, J. O’Rourke and I. Swanson, Frobenius numbers of numerical semigroups generated by three consecutive squares or cubes, *Semigroup Forum* **91** (2015), No. 1, 238–259.
- [22] J. M. Marin, J. R. Alfonsín, and M. P. Revuelta, On the Frobenius number of Fibonacci numerical semigroups, arXiv preprint math/0606717 (2006).
- [23] D. C. Ong, and V. Ponomarenko, The Frobenius number of geometric sequences, *Integers*, **8.1** (2008), Article-A33.
- [24] S. Plouffe, *Approximations of generating functions and a few conjectures*, Dissertation, Université du Québec à Montréal, 1992; arXiv:0911.4975v1 (2009).
- [25] J.L. Ramírez Alfonsín, Complexity of the Frobenius problem, *Combinatorica* **16** (1996), 143147.
- [26] J.L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford Lecture Ser. Math. Appl., vol. 30, Oxford Univ. Press, Oxford, 2005,
- [27] J.C. Rosales, M.B. Branco, D. Torrão, The Frobenius problem for Mersenne numerical semigroups, *Math. Z.* **286** (2017) 741749
- [28] A. M. Robles-Perez and J. C. Rosales, The Frobenius number for sequences of triangular and tetrahedral numbers, *J. Number Theory* **186** (2018), 473–492.
- [29] J. C. Rosales and P. A. Garcia-Sanchez, *Numerical semigroups*, Developments in Mathematics, 20. Springer, New York, 2009.
- [30] E. S. Selmer, On the linear diophantine problem of Frobenius, *J. Reine Angew. Math.* **293/294** (1977), 1–17.
- [31] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, available at oeis.org. (2024).
- [32] J. J. Sylvester, On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order, *Amer. J. Math.* **5** (1882), 119136.
- [33] J. J. Sylvester, Problem 7382, in W. J. C. Miller, ed., *Mathematical Questions, with their Solutions, from the “Educational Times”*, 41 (1884), p. 21. Solution by W. J. Curran Sharp.

- [34] A. Tripathi, and S. Vijay, On a generalization of the coin exchange problem for three variables, *Journal of Integer Sequences* 9.2 (2006).
- [35] A. Tripathi, Formulae for the Frobenius number in three variables, *Journal of Number Theory* 170 (2017): 368-389.
- [36] A. Tripathi, The coin exchange problem for arithmetic progressions, *Amer. Math. Monthly* 101 (10), (1994) 779781.
- [37] A. Tripathi, The number of solutions to $ax + by = n$, *Fibonacci Quart.* **38** (2000), 290–293.
- [38] Yin, R., Mu, J., and T. Komatsu, The p -Frobenius Number for the Triple of the Generalized Star Numbers, *Symmetry* 16.8 (2024): 1090.
- [39] D.X. Zheng, A note on the Frobenius problem for linear forms, *Sichuan Daxue Xuebao* 29 (2) (1992), 188192 (in Chinese).

INSTITUTE OF MATHEMATICS, HENAN ACADEMY OF SCIENCES, ZHENGZHOU 450046, CHINA AND
DEPARTMENT OF MATHEMATICS, INSTITUTE OF SCIENCE TOKYO, 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO
152-8551, JAPAN
Email address: komatsu@zstu.edu.cn

DEPARTMENT OF MATHEMATICS, SHIV NADAR INSTITUTE OF EMINENCE, GAUTAM BUDDHA NAGAR - 201314,
INDIA
Email address: rg796@snu.edu.in

DEPARTMENT OF MATHEMATICS, SHIV NADAR INSTITUTE OF EMINENCE, GAUTAM BUDDHA NAGAR - 201314,
INDIA
Email address: neha.gupta@snu.edu.in