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EXISTENCE OF A GROUP STRUCTURE IN ZF SET THEORY

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Abstract

In terms of ZF set theory and without using the axiom of choice, we give a necessary and sufficient condition so that, given a non-empty set A, there exists an inner binary operation $(\cdot) : A \times A \to A$ and an element $e \in A$, such that $(A, \cdot, e)$ is a group.

0. Notation

- Let A be a non-empty set. We denote by $\Delta_{A\times A} = \{(a,a) : a \in A\}$ the diagonal of $A \times A$.
- If $H : X \leftrightarrow Y$ is a bijection from X to Y we denote by $H^* : Y \leftrightarrow X$ the inverse bijection of H.
- If $F \subseteq X \times X$ and $G \subseteq X \times X$ are binary relations in X, we set: $F \circ G = \{(x,z) :$ there exists $y \in X$ that $(x,y) \in F$ and $(y,z) \in G\}$.
- Let $(\sim)$ be an equivalence relation in the set $A \times A$. Then for any $(a,b) \in A \times A$, $\overline{(a,b)}$ denotes the equivalence class of (a,b).

1. Introduction

It is well known in the mathematical community that in ZFC set theory every set can become a group [2]. In the weaker case of ZF set theory we need to characterize those sets which can become groups.

Our approach is based on the following definition:

Definition 1.1. *If A is a non-empty set, $\mathscr{R}$ a partition of $A \times A$ into graphs of functions $f : A \to A$, $\Delta_{A\times A} \in \mathscr{R}$, and $(\mathscr{R}, \circ, \Delta_{A\times A})$ is a group, then $(A, \mathscr{R}, \circ, \Delta_{A\times A})$ is called a "**shadow (or sketched) group structure over A**".*

Remark 1.1. *The term "shadowed" has been extensively used in the literature of Fuzzy Sets and related structures, in a very different sense, in order to express various fuzzifications of the notion of a set (see for example [1]). Here, the term "shadow group structure" refers to a set-theoretical description in the classical context of Set Theory.*

Remark 1.2. *If $(A, \mathscr{R}, \circ, \Delta_{A\times A})$ is a shadow group structure, then the members of $\mathscr{R}$ are graphs of bijections. Indeed, since every function graph in $\mathscr{R}$ has an inverse in $\mathscr{R}$, it is in fact the graph of a bijection of A onto itself.*

2. A necessary and sufficient condition

Suppose that $(A, \mathscr{R}, \circ, \Delta_{A\times A})$ is a shadow group structure over A. We consider the equivalence relation $(\sim)$ induced by $\mathscr{R}$. We choose a fixed $e \in A$. Let $H : A \to \mathscr{R}$ the function defined by $H(a) = \overline{(a,e)}$, i.e., H sends each $a \in A$ to the bijection graph which forms the equivalence class of the pair (a,e) in $\mathscr{R}$.

Theorem 2.1. *The function $H : A \to \mathscr{R}$ is a bijection.*

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Proof. It is enough to prove that

(i) H is an injection.
(ii) H is a surjection.

For (i): Suppose that H is not an injection. Then there would exist at least two elements $x \neq y$ in A with $H(x) = H(y)$, i.e., $\overline{x,e} = \overline{y,e}$ which is absurd because $\overline{x,e}$ and $\overline{y,e}$ are graphs of the same bijection.

For (ii): Let $\overline{x,y}$ be given in $\mathscr{R}$. The equivalence class $\overline{x,y}$ is the graph of a bijection $A \leftrightarrow A$. If the inverse of this bijection sends e to $w \in A$, then

$$H(w) = \overline{w,e} = \overline{x,y},$$

since the last equation is equivalent to $\overline{x,y}(w) = e$.

Observe that for every $a \in A$,

$$H(e) = \overline{e,e} = \Delta_{A\times A} = \overline{a,a}.$$

□

We also define a binary inner operation $(\cdot)$ in A as follows:

$$a \cdot b = H^{*}(\overline{a,e} \circ \overline{b,e}),$$

where H^{*} is the inverse of $H : A \longrightarrow \mathscr{R}$. We call the operation $(\cdot)$ "*the binary operation in A induced by* $(A,\mathscr{R},\circ,\Delta_{A\times A})$".

Theorem 2.2. *The structure $(A,\cdot,e)$ is a group.*

Proof. First we show that the operation $(\cdot)$ is associative. Let $a,b,c \in A$. Then we have:

$$H((a\cdot b)\cdot c) = \overline{a\cdot b,e} \circ \overline{c\cdot e} = (\overline{a,e} \circ \overline{b,e}) \circ \overline{c,e} = \overline{a,e} \circ \overline{b\cdot c,e} = H(a\cdot(b\cdot c)).$$

Since H is a bijection we have

$$(a\cdot b)\cdot c = a\cdot(b\cdot c).$$

Also if $a \in A$ then

$$H(a\cdot e) = \overline{a,e} \circ \overline{e,e} = \overline{a,e} \circ \Delta_{A\times A} = \overline{a,e} = H(a).$$

Hence $a\cdot e = a$. Also, $H(e\cdot a) = \overline{a,a} \circ \overline{a,e} = H(a)$.

Finally for every $a \in A$ we set

$$a^{-1} = H^{*}(\overline{e,a}).$$

Then

$$\overline{a^{-1},e} = H(a^{-1}) = H(H^{*}(\overline{e,a})) = \overline{e,a}$$

and

$$H(a\cdot a^{-1}) = \overline{a,e} \circ \overline{a^{-1},e} = \overline{a,e} \circ \overline{e,a} = \overline{a,a} = \overline{e,e} = H(e).$$

Hence, $a\cdot a^{-1} = e$, and similarly $a^{-1}\cdot a = e$, so a^{-1} is the inverse element of a.

□

Theorems 2.1 and 2.2 show that the existence of a shadow group structure in a set A is a sufficient condition in order that A becomes a group. Now we prove that this condition is also necessary.

Theorem 2.3. *Let A be a non-empty set. The following are equivalent:*

(i) There exists an inner binary operation $(\cdot) : A\times A \longrightarrow A$ and $e \in A$ such that $(A,\cdot,e)$ is a group.
(ii) There exists a shadow group structure $(A,\mathscr{R},\circ,\Delta_{A\times A})$ over A.

Proof. $(i) \implies (ii)$: If $(A,\cdot,e)$ is a group we define in $A\times A$ the equivalence relation $(\sim)$ by $(x,y)\sim(z,w)$ if and only if $x\cdot y^{-1} = z\cdot w^{-1}$, where $x,y,z,w\in A$ and $\mathscr{R} = A\times A/_{\sim}$. We shall prove that $(A,\mathscr{R},\circ,\Delta_{A\times A})$ is a shadow group structure.

Indeed we prove first that $(\circ)$ is an inner operation. We consider the auxiliary function $H^* : \mathscr{R}\to A$ defined by the formula $H^*(\overline{a,b}) = a\cdot b^{-1}$. This is a well defined function, which is a bijection. Thus, every equivalence class $\overline{a\cdot b}$ can be viewed as a bijection of A onto itself defined as follows: to each $x\in A$ corresponds a unique $y\in A$ such that $y = ab^{-1}x$, and vice-versa, each $y\in A$ is the image of a unique x by the same correspondence, since $x = (ab^{-1})^{-1}y$.

Consider $\overline{a,b}\in\mathscr{R}$ and $\overline{c,d}\in\mathscr{R}$, as bijections of A onto A. We prove that $\overline{a,b}\circ\overline{c,d}\in\mathscr{R}$. We have:

$$H^*(a\cdot b^{-1},e) = a\cdot b^{-1}\cdot e^{-1} = a\cdot b^{-1} = H^*(\overline{a,b}).$$

Also,

$$H^*(\overline{e,d\cdot c^{-1}}) = e\cdot(d\cdot c^{-1})^{-1} = c\cdot d^{-1} = H^*(\overline{c,d}).$$

From $H^*(\overline{a\cdot b^{-1},e}) = H^*(\overline{a,b})$, we get $\overline{a\cdot b^{-1},e} = \overline{a,b}$ and from $H^*(\overline{e,d\cdot c^{-1}}) = H^*(\overline{c,d})$, we get $(\overline{e,d\cdot c^{-1}}) = (\overline{c,d})$.

Hence,

$$\overline{a,b}\circ\overline{c,d} = \overline{a\cdot b^{-1},e}\circ\overline{e,d\cdot c^{-1}} = \overline{a\cdot b^{-1},d\cdot c^{-1}}\in\mathscr{R}.$$

The rest of proving that $(A,\mathscr{R},\circ,\Delta_{A\times A})$ is a shadow group structure follows easily, since in fact $\mathscr{R}$ is the set of bijection graphs.

$(ii) \implies (i)$: From the theorems 2.1 and 2.2. □

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REFERENCES

[1] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[2] András Hajnal and Andor Kertész. Some new algebraic equivalents of the axiom of choice. *Publ. Math. Debrecen*, 19:339–340, 1972.

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