BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 68, 2024 (1–9)

SOME SHARP LOWER BOUNDS FOR THE BIPARTITE TURÁN NUMBER OF THETA GRAPHS

STEFANOS THEODORAKOPOULOS

ABSTRACT. We extend Conlon's random algebraic construction to show that for any odd number $k \geq 3$ exists a natural number c_k (the same as Conlon's) such that $\mathrm{ex}(n^a, n, \theta_{k,c_k}) = \Omega_{k,a}((n^{1+a})^{\frac{k+1}{2k}})$, with $a \in [\frac{k-1}{k+1}, 1)$. Given a graph H, we denote by $\mathrm{ex}(n, m, H)$ the maximum number of edges an $H-$free bipartite graph can have when the cardinalities of its parts are n and m. Also, we denote with $\theta_{k,l}$ the graph where two vertices are connected through l disjoint paths of length k.

1. INTRODUCTION

In extremal graph theory, a classical problem is to determine $\mathrm{ex}(n; H)$ for a given graph H, where $\mathrm{ex}(n; H)$ is defined as the biggest number of edges a graph G with $|V(G)| = n$, not containing a subgraph isomorphic to H, can possible have. In practice, the search for lower bounds of the extremal function $\mathrm{ex}(n; H)$ that are as big as possible has proved to be surprisingly difficult. It is important to note that when someone tries to create an H-free graph G, the size of G plays a major role to how delicate the said construction needs to be, because the smaller the size of G the easier it is for H to appear, as the edges have less "space" to move. Of great importance is the case where someone has a bipartite graph $G = (A, B)$, with $|A| = n, |B| = m, n \geq m$, and tries to find out how many edges G can have without containing a graph H. Then we talk about the asymmetric bipartite Turán number $\mathrm{ex}(m, n, H)$ of H. A well studied occasion is when $H = C_{2k}$, the cycle of length $2k$ for some natural number k. The C_{2k} graphs belong to the more general family of $\theta_{k,\ell}$ graphs (note that $C_{2k} = \theta_{k,2}$). Conlon in [3], by building upon a paper of Bukh [1], shows that for every natural number $k \geq 2$ there exists a natural number $\ell := \ell(k)$ such that, for every n, there is a balanced bipartite graph with n vertices and $\Omega_k\left(n^{1+\frac{1}{k}}\right)$ edges with at most ℓ paths of length k between any two vertices. A result of Faudree and Simonovits [4] implies that the bound on the number of edges is tight up to the implied constant. We extend on this method of Conlon's to show that for any odd number $k \geq 3$ and rational number a with $\frac{k-1}{k+1} \leq a < 1$ there exists an unbalanced bipartite graph with $|A| = n, |B| = O_k(n^a)$ and $|E(G)| = \Omega_{k,a}\left(\left(n^{1+a}\right)^{\frac{k+1}{2k}}\right)$ such that between any two vertices

Date: Received 24 March 2024. Accepted 26 June 2024.
2020 *Mathematics Subject Classification.* 05C35, 05D40.
Key words and phrases. unbalanced; bipartite graph; Turán number; sharp bound; theta graph.

there exist at most ℓ paths of length k.

Remark 1.1. *The present work was part of a thesis the author submitted at the University of Warwick for the completion of his MASt degree in March of 2019. Furthermore, it justifies the intuition described by Tao, Longbrake and Jie after [6][Proposition 5.1].*

2. Preliminaries

To begin let q be a prime and $\mathbb{F}_q = \mathbb{Z}_q$ be the field of order q. We will talk about polynomials over $\mathbb{F}_q^t$ for a given natural number t, writing any such polynomial as $f(x)$ where $x = (x_1, \ldots, x_t) \in \mathbb{F}_q^t$. Let d be a natural number. We define as $\mathbb{P}_d$ the set of polynomials in $\mathbb{F}_q^t$ of degree at most d. That is, the set of linear combinations over $\mathbb{F}_q$ of monomials of the form $x_1^{a_1}x_2^{a_2}\ldots x_t^{a_t}$ with $\sum_{i=1}^t a_i \leq d$. By a random polynomial, we just mean a polynomial chosen uniformly from the set $\mathbb{P}_d$. One may produce such a random polynomial by choosing the coefficients of the monomials above to be random elements of $\mathbb{F}_q$.

Note that

$$\mathbb{P}_d \cong \mathbb{F}_q^{\frac{t^{d+1}-1}{t-1}}.$$

So, we have the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{P}_d, \mathcal{F} = \mathcal{P}(\mathbb{P}_d)$ and $\mathbb{P}$ the uniform probability distribution. It is obvious that every function which has $\mathbb{P}_d$ as domain is a random variable.

The following two lemmas are taken from [3].

Lemma 2.1. *If f is a randomly chosen polynomial from $\mathbb{P}_d$ then, for any fixed $x \in \mathbb{F}_q^t$ we have*

$$\mathbb{P}[f(x) = 0] = \frac{1}{q}.$$

Lemma 2.2. *Assume that $q > \binom{m}{2}$ and $d \geq m-1$. Then, if f is a randomly chosen polynomial from $\mathbb{P}_d$ and $x_1, \ldots, x_m$ are m distinct points from $\mathbb{F}_q^t$, we have*

$$\mathbb{P}\left[f(x_1) = \ldots = f(x_m) = 0\right] = \frac{1}{q^m}.$$

To introduce the final tools that we are going to use we must first give the following definition.

Definition 2.3. *Given a field $\mathbb{F}_q$ we denote its algebraic closure with $\overline{\mathbb{F}}_q$. A variety over $\overline{\mathbb{F}}_q$ is a set W of the form*

$$W := \left\{x \in \overline{\mathbb{F}}_q^t : f_1(x) = \ldots = f_s(x) = 0\right\},$$

where $f_1, ..., f_s$ are polynomials with domain $\overline{\mathbb{F}}_q^t$ that take values in $\mathbb{F}_q$. We write $W(\mathbb{F}_q) = W \cap \mathbb{F}_q^t$ and if the coefficients of these polynomials are in $\mathbb{F}_q$ we say that W is defined over $\mathbb{F}_q$. Furthermore, we say that W has complexity $M \in \mathbb{N}$ if s, t and the degrees of $f_1, ..., f_s$ are all bounded by M. Additionally, we say that a variety is absolutely irreducible if it is irreducible[1] over $\overline{\mathbb{F}}_q$. Finally, the dimension $\dim W$ is the maximum integer d such that there exists a chain of absolutely irreducible subvarieties of W of the form

$$\emptyset \subset \{p\} \subset W_1 \subset W_2 \ldots \subset W_d \subset W,$$

where p is a point.

The next is the known Lang–Weil bound, see [7].

Lemma 2.4. *Suppose that W is a variety over $\overline{\mathbb{F}}_q$ of complexity at most M. Then*

$$\left|W\left(\mathbb{F}_q\right)\right| = O_M\left(q^{\dim W}\right).$$

Moreover, if W is defined over $\mathbb{F}_q$ and is absolutely irreducible, then

$$\left|W\left(\mathbb{F}_q\right)\right| = q^{\dim W}\left(1 + O_M\left(q^{-\frac{1}{2}}\right)\right).$$

The result below is standard in algebraic geometry (see Bump [2] for example).

Lemma 2.5. *Suppose that W is an absolutely irreducible variety over $\overline{\mathbb{F}}_q$ of complexity at most M and $\dim W \geq 1$. Then, for any polynomial $g : \overline{\mathbb{F}}_q^t \longmapsto \overline{\mathbb{F}}_q$,*

$W \subseteq \{x : g(x) = 0\}$ or $W \cap \{x : g(x) = 0\}$ is a variety of dimension less than $\dim W$.

The last preliminary result is taken again from [3].

Lemma 2.6. *Suppose that W is a variety over $\overline{\mathbb{F}}_q$ of complexity at most M which is defined over $\mathbb{F}_q$. Then, there are $O_M(1)$ absolutely irreducible varieties $Y_1, \ldots, Y_s$, each of which is defined over $\mathbb{F}_q$ and has complexity $O_M(1)$, such that $\bigcup_{i=1}^s Y_i\left(\mathbb{F}_q\right) = W\left(\mathbb{F}_q\right)$.*

For completeness we also remind Bertrand's Postulate.

For every natural number $n > 1$ there is at least one prime p such that
$$n < p < 2n.$$

3. Construction

Theorem 3.1. *Let $k \geq 3$ be an odd number and $a \in \mathbb{Q}$ where $\frac{k-1}{k+1} \leq a < 1$. Then there exists a natural number c_k such that for every $n \in \mathbb{N}$ sufficiently large, there exists a θ_{k,c_k}–free bipartite graph $G = (A, B)$ with $|A| = n, |B| = O_k(n^a)$ and*

$$|E(G)| = \Omega_{k,a}\left(\left(n^{1+a}\right)^{\frac{k+1}{2k}}\right). \tag{1}$$

[1]We say that a variety is irreducible if it cannot be written as the union of two proper subvarieties.

Proof. Let q be a sufficiently large prime, $k = 2x+1$, $a = \frac{\tau}{\lambda}$, $\gamma = x(\lambda+\tau)$, $t = k(\lambda+\tau)$ and $d = kr$, with $\gcd(\tau, \lambda) = 1$ and r be a natural number that will be determined later. We define the probability space $(\Omega, F, \mathbb{P})$ where $\Omega = \mathbb{P}_d^\gamma$, $F = \mathcal{P}\left(\mathbb{P}_d^\gamma\right)$ and $\mathbb{P}$ the uniform probability distribution. Let $f_1, \ldots, f_\gamma : {\mathbb{F}_q}^{k\lambda} \times {\mathbb{F}_q}^{k\tau} \longmapsto \mathbb{F}_q$ be independent random polynomials in $\mathbb{P}_d$. We construct the bipartite graph G with $A = {\mathbb{F}_q}^{k\lambda}$ and $B = {\mathbb{F}_q}^{k\tau}$ where two vertices u, v are connected if and only if $f_i(u, v) = 0 \ \forall i \in \{1, \ldots, \gamma\}$. Since $f_1, \ldots, f_\gamma$ were chosen independently, by Lemma 2.2 the probability a given edge (u, v) is in G is $q^{-\gamma}$. Therefore $|E(G)| : \mathbb{P}_d^\gamma \longmapsto \mathbb{N}$ is a random variable and if we define $e_{(u,v)} : \mathbb{P}_d^\gamma \longmapsto \{0, 1\}$, where $e_{(u,v)}\left(f_1, \ldots, f_\gamma\right) := \begin{cases} 0 & (u, v) \notin E(G) \\ 1 & (u, v) \in E(G) \end{cases}$, the expected number of edges, is

$$\mathbb{E}[|E(G)|] = \mathbb{E}\left[\sum_{(u,v)\in {\mathbb{F}_q}^{k\lambda}\times {\mathbb{F}_q}^{k\tau}} e_{(u,v)}\right] = \sum_{(u,v)\in {\mathbb{F}_q}^{k\lambda}\times {\mathbb{F}_q}^{k\tau}} \mathbb{E}\left[e_{(u,v)}\right] = q^{k(\lambda+\tau)-\gamma} = q^{k(\lambda+\tau)\frac{k+1}{2k}}.$$

Suppose now that w_1, w_2 are two fixed vertices in A, B respectively and let S be the set of paths of length k between them. We are going to estimate the r-th moment of $|S|$. At this point it is important to note that $|S|^r : \mathbb{P}_d^\gamma \longmapsto \mathbb{N}$, is a random variable that counts the number of ordered collections of r paths of length k in G between w_1, w_2 without any restrictions, allowing overlapping or identical paths. Since the total number m of edges in any such collection of r paths is at most $d = kr$ and q is a sufficiently large prime, Lemma 2.2 tells us that the probability that a particular collection of paths is in G is $q^{-\gamma m}$, where we again used the fact that $f_1, \ldots, f_\gamma$ were chosen independently. Within the complete bipartite graph between A and B, let $P_{r,m}$ be the number of ordered collections of r paths, each of length k, from w_1 to w_2 whose union has m edges. So define the random variables $|S|_m^r : \mathbb{P}_d^\gamma \longmapsto \mathbb{N}$ that count exactly that and we have

$$|S|^r = \sum_{m=1}^{kr} |S|_m^r.$$

We needed to distinguish the collection of paths by the number of the edges that they contain because the probability of their existence depends on m. Then as before, we get

$$\mathbb{E}\left[|S|^r\right] = \sum_{m=1}^{kr} P_{r,m} q^{-\gamma m}.$$

To finish the estimation we need to consider the size of $P_{r,m}$. Before that, fix some $m \in \{1, \ldots, kr\}$ and some $(n_1, n_2) \in \{1, \ldots, rx\} \times \{1, \ldots, rx\}$. Assume that there is a collection of r paths of length k from w_1 to w_2, $(p_1, \ldots, p_r)$, that are defined from n_1 inner[2] vertices of A and n_2 inner vertices of B and are constructed by m edges. We

[2]Inner with respect to the paths from w_1 to w_2.

want to show that $k\lambda n_1 + k\tau n_2 \le \gamma m$ or, equivalently,

$$(2x+1)\lambda n_1 + (2x+1)\tau n_2 \le x(\lambda+\tau)m. \tag{2}$$

For all $j \in \{1,\dots,r\}$ let m_j be the number of edges that belong to $p_j \setminus \bigcup_{i=1}^{j-1} p_i$ and similarly $n_{1,j}, n_{2,j}$ the number of inner vertices of A, B that belong to $p_j \setminus \bigcup_{i=1}^{j-1} p_i$. By definition we have

$$\sum_{j=1}^{r} m_j = m, \quad \sum_{j=1}^{r} n_{1,j} = n_1, \quad \sum_{j=1}^{r} n_{2,j} = n_2.$$

We will prove the desired inequality by proving that $\forall j \in \{1,\dots,r\}$

$$(2x+1)\lambda n_{1,j} + (2x+1)\tau n_{2,j} \le x(\lambda+\tau)m_j.$$

Claim

For a given $j \in \{1,\dots r\}$ at least one of the following is true :

- $m_j \ge 2n_{1,j}+1$ and $m_j \ge 2n_{2,j}+1$,
- $m_j \ge 2n_{1,j}+2$ and $m_j \ge 2n_{2,j}$,
- $m_j \ge 2n_{1,j}$ and $m_j \ge 2n_{2,j}+2$.

Proof of the claim. To begin take the case where the new m_j edges create a single path. There are 3 possibilities. In the first the path starts and ends in different parts of G and we get the first set of inequalities. The second possibility is when the path starts at A and finishes at A and the third possibility is when the path starts at B and finishes at B. It is easy to see that in those we get the second and third set of inequalities respectively. Now consider the general case where the new m_j edges create multiple disjoint paths. To see that in the case of multiple paths they need to be disjoint note that we can think of those paths as the subgraph we get from the path p_j after we remove from it the edges of the previous paths $\bigcup_{i=1}^{j-1} p_i$. First we examine every path separately and map it with its corresponding set of inequalities. The, if we add up with respect to the paths, it is fairly obvious that each of these sets of inequalities remains valid also for $m_j, n_{1,j}, n_{2,j}$. □

Now we are ready to show that (2) holds. Note that $x \ge n_{1,j}, n_{2,j} \forall j \in \{1,\dots,r\}$, hence in the first instance we have

$$\begin{aligned} x(\lambda+\tau)m_j &= x\lambda m_j + x\tau m_j \\ &\ge x\lambda\left(2n_{1,j}+1\right) + x\tau\left(2n_{2,j}+1\right) \\ &\ge (2x+1)\lambda n_{1,j} + (2x+1)\tau n_{2,j}. \end{aligned}$$

In the second instance, because $\lambda > \tau$ we have

$$\begin{aligned} x(\lambda+\tau)m_j &= x\lambda m_j + x\tau m_j \\ &\ge x\lambda\left(2n_{1,j}+2\right) + x\tau 2n_{2,j} \\ &\ge (2x+1)\lambda n_{1,j} + (2x+1)\tau n_{2,j}. \end{aligned}$$

In the final instance we have

$$
\begin{aligned}
x(\lambda+\tau)m_j &= x\lambda m_j + x\tau m_j \\
&\geq x\lambda 2n_{1,j} + x\tau\left(2n_{2,j}+2\right) \\
&= 2x\lambda n_{1,j} + (2x+1)\tau n_{2,j} + \left(2x - n_{2,j}\right)\tau. \qquad (3)
\end{aligned}
$$

Before we continue we need to consider the two possible subcases. The first is that the new m_j edges create a single path that starts at B and finishes at B and the second is when the new m_j edges create multiple disjoint paths.

In the second subcase if one of the paths starts and ends in different parts of G or starts and ends at A we can go to one of the first two instances and we are done. If they create more than one paths that start and end in B, then for every such path[3] we get the corresponding inequality (3), and by summing all those inequalities we can substitute 2 by at least 4 in (3). In other words, assume that we get $M \geq 2$ paths that start and end in B, then for every $i \in \{1, ..., M\}$ we have

$$x(\lambda+\tau)m_j^i \geq 2x\lambda n_{1,j}^i + (2x+1)\tau n_{2,j}^i + \left(2x - n_{2,j}^i\right)\tau,$$

where

$$\sum_{i=1}^{M} m_j^i = m_j, \sum_{i=1}^{M} n_{1,j}^i = n_{1,j}, \sum_{i=1}^{M} n_{2,j}^i = n_{2,j}.$$

So, by summing up with respect to i, we have

$$
\begin{aligned}
x(\lambda+\tau)m_j &\geq 2x\lambda n_{1,j} + (2x+1)\tau n_{2,j} + (2Mx - n_{2,j})\tau \\
&\geq 2x\lambda n_{1,j} + (2x+1)\tau n_{2,j} + 3x\tau.
\end{aligned}
$$

But $3\tau \geq \lambda$, therefore again we get what we want.

Finally we need only to consider the first subcase. So now we have $n_{1,j} = \frac{m_j}{2}, n_{2,j} + 1 = \frac{m_j}{2}$ and want to prove that $\left(2x - n_{2,j}\right)\tau \geq \lambda n_{1,j}$, or, equivalently,

$$a \geq \frac{n_{1,j}}{2x - n_{2,j}} = \frac{\frac{m_j}{2}}{2x + 1 - \frac{m_j}{2}} = \frac{1}{\frac{2x+1}{\frac{m_j}{2}} - 1}.$$

But for all j we have $m_j \leq 2x$, so

$$\frac{1}{\frac{2x+1}{\frac{m_j}{2}} - 1} \leq \frac{1}{\frac{2x+1}{x} - 1} = \frac{x}{x+1} = \frac{k-1}{k+1}.$$

Thus, the proof of (2) is complete.

Now, denote with Γ_m the pairs $(n_1, n_2) \in \{1, \ldots, rx\} \times \{1, \ldots, rx\}$ such that there exists a collection of r paths of length k from w_1 to w_2, $(p_1, \ldots, p_r)$, that are defined from n_1 inner vertices of A and n_2 inner vertices of B and are constructed by m edges. For

[3] we remind that they are disjoint

every such pair (n_1, n_2) there are at most $q^{k\lambda n_1+k\tau n_2} = q^{(2x+1)\lambda n_1+(2x+1)\tau n_2}$ such collections, so

$$P_{r,m} \leq \sum_{(n_1,n_2)\in\Gamma_m} q^{(2x+1)\lambda n_1+(2x+1)\tau n_2}.$$

Note that $|\Gamma_m| \leq x^2r^2$ for every $m \in \{1, \ldots, rk\}$. Now we return to the expectation of $|S|^r$ and get

$$\begin{aligned}\mathbb{E}\left[|S|^r\right] = \sum_{m=1}^{kr} P_{r,m}q^{-\gamma m} &\leq \sum_{m=1}^{kr} \sum_{(n_1,n_2)\in\Gamma_m} q^{(2x+1)\lambda n_1+(2x+1)\tau n_2-x(\lambda+\tau)m}\\ &\leq \sum_{m=1}^{kr} |\Gamma_m| \leq \sum_{m=1}^{kr} x^2r^2 = kx^2r^3 = k\left(\frac{k-1}{2}\right)^2 r^3.\end{aligned}$$

By Markov's inequality we conclude that $\mathbb{P}[|S| \geq s] = \mathbb{P}\left[|S|^r \geq s^r\right] \leq k\left(\frac{k-1}{2}\right)^2 \frac{r^3}{s^r}$. We need to note that the set of paths S is a subset of

$$\begin{aligned}T := \{(x_1, \ldots, x_{k-1}) : x_{2i-1} \in \mathbb{F}_q^{k\tau}, x_{2i} \in \mathbb{F}_q^{k\lambda} \;\; \forall i \in \{1, \ldots, x\} \;\; \text{and}\\ f_j(w_1, x_1) = f_j(x_1, x_2) = \ldots = f_j(x_{k-1}, w_2) = 0, \;\; \forall j \in \{1, \ldots, \gamma\}\}.\end{aligned}$$

Unfortunately T may contain degenerate walks as well as the paths we are interested in, so we proceed to the following analysis. If T contains a degenerate walk $w_1, x_1, \ldots, x_{k-1}, w_2$ it must be one of the following cases: either $w_1 = x_b$ for some $1 \leq b \leq k-1$ or $x_a = x_b$ for some $1 \leq a < b \leq k-1$ or $x_a = w_2$ for some $1 \leq a \leq k-1$. Let us then define the sets :

- $T_{0b} := T \cap \{(x_1, \ldots, x_{k-1}) : w_1 = x_b\}$ for some $1 \leq b \leq k-1$,
- $T_{ab} := T \cap \{(x_1, \ldots, x_{k-1}) : x_a = x_b\}$ for some $1 \leq a < b \leq k-1$,
- $T_{ak} := T \cap \{(x_1, \ldots, x_{k-1}) : x_a = w_2\}$ for some $1 \leq a \leq k-1$.

Since T is defined over $\mathbb{F}_q$ and has complexity bounded in terms of k, Lemma 2.6 tells us that there are $O_k(1)$ absolutely irreducible varieties $Y_1, \ldots, Y_s$, each of which is defined over $\mathbb{F}_q$ and has complexity $O_k(1)$, such that $\bigcup_{i=1}^{s} Y_i\left(\mathbb{F}_q\right) = T\left(\mathbb{F}_q\right)$. If $\dim Y_i \geq 1$, Lemma 2.5 tells us that either there exist a and b[4] such that $Y_i \subseteq T_{ab}$ or the dimension of $Y_i \cap T_{ab}$ is smaller than the dimension of Y_i for all a and b. If $Y_i \subseteq T_{ab}$ for some a and b, the component does not contain any non-degenerate paths and may be removed from consideration. If instead the dimension of $Y_i \cap T_{ab}$ is smaller than the dimension of Y_i for all a and b, the Lang-Weil bound, Lemma 2.4, tells us that for q sufficiently large

$$|S| \geq \left|Y_i\left(\mathbb{F}_q\right)\right| - \sum_{a,b} |Y_i \cap T_{ab}| \geq q^{\dim Y_i} - O_k\left(q^{\dim Y_i - \frac{1}{2}}\right) - O_k\left(q^{\dim Y_i - 1}\right) \geq \frac{q}{2}.$$

On the other hand, if $\dim Y_i = 0$ for every Y_i which is not contained in some T_{ab}, Lemma 2.4 tells us that $|S| \leq \sum |Y_i| = O_k(1)$, where the sum is taken over all i for which $\dim Y_i = 0$. Putting everything together, we see that that there exists

[4] a, b taking values as described in the definition of T_{ab}.

a constant $c_k - 1$, depending only on k, such that either $|S| \leq c_k - 1$ or $|S| \geq q/2$. Therefore, by the consequence of Markov's inequality noted earlier we get

$$\mathbb{P}\left[|S| > c_k - 1\right] = \mathbb{P}\left[|S| \geq \frac{q}{2}\right] \leq k\left(\frac{k-1}{2}\right)^2 \frac{r^3}{\left(\frac{q}{2}\right)^r}.$$

Call a pair of vertices (w_1, w_2) *bad* if $w_1 \in A, w_2 \in B$ and there are more than $c_k - 1$ paths of length k between them. Let $\Lambda : \mathbb{P}_d^\gamma \longmapsto \mathbb{N}$ be the random variable counting the number of bad pairs. So

$$\mathbb{E}[\Lambda] \leq \sum_{(w_1,w_2)\in A\times B} \mathbb{P}\left[|S| > c_k - 1\right] = q^{k(\lambda+\tau)} k\left(\frac{k-1}{2}\right)^2 \frac{r^3}{\left(\frac{q}{2}\right)^r}.$$

Now remove a vertex from B for every bad pair. Since each vertex has degree at most n the total number of edges removed is at most Λn. From all the above the expected number of edges for the final graph G' would be

$$\mathbb{E}\left[|E(G')|\right] \geq q^{k(\lambda+\tau)\frac{k+1}{2k}} - q^{k(\lambda+\tau)+k\lambda-r} k\left(\frac{k-1}{2}\right)^2 \frac{r^3}{\left(\frac{1}{2}\right)^r}.$$

Hence, if we define

$$r := \frac{3k-1}{2}\lambda + \frac{k-1}{2}\tau + 1,$$

there are choices of $f_1, \ldots, f_\gamma$ with the desired properties. As stated, this result only holds when q is a prime and $n = q^{kl}$. To generalize this let n be a sufficiently large natural number, then by Bertrand's postulate we have that exists a prime p_n such that

$$\left\lfloor \frac{n^{\frac{1}{kl}}}{2} \right\rfloor < p_n < 2\left\lfloor \frac{n^{\frac{1}{kl}}}{2} \right\rfloor \;\Rightarrow\; \frac{n^{\frac{1}{kl}}}{4} < p_n < n^{\frac{1}{kl}}$$
$$\Rightarrow\; \frac{1}{4^{kl}} n < p_n^{kl} < n.$$

Hence, by applying the previous result to p_n and adding to part A the remaining vertices until we reach $|A| = n$ we complete the proof. □

From the techniques of the proof it is obvious that a somewhat stronger result was proved because the c_k paths need not be internally disjoint. On the other hand, the value of c_k becomes undesirably big.

4. Conclusion

A direct consequence from our construction is that $\mathrm{ex}\left(n^a, n, \theta_{k,c_k}\right) = \Omega_{k,a}\left(\left(n^{1+a}\right)^{\frac{k+1}{2k}}\right)$. Combining this with Jiang et al. [5][Theorem 1.11], which states that $\mathrm{ex}\left(n^a, n, \theta_{k,c_k}\right) = O_{k,a}\left(\left(n^{1+a}\right)^{\frac{k+1}{2k}} + n + n^a\right)$ we conclude our final result.

Theorem 4.1. *Let $k \geq 3$ be an odd number and $a \in \mathbb{Q}$, where $\frac{k-1}{k+1} \leq a < 1$. Then, there exists a natural number c_k such that for every $n \in \mathbb{N}$ sufficiently large we have*

$$\mathrm{ex}\,(n^a, n, \theta_{k,c_k}) = \Theta_{k,a}\Big(\big(n^{1+a}\big)^{\frac{k+1}{2k}}\Big). \tag{4}$$

Acknowledgements

I would like to sincerely thank Dr. Hong Liu for his valuable comments in the process of writing this paper.

References

[1] Boris Bukh. Random algebraic construction of extremal graphs. *Bulletin of the London Mathematical Society*, 47, 09 2014. doi: 10.1112/blms/bdv062.
[2] Daniel Bump. *Algebraic geometry*. World Scientific Publishing Company, 1998.
[3] David Conlon. Graphs with few paths of prescribed length between any two vertices. *Bulletin of the London Mathematical Society*, 51(6):1015–1021, 2019.
[4] Ralph J Faudree and Miklós Simonovits. On a class of degenerate extremal graph problems. *Combinatorica*, 3:83–93, 1983.
[5] Tao Jiang, Jie Ma, and Liana Yepremyan. On turán exponents of bipartite graphs. *Combinatorics, Probability and Computing*, 31(2):333344, 2022. doi: 10.1017/S0963548321000341.
[6] Tao Jiang, Sean Longbrake and Jie Ma. Bipartite-ness under smooth conditions. *Combinatorics, Probability and Computing*, 32(4):546558, 2023. doi: 10.1017/S0963548323000019.
[7] Serge Lang and André Weil. Number of points of varieties in finite fields. *American Journal of Mathematics*, 76(4):819–827, 1954.

Department of Mathematics, NTUA, Zografou Campus, 15780 Athens, Greece
Email address: steftheodorakopoulos@mail.ntua.gr