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SHARP LOWER BOUNDS FOR THE VECTOR ALLEN-CAHN ENERGY AND QUALITATIVE PROPERTIES OF MINIMIZERS UNDER NO SYMMETRY HYPOTHESES

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ABSTRACT. We study vector minimizers u^ϵ of $J^\epsilon_\Omega(u) = \int_\Omega(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u))dx$, $W > 0$, on $\mathbb{R}^m \setminus \{a_1, \dots, a_N\}$, $m \geq 1$, for bounded domains $\Omega \subset \mathbb{R}^n$, with certain geometrical features and $u = g_\epsilon$ on $\partial\Omega$. We derive a sharp lower bound (as $\epsilon \to 0$) with the additional feature that it involves half of the gradient and part of the domain. Based on this we derive very precise (in ϵ) pointwise estimates up to the boundary for $\lim_{\epsilon\to 0} u^\epsilon = u^0$. Depending on the geometry of Ω, u^ϵ exhibits either boundary layers of internal layers. We do not impose symmetry hypotheses and we do not employ Γ-convergence techniques.

1. INTRODUCTION

The object of study in the present paper is the system

$$\Delta u - W_u(u) = 0, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$ is a C^2 phase transition potential. That is: W is nonnegative and vanishes only on a finite set $\{W = 0\} = A = \{a_1, \dots, a_N\}$ for some distinct points $a_1, \dots, a_N \in \mathbb{R}^m$ that represent the phases of a substance that can exist in $N \geq 2$ different equally preferred states. We assume that the zeros $a_1, \dots, a_N$ are nondegenerate in the sense that the Jacobian matrix $\partial^2 W(a)$, $a \in A$, is positive definite. Finally we assume that

$$\liminf_{|z|\to+\infty} W(z) > 0. \tag{1.2}$$

System (1.1) is the Euler-Lagrange equation corresponding the Allen-Cahn energy

$$J_\Omega(v) = \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 + W(v)\Big)dx. \tag{1.3}$$

We are interested in the class of solutions that connect in some way the phases or a subset of them.

The scalar case $m = 1$ has been extensively studied: here $N = 2$ is the natural choice. The reader may consult [25, 21, 12] where further references can be found.

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A well known conjecture of De Giorgi (1978) and its solution about thirty years later, played a significant role in the development of a large part of this work.

The vector case $m \geq 2$ by comparison has been studied very little. We note that for the coexistence of three or more phases a vector order parameter is necessary and so there is physical interest for the system. On the geometric side (1.3), when rescaled as in (1.4) below, it produces minimal partitions $\{P_j\}_1^N$ (see (1.7)) that exhibit *junctions*, that is singularities with certain structures, that do not exist for $m = 1$.

For $m \geq 2$ (1.1) has been mainly studied in the class of equivariant solutions with respect to point reflection groups beginning with Bronsard, Gui and Schatzman [10] and later Gui and Schatzman [18], and has been significantly extended and generalized in various ways by the authors, including also finite and discrete groups [8]. We refer also to Chapters 6 and 7 in [5] and the reference therein. The only related works that do not require symmetry concern the case $N = 2$, see [22, 20, 24].

We will be focusing on the rescaled functional

$$J_\Omega^\epsilon(v) = \int_\Omega \Big(\frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon}W(v)\Big)dx, \tag{1.4}$$

where Ω is an open, bounded, $C^{1,\alpha}$, for some $\alpha \in (0,1)$, smooth connected set and we consider the minimization problem

$$\min J_\Omega^\epsilon(v), \quad v = g_\epsilon \ \text{ on } \partial\Omega, \tag{1.5}$$

where g_ϵ is a given map that may depend on ϵ.

The rescaled problem (1.5) is also useful for constructing entire solutions of (1.1) over $\mathbb{R}^n$, and although this point of view is not exploited in the present paper, it provides one of the motivations behind this work. We revisit this point later in the introduction.

We are interested in uniformly (with respect to ϵ) pointwise bounded global minimizers connecting minima of W, and for this reason we adopt the simple hypothesis

$$\begin{aligned} &W_u(u)\cdot u > 0, \quad \text{for } |u| > M, \ \text{some } M, \\ &|g_\epsilon| \leq M, \end{aligned} \tag{1.6}$$

which together with the previously mentioned assumptions $W \in C^2(\mathbb{R}^m;\mathbb{R})$, $W \geq 0$, $\{W = 0\} = \{a_1,\dots,a_N\}$, $\partial^2 W(a_i)$ positive definite, $i = 1,\dots,N$, will comprise the hypothesis on W.

A major general tool for the study of the minimizers u_ϵ of (1.5) is the limiting functional as $\epsilon \to 0$ given by the weighted perimeter functional

$$\begin{aligned} E(\mathcal{P}) &= \sum_{i\neq j}^{N} \sigma_{i,j}\mathcal{H}^{n-1}(\partial D_i \cap \partial D_j), \ \ \mathcal{P} = \{D_j\}_{j=1}^N, \ \text{ a partition of } \Omega, \\ \sigma_{i,j} &= \inf \int_{\mathbb{R}} \Big(\frac{1}{2}|\dot{U}|^2 + W(U)\Big) ds, \ \ U(-\infty) = a_i, \ U(+\infty) = a_j. \end{aligned} \tag{1.7}$$

Baldo [7] established for the vector case $m \geq 2$, for the mass constrained problem, that $\Gamma - \lim_{\epsilon\to 0} J^\epsilon_\Omega = E(\mathcal{P})$, from which it follows that along subsequences

$$\lim_{\epsilon\to 0} \|u_\epsilon - u_0\|_{L^1(\Omega;\mathbb{R}^m)} = 0, \tag{1.8}$$

where $u_0 = \sum_{j=1}^N a_j \chi_{D_j}$ is a minimizer of E under the same constraint.

The other major tool for the study of the minimizers $\{u_\epsilon\}$ as $\epsilon \to 0$ is the Caffarelli-Cordoba [11] density estimate originally derived for $m = 1$. This is independent from Γ-convergence and complements it by upgrading (1.8) to uniform convergence over compacts in D_j. It has been extended to the vector case in [4]. We refer to Chapter 5, in [5].

A major difficulty that one faces in implementing these general results and their variants is that the convergence in (1.8) does not come with an estimate in ϵ. This also is the major obstruction for utilizing the rescaled problem in constructing entire solutions to (1.1).

It is only under symmetry conditions that such exponential estimates similar to (1.12) below have been obtained in generality. More specifically we have the following general result (Theorem 6.1 in [5])

Theorem 1.1. *(*[3, 2, 14]*)*
Under the hypotheses

H_1**:** $W \in C^2(\mathbb{R}^m;\mathbb{R})$, $W \geq 0$, $\{W = 0\} = \{a_1, \dots, a_N\}$, $\partial^2 W(a_i)$ *positive definite,* $i = 1, \dots, N$.

H_2**:** *G a finite point reflection group acting on $\mathbb{R}^m$;*

$$W(gu) = W(u), \ \ \text{any } g \in G, u \in \mathbb{R}^m.$$

Moreover there exists $M > 0$ such that $W(su) \geq W(u)$, for $s \geq 1$ and $|u| = M$.

H_3**:** *Let $F \subset \mathbb{R}^m$ be a fundamental region of G. Assume that $\overline{F}$ contains a single global minimum, say a_1. Let G_{a_1} be the stabilizer of a_1 and set*

$$D := \mathrm{Int}(\cup_{g\in G_{a_1}} g\overline{F}).$$

Note that a_1 is also the unique global minimum of G in D. Moreover

$$\frac{|G|}{|G_{a_1}|} = N.$$

Then there is an equivariant entire solution of

$$\Delta u - W_u(u), \quad u(gx) = gu(x), \ \ \text{any } x \in \mathbb{R}^n, \ g \in G.$$

The solution u is a minimizer in the equivariance class and has the following properties:

(i) $|u(x) - a_1| \leq Ke^{-k\,d(x,\partial D)}$, $x \in D$, $k, K,$*positive constants.*
(ii) $u(\overline{F}) \subset \overline{F}$, $u(D) \subset D$ *(positivity).*
(iii) $\lim_{\lambda\to+\infty} u(\lambda g\nu) = ga_1$, *for* $g \in G$, *uniformly for* ν *in compact subsets of* $D \cap \mathbb{S}^{n-1}$ *(connecting the minima).*

The proof of this theorem is based on minimizing in the equivariant class

$$J_{B_R}(u) = \int\limits_{B_R} \Big(\frac{1}{2}|\nabla u|^2 + W(u)\Big)dx$$

over balls centered at the origin, and taking the limit

$$u(x) = \lim_{R\to+\infty} u_R(x).$$

Uniform (in R) estimates at infinity are crucial for guaranteeing the non triviality of u. A fundamental step is the derivation of the estimate

$$|u_R(x) - a_1| \leq Ke^{-k\,d(x,\partial D_R)}, \;\; x \in D_R := D \cap B_R, \; R \geq R_0,$$

or equivalently

$$|u_\epsilon(y) - a_1| \leq Ke^{-\frac{k}{\epsilon}d(y,\partial D_1)}, \; y \in D_1 \; \epsilon = \frac{1}{R}, \; y = \epsilon x, \; \epsilon < \epsilon_0 = \frac{1}{R_0}.$$

We note that the pointwise estimate in (1.12) below is weaker since it only locates the internal layer within $O(\epsilon^{\frac{1}{4}})$ neighborhood of the sharp interface, and by itself is not sufficient to produce a connecting entire solution. We believe that this is the effect of the lack of symmetry. However what is crucial for a successful rescaling is the optimal thickness $O(\epsilon)$ of the layer.

In the present paper we illustrate in term of various simple examples how the derivation of sharp lower bounds for $J^\epsilon_\Omega(u_\epsilon)$ allows extraction of pointwise estimates for $u_\epsilon \to u_0$ all the way up to the boundary of the partition. We do not make any symmetry assumptions. We do not utilize the limiting problem, but we do utilize the vector Caffarelli-Cordoba density estimate.

We now describe the content of the paper in more detail. We consider two examples where besides an upper bound it is also possible to derive a sharp lower bound for the energy of a minimizer of problem (1.5). We show that the knowledge of sharp lower and upper bounds together with the vector Caffarelli-Cordoba density estimate [11], [4] allow for an accurate description of the fine structure of minimizers for all $\epsilon > 0$ sufficiently small. A key point of the analysis is that the lower bound is obtained by considering only the energy in a proper subset of Ω.

In our first example Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $g_\epsilon \equiv z$ where $z \in \mathbb{R}^m \setminus A$, $A = \{W = 0\}$, is a fixed vector. We prove

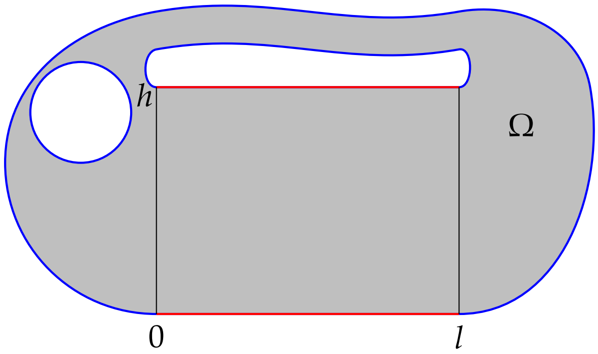

FIGURE 1. An example of Ω.

Theorem 1.2. *Let u ($u = u_\epsilon$) be a minimizer of problem* (1.5) *with $g_\epsilon \equiv z$, $z \in \mathbb{R}^m \setminus A$. Then there exist $a_z \in A$ and positive constants k, K and C such that*

$$|u(x) - a_z| \leq K e^{-\frac{k}{\epsilon}(d(x,\partial\Omega) - C\epsilon^{\frac{1}{3(n-1)}})^+}, \quad x \in \Omega,$$

where $r^+ = \max\{r, 0\}$.

The proof of this theorem is based on the fact that one can show (see Theorem 3.3) that most of the energy of a minimizer is contained in a tiny neighborhood of $\partial\Omega$. This allows for a transparent use of the density estimate in combination with the sharp upper and lower energy bounds.

In our second example we consider a domain $\Omega \subset \mathbb{R}^2$ as introduced after (1.4) and with the additional geometrical property

$$\Omega \cap [0, l] \times [0, h] = [0, l] \times (0, h) =: R. \tag{1.9}$$

In Figure 1 we give an example of a set Ω that satisfies (1.9). In (1.9) $l > 0$ and $h > 0$ are free parameters and our interest is in describing the fine structure of minimizers as a function of the ratio h/l. We choose the Dirichlet data $u = g_\epsilon$ on $\partial\Omega$, g_ϵ a $C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^m)$, some $\alpha \in (0, 1)$, with the feature that g_ϵ converges, as $\epsilon \to 0$, in a controlled manner to a step map taking values a_-, a_+ in A. Here $a_- \in A$ can be fixed arbitrarily while a_+ is chosen via a minimizing process (see Lemma 2.2) that determines a minimizing connection $\bar{u} : \mathbb{R} \to \mathbb{R}^m$ between a_- and a_+. We assume

$$\begin{aligned} & g_\epsilon(x, 0) = g_\epsilon(x, h) = a_+, \quad x \in (C_0\epsilon, l - C_0\epsilon), \\ & g_\epsilon(x, y) = a_-, \quad (x, y) \in \partial\Omega \setminus (0, l) \times \{0, h\}, \\ & |g_\epsilon(x, y)| \leq C, \ |g_{\epsilon,x}(x, y)| \leq \frac{C}{\epsilon}, \quad x \in (0, C_0\epsilon) \cup (l - C_0\epsilon, l), \quad y = 0, h, \\ & |g_{\epsilon,x}(\cdot, y)|_\alpha \leq \frac{C}{\epsilon^{1+\alpha}}, \quad y = 0, h. \end{aligned} \tag{1.10}$$

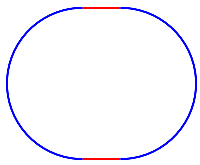

FIGURE 2. The case $l < h$. Blue = a_-, red= a_+.

We show that the structure of global minimizers u_ϵ for $0 < \epsilon << 1$ depends drastically on whether $l < h$ of $l > h$, that is on the geometry of Ω. Figures 2 and 3 are simple illustrations of the cases $h/l > 1$ and $h/l < 1$.

We state our main results

Theorem 1.3. ($l < h$, *The Boundary Layer Case*) *There is* $\epsilon_0 > 0$ *such that, if* u_ϵ, $\epsilon \in (0, \epsilon_0]$ *is a minimizer of* (1.5), *then*

$$\begin{aligned} &2\sigma l - C\epsilon \leq \int_R \Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial y}|^2 + \frac{1}{\epsilon}W(u_\epsilon)\Big)dxdy \leq J_\Omega^\epsilon(u_\epsilon) \leq 2\sigma l + C\epsilon|\ln\epsilon|^3, \\ &|u_\epsilon(z) - a_-| \leq Ke^{-\frac{k}{\epsilon}(d(z,\partial^+\Omega) - C\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}})^+}, \quad z \in \Omega, \end{aligned} \tag{1.11}$$

where σ *is the action of the connecting orbit between* a_- *and* a_+ *(cfr.* (1.7)*) and* C, k *and* K *are positive constants, and* $\partial^+\Omega = (0, l) \times \{0, h\}$.

These estimates imply that u_ϵ converges uniformly in compacts in $\overline{\Omega} \setminus \partial^+\Omega$ to a_-. Furthermore Theorem 1.3 is complemented by Theorem 4.8 below, which establishes the existence of a boundary layer at $\partial^+\Omega$ strictly thicker than $O(\epsilon)$. See (1.14) for explanations.

Theorem 1.4. ($l > h$, *The Internal Layer Case*) *There is* $\epsilon_0 > 0$ *such that, if* u_ϵ, $\epsilon \in (0, \epsilon_0]$ *is a minimizer of* (1.5), *then*

$$\begin{aligned} &2\sigma h - C\epsilon \leq \int_D \Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial x}|^2 + \frac{1}{\epsilon}W(u_\epsilon)\Big)dxdy \leq J_\Omega^\epsilon(u_\epsilon) \leq 2\sigma h + C\epsilon, \\ &|u_\epsilon(z) - a_-| \leq Ke^{-\frac{k}{\epsilon}(d(z,R) - C\epsilon^{\frac{1}{4}})^+}, \quad z \in \Omega \setminus R, \\ &|u_\epsilon(z) - a_+| \leq Ke^{-\frac{k}{\epsilon}(d(z,\Omega\setminus R) - C\epsilon^{\frac{1}{4}})^+}, \quad z \in R, \end{aligned} \tag{1.12}$$

where D *is strictly contained in* Ω, *and* K, k *and* C *positive constants.*

These estimates imply that, as $\epsilon \to 0$, u_ϵ converges to the step map u_0:

$$u_0 = \begin{cases} a_-, & \text{in } \Omega \setminus R, \\ a^+, & \text{in } R, \end{cases}$$

FIGURE 3. Case $l > h$. Blue = a_-, red= a_+.

and the convergence is uniform in compacts in $\Omega \setminus \{0, l\} \times [0, h]$ and all the way up to $\partial\Omega$. In particular there are no boundary layers. On the other hand there are two internal layers located within a distance $O(\epsilon^{\frac{1}{4}})$ from the discontinuity sets $\{0\}\times(0, h)$, $\{l\} \times (0, h)$.

At a formal level it is not difficult to understand why minimizers favour the boundary layer or the internal layer by considering the limiting problem and assuming that the Dirichlet data are preserved and comparing the interface energies. We refer to [6] where a scalar problem with ϵ-dependent Dirichlet data similar to ours is considered and the Γ-limit is justified rigorously and also to Theorem 7.10 in [9].

The lower bound estimates have different features that play different roles in the derivation of the pointwise estimates.

Fist note that they involve only part of the gradient. For instance in the internal layer case via the lower and upper bounds we obtain

$$\int_{\Omega} |\frac{\partial u_\epsilon}{\partial y}|^2 dxdy \leq C, \tag{1.13}$$

which implies that, for small $\epsilon > 0$, the interfaces are almost orthogonal to the x axis. We note that this point is reminiscent of an estimate in Alama, Bronsard and Gui [1] and also in Schatzman [22] where, however, the setup and the arguments are entirely different (cfr. [5] Theorem 8.5 and Lemma 9.4). In the boundary layer case the presence of the logarithm term (that we don't think can be removed) does not allow the derivation of a bound independent of ϵ for $\int_\Omega |\frac{\partial u_\epsilon}{\partial x}|^2 dxdy$. However it serves a similar purpose (Lemma 4.9) for establishing the width of the boundary layer as we explain later in this introduction.

The other feature of the lower bounds is that they are based on proper subsets of Ω. This fact, in conjunction with the upper bound can be used to obtain certain refinements and estimates up to the boundary in the remaining part of Ω via the density estimate.

At a formal level it is not difficult to guess that the boundary layer is strictly thicker than $O(\epsilon)$ since there is no connecting orbit in the half line. For establishing this, one needs to show that the problem in the half space

$$\begin{aligned} &\Delta U^0 = W_u(U^0), \quad (\xi, \eta) \in \mathbb{R} \times (0, \infty), \\ &U^0(\xi, 0) = a_+, \quad \xi \in \mathbb{R}, \end{aligned} \tag{1.14}$$

has the unique solutions $U^0 \equiv a_+$. In the scalar case $m = 1$, and in the whole plane (on $\mathbb{R}^n$ generally) it was established by Modica [19] that, if the entire solution U has a point $x_0 \in \mathbb{R}^n$ such that $W(U(x_0)) = 0$, then necessarily $U \equiv a$ where $a = U(x_0) \in A$. He obtained this as a corollary of the so called Modica Inequality

$$\frac{1}{2}|\nabla U|^2 \leq W(U),$$

that is valid for L^∞ solutions of $\Delta U = W_u(U)$, $U : \mathbb{R}^n \to \mathbb{R}$.

Farina and Valdinoci [13] have extended the Modica inequality to solutions satisfying $\Delta U = W_u(U)$ in the half space $\mathbb{R}^{n-1} \times [0, \infty)$ provided that $U \in C^2(\mathbb{R}^{n-1} \times [0, \infty)) \cap L^\infty$. Utilizing this result we settle the question following (1.14) for $m = 1$.

However for $m \geq 2$, P. Smyrnelis [23] (cfr. 3.3 in [5]) has pointed out that the Modica inequality is not generally valid and thus a different approach is required. For this purpose we can employ an appropriate Hamiltonian Identity of Gui [17] (e.g. 3.4 in [5]) together with the analog of (1.13) mentioned above to deduce that

$$\int_{\mathbb{R}} |\frac{\partial U^0}{\partial \eta}(\xi, 0)|^2 d\xi = 0, \tag{1.15}$$

and thus conclude via a unique continuation theorem in [15], that $U^0(\xi, \eta) = a_+$.

The following simple estimate on the 1-dimensional energy is basic and is implemented in several places throughout the paper

$$J_{(s_1,s_2)}(v) = \int_{s_1}^{s_2} \Big(\frac{|v'|^2}{2} + W(v)\Big) ds \geq \sigma - C_W \frac{1}{2}(\delta_-^2 + \delta_+^2), \tag{1.16}$$

where C_W is a constant depending only on W and $|v(s_1) - a_-| \leq \delta_-$ and $|v(s_2) - a_+| \leq \delta_+$ (cfr. Lemma 2.3 below).

The pointwise estimates (1.12) and (1.11) are obtained via linear elliptic theory from an estimate of the type

$$|u_\epsilon(z) - a| \leq \delta, \ \ z \in V \subset \Omega, \text{ some open } V, \tag{1.17}$$

where $\delta > 0$ is a small number.

A route for obtaining (1.17) in a neighborhood of the part of $\partial\Omega$ where $u = a$ is by constructing a V with ∂V partially coinciding with $\partial\Omega$ and such that

$$|u_\epsilon(z) - a| \leq \delta, \ \ z \in \partial V.$$

Then we conclude via Theorem 4.1 in [5]. It should be noted that this argument utilizes the fact that Ω is $2d$ and ∂V is a curve along which (1.16) can be implemented.

Finally a few comments on possible extensions of our results are in order. We have studied the case $l < h$ and $l > h$ and established a significant qualitative difference between the global minimizers. We expect that both types of solutions exist independently of the relation between l and h but as local minimizers. The case $l = h$ is open. Here we expect two types of global minimizers. Extending the results to genuinely higher dimensional examples is not straightforward. A minimizer of the type of Theorem 1.3 should require that the boundary $\partial\Omega$ or part of it is a minimal surface. The lower bound also should be significantly harder since the sections will not be one-dimensional.

The remaining part of the paper is structured as follows: in §2 we present the basic lemmas. In §3 we present a different Example 1. In §4 we study the example covered in Theorems 1.3 and 1.4 above. First we introduce the hypothesis in §4.1. Then in §4.2 we consider the boundary layer case and in §4.3 the internal layer case.

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2. Basic Lemmas

Lemma 2.1. *The nondegeneracy assumptions on the zeros of* W *imply the existence of* $\delta_W > 0$*, and constants* $c_W, C_W > 0$ *such that*

$$\Rightarrow \quad \frac{1}{2}c_W^2|u-a|^2 \leq W(u) \leq \frac{1}{2}C_W^2|u-a|^2; \ |u-a| \leq \delta_W, \ a \in A. \tag{2.1}$$

Moreover, if

$$\begin{aligned} &\min_{a\in A}|u-a| \geq \delta, \\ \text{then} \quad & \\ &\tfrac{1}{2}c_W^2\delta^2 \leq W(u). \end{aligned} \tag{2.2}$$

For a map $v : (s_1, s_2) \to \mathbb{R}^m$ in H^1_{loc}, with $-\infty \leq s_1 < s_2 \leq +\infty$, we define

$$J_{(s_1,s_2)}(v) = \int_{s_1}^{s_2} \Big(\frac{|v'|^2}{2} + W(v)\Big)ds.$$

For $J_{(-\infty,+\infty)}(v)$ we also use the notation $J_{\mathbb{R}}(v)$.

Lemma 2.2. *Given* $a_- \in A$ *there exists* $a_+ \in A' = A \setminus \{a_-\}$ *and a map* $\overline{u} : \mathbb{R} \to \mathbb{R}^m$ *which satisfy*

$$\begin{aligned} &\lim_{s\to-\infty}\overline{u}(s) = a_-, \qquad \lim_{s\to+\infty}\overline{u}(s) = a_+, \\ &J_{\mathbb{R}}(\overline{u}) = \min_v J_{\mathbb{R}}(v), \end{aligned}$$

where the minimization is taken on the set of $v \in H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$ *that satisfy*

$$\lim_{s\to-\infty} v(s) = a_-, \qquad \lim_{s\to+\infty} d(v(s), A') = 0.$$

Given $z \in \mathbb{R}^m \setminus A$ there exist $a_z \in A$ and a map $\overline{u}_z \in H^1_{\mathrm{loc}}([0,+\infty);\mathbb{R}^m)$ that satisfies

$$\overline{u}_z(0_+) = z, \qquad \lim_{s\to+\infty} \overline{u}_z(s) = a_z \in A,$$
$$J_{(0,+\infty)}(\overline{u}_z) = \min J_{(0,+\infty)}(v),$$

where the minimization is taken on the set of $v \in H^1_{\mathrm{loc}}([0,+\infty);\mathbb{R}^m)$ that satisfy

$$v(0) = z, \qquad \lim_{s\to+\infty} d(v(s), A) = 0.$$

Proof. See for example Theorem 2.1 in [5]) and its proof. □

We set $\sigma = J_{\mathbb{R}}(\overline{u})$ and $\sigma_z = J_{(0,+\infty)}(\overline{u}_z)$. That is: σ is the energy of the map $\overline{u}$ that connects a_- to a_+ and σ_z is the energy of the map $\overline{u}_z$ that connects z to a_z.

Set $\Gamma_0(a_\pm) = \{a_\pm\}$ and $\Gamma_\delta(a_\pm) = \partial B_\delta(a_\pm)$ for $\delta > 0$.

Lemma 2.3. *Let $a_\pm$, a_z, $\overline{u}$ and $\overline{u}_z$ be as in Lemma 2.2.*

(i) Let $\delta_\pm \in [0, \delta_W]$, δ_W as in Lemma 2.1, and let $v : (s_-, s_+) \to \mathbb{R}^m$ be a smooth map such that

$$\lim_{s\to s_\pm} d(v(s), \Gamma_{\delta_\pm}(a_\pm)) = 0. \tag{2.3}$$

Then

$$J_{(s_-,s_+)}(v) \geq \sigma - \frac{1}{2}C_W(\delta_-^2 + \delta_+^2). \tag{2.4}$$

(ii) Let $v \in C^1((0,s_+);\mathbb{R}^m) \cap C([0,s_+);\mathbb{R}^m)$,

$$\begin{aligned} &v(0) = z,\\ &\lim_{s\to s_+} d(v(s), \Gamma_{\delta_+}(a_z)) = 0. \end{aligned} \tag{2.5}$$

Then

$$J_{(0,s_+)}(v) \geq \sigma_z - \frac{1}{2}C_W\delta_+^2. \tag{2.6}$$

Proof. 1. For $\delta_- = \delta_+ = 0$ (2.4) is just the statement of the minimality of $\overline{u}$. Therefore we can assume that either δ_- or δ_+ or both are positive. From (2.3) and $\delta_+ > 0$, if $s_+ = +\infty$, it follows $\int_{s_-}^{s_+} W(v)ds = +\infty$ and (2.4) holds trivially. The same is true if $\delta_+ > 0$, $s_+ < +\infty$ and $\lim_{s\to s_+} v(s)$ does not exist. Indeed in this case we have $\int_{s_-}^{s_+} |\dot{v}|^2 ds = +\infty$. It follows that, if $\delta_+ > 0$, we can assume $s_+ < +\infty$ and moreover that

$$\lim_{s\to s_+} v(s) = v_+, \tag{2.7}$$

for some $v_+ \in \Gamma_{\delta_+}(a_+)$. An analogous conclusion applies to the case $\delta_- > 0$.

2. If both δ_- and δ_+ are positive and $w_\pm$ is a test map that connects $v_\pm$ to $a_\pm$, the minimality of $\overline{u}$ implies

$$J_{(s_-,s_+)}(v) \geq \sigma - J(w_-) - J(w_+),$$

where $J(w_\pm)$ is the energy of $w_\pm$. This yields (2.4) provided we show that $w_\pm$ can be chosen so that

$$J(w_\pm) \leq \frac{1}{2}C_W\delta_\pm^2.$$

3. We choose

$$w_+ = (1 - \frac{\gamma(s)}{\delta_+})a_+ + \frac{\gamma(s)}{\delta_+}v_+, \quad \gamma(s) = \delta_+ e^{-C_W(s-s_+)}.$$

If $\delta_\pm \leq \delta_W$, it follows from (2.1) that

$$\frac{1}{2}\int_{s_+}^{+\infty} |\dot{w}_+|^2 ds = \frac{C_W^2}{2}|v_+ - a_+|^2 \int_{s_+}^{+\infty} e^{-2C_W(s-s_+)} ds = \frac{1}{4}C_W\delta_+^2,$$

$$\int_{s_+}^{+\infty} W(w_+) ds \leq \frac{C_W^2}{2}|v_+ - a_+|^2 \int_{s_+}^{+\infty} e^{-2C_W(s-s_+)} ds = \frac{1}{4}C_W\delta_+^2.$$

This and the analogous computation for $J(w_-)$ establish (2.4) for δ_- and δ_+ positive. Clearly (2.4) is valid also if δ_- or δ_+ vanishes. The proof of (2.6) is analogous. The proof is complete. □

We also define σ^* by setting

$$\sigma^* = \inf_v J_{(s_1,s_2)}(v),$$

where $v \in H^1_{\mathrm{loc}}((s_1, s_2); \mathbb{R}^m)$ is a map that satisfies $v(s_1) = z$ and $\lim_{s\to s_2} v(s) \in A \setminus \{a_z\}$.

Note that, if the extreme $a_z \in A$ of $\overline{u}_z$ is uniquely determined, then we have

$$\sigma_z < \sigma^*.$$

The same is true if a_z is not unique provided the set of v is restricted to the maps that satisfy $\lim_{s\to s_2} d(v(s_2), A \setminus A_z) = 0$, where $A_z \subset A$ is the set of possible a_z's.

Lemma 2.4. *Minimizers u_ϵ of problem* (1.5) *(actually $H^1(\Omega;\mathbb{R}^m)$ critical points) satisfy the estimates*

$$\begin{aligned} &\|u_\epsilon\|_{L^\infty} < M, \\ &\|\nabla u_\epsilon\|_{L^\infty} < \frac{C''}{\epsilon}. \end{aligned} \tag{2.8}$$

Proof. By linear elliptic theory $u_\epsilon \in C^2(\Omega;\mathbb{R}^m)$. Set $v_\epsilon = |u_\epsilon|^2$. Then

$$\Delta v_\epsilon = 2W_u(u_\epsilon)\cdot u_\epsilon + 2|\nabla u_\epsilon|^2 > 0, \quad \text{for } |u_\epsilon| > M. \tag{2.9}$$

Hence $\max |u_\epsilon|^2 \leq M^2$ if v_ϵ attains its max in the interior of Ω. On the other hand by (2.9) $\max_{\partial\Omega} |u_\epsilon| \leq M$ and $(2.8)_1$ follows. For the gradient bound rescale the solution of the Euler-Lagrange equation

$$\epsilon\Delta u_\epsilon = \frac{1}{\epsilon}W_u(u_\epsilon),$$

by $\frac{z}{\epsilon}$ and denote by $\tilde{u}, \tilde{g}$ the rescaled u_ϵ, g_ϵ. Then by linear theory (see (8.87) in [16])

$$|\tilde{u}|_{1+\alpha} \leq C'(\|\tilde{u}\|_{L^\infty} + |\tilde{g}|_{1,\alpha}) \leq C''.$$

Hence

$$\|\nabla\tilde{u}\|_{L^\infty} \leq C'',$$

and therefore $(2.8)_2$ follows. □

3. Example 1

We consider the problem

$$\begin{aligned}&\min_{u\in\mathscr{A}} J_\Omega^\epsilon(u),\quad J_\Omega^\epsilon(u) = \int_\Omega \Big(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\Big)dx,\\ &\mathscr{A} = \{u \in H^1(\Omega;\mathbb{R}^m) : u = z,\ x \in \partial\Omega\},\end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded smooth domain and $z \in \mathbb{R}^m \setminus A$ is a fixed vector.

We let $u = u_\epsilon$ a minimizer of (3.1). We now introduce a special system of coordinates in a neighborhood of $\partial\Omega$ that we use for deriving upper and lower bounds for $J_\Omega^\epsilon(u_\epsilon)$.

For each $p \in \partial\Omega$ we let ν_p the unit exterior normal to $\partial\Omega$ at p. The smoothness of Ω implies that there is $h_0 > 0$ such that

$$x(p,h) = p - h\nu_p,\quad p \in \partial\Omega,\ h \in (-h_0,h_0)$$

defines a diffeomorphism of $\partial\Omega \times (-h_0,h_0)$ onto $\{x \in \mathbb{R}^n : d(x,\partial\Omega) < h_0\}$. We have

$$\frac{\partial x(p,h)}{\partial(p,h)} = \begin{pmatrix} 1-hk_1 & 0 & \cdots & 0\\ 0 & 1-hk_2 & 0\cdots & 0\\ \cdots & & & \\ 0 & \cdots 0 & 1-hk_{n-1} & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}$$

and

$$\det\Big(\frac{\partial x(p,h)}{\partial(p,h)}\Big) = \Pi_{j=1}^{n-1}(1-hk_j),$$

where $k_1,\ldots,k_{n-1}$ are the principal curvatures of $\partial\Omega$ at p. We let σ_z, σ^* and $\overline{u}_z$ as in §2. We have

Lemma 3.1. (Upper Bound) *There is a constant* $C_0 > 0$ *such that*

$$J_\Omega^\epsilon(u_\epsilon) \leq \sigma_z\mathcal{H}^{n-1}(\partial\Omega) + C_0\epsilon,$$

for a minimizer u_ϵ *of* (3.1).

Proof. We define a comparison map $\tilde{u} : \overline{\Omega} \to \mathbb{R}^m$ by setting

$$\begin{aligned}&\tilde{u}(x(p,h)) = \overline{u}_z\Big(\frac{h}{\epsilon}\Big),\quad p \in \partial\Omega,\ h \in [0,h_0),\\ &\tilde{u}(x) = \overline{u}_z\Big(\frac{h_0}{\epsilon}\Big),\quad x \in \Omega,\ d(x,\partial\Omega) \geq h_0.\end{aligned}$$

If $e_1, ..., e_{n-1}, \nu_p$ is an orthonormal basis with $e_1, ..., e_{n-1}$ tangent to $\partial\Omega$ at p in the direction of the principal curvatures, we have

$$\tilde{u}_{x_j}(x(p,h)) = 0, \quad h \in (0, h_0), \; j = 1, \ldots, n-1,$$
$$\tilde{u}_{x_n}(x(p,h)) = \frac{1}{\epsilon}\overline{u}_z'(\frac{h}{\epsilon}), \quad h \in (0, h_0).$$

It follows that

$$J^\epsilon_{\{d(x,\partial\Omega)<h_0\}}(\tilde{u}) = \frac{1}{\epsilon}\int_{\partial\Omega}\int_0^{h_0}\Big(\frac{1}{2}|\overline{u}_z'(\frac{h}{\epsilon})|^2 + W(\overline{u}_z(\frac{h}{\epsilon}))\Big)|\Pi_{j=1}^{n-1}(1 - hk_j)|dhdp$$
$$= \int_{\partial\Omega}\int_0^{\frac{h_0}{\epsilon}} |\overline{u}_z'(s)|^2|\Pi_{j=1}^{n-1}(1 - \epsilon s k_j)|dsdp \le \sigma_z \mathcal{H}^{n-1}(\partial\Omega) + \frac{1}{2}C_0\epsilon,$$

where $C_0 > 0$ is a constant and we have used $\frac{1}{2}|\overline{u}_z'(s)|^2 = W(\overline{u}_z(s))$. We also have

$$J^\epsilon_{\{d(x,\partial\Omega)\ge h_0\}}(\tilde{u}) = \frac{1}{\epsilon}W(\overline{u}_z(\frac{h_0}{\epsilon}))\int_{\{d(x,\partial\Omega)\ge h_0\}} dx$$
$$\le \frac{1}{2\epsilon}C_W^2|\overline{u}_z(\frac{h_0}{\epsilon}) - a_z|^2|\Omega| \le \frac{1}{2\epsilon}C_W^2\overline{K}^2 e^{-\frac{2\bar{k}h_0}{\epsilon}}|\Omega| \le \frac{1}{2}C_0\epsilon,$$

where we have used (2.1) and $|\overline{u}_z(s) - a_z| \le \overline{K}e^{-\bar{k}s}$ for some $\bar{k}, \overline{K} > 0$. □

Next we introduce some basic lemmas and the lower bound.

Lemma 3.2. *Let u_ϵ a minimizer of* (3.1) *and $\delta > 0$ a small number. Set $\Omega_\delta = \{x \in \Omega : \min_{a\in A}|u_\epsilon - a|\} \le \delta\}$. Then there is a constant $C_1 > 0$ such that*

$$|\Omega_\delta| \ge |\Omega|(1 - \frac{C_1}{\delta^2}\epsilon),$$

where $|\cdot|$ stands for the n-dimensional Lebesgue measure.

Proof. Let $\Omega_\delta^c = \overline{\Omega} \setminus \Omega_\delta$. Then Lemma 3.1 and (2.2) imply

$$\frac{1}{2}\frac{c_W^2\delta^2}{\epsilon}|\Omega_\delta^c| \le \frac{1}{\epsilon}\int_\Omega W(u_\epsilon)dx \le J^\epsilon_\Omega(u_\epsilon) \le \sigma_z\mathcal{H}^{n-1}(\partial\Omega) + C_0\epsilon.$$

It follows that there is $C_1 > 0$ such that

$$|\Omega_\delta^c| \le \frac{C_1}{\delta^2}|\Omega|\epsilon,$$
$$|\Omega_\delta| \ge |\Omega|(1 - \frac{C_1}{\delta^2}\epsilon).$$

□

Theorem 3.3. (The Lower Bound) *Let u_ϵ a minimizer of* (3.1). *Then there is a constant $C > 0$ such that*

$$J^\epsilon_\Omega(u_\epsilon) \geq \sigma_z \mathcal{H}^{n-1}(\partial\Omega)(1 - C\epsilon^{\frac{1}{3}}). \tag{3.2}$$

Proof. For each $p \in \partial\Omega$ define h_p by setting

$$\begin{aligned} &h_p = h_0, \quad \text{if } p - h\nu_p \in \Omega^c_\delta,\ h \in [0, h_0],\\ &h_p = \max\{h \in (0, h_0] : p - s\nu_p \in \Omega^c_\delta,\ s \in [0, h)\}, \quad \text{otherwise.} \end{aligned}$$

Let $\alpha \in [0, 1)$ a number to be chosen later and let

$$\mathscr{S}^\alpha = \{p \in \partial\Omega : h_p \geq \epsilon^\alpha\}.$$

We have

$$\int_{\mathscr{S}^\alpha}\int_0^{\epsilon^\alpha} |\Pi_j^{n-1}(1 - hk_j)| dh dp \leq \int_{\mathscr{S}^\alpha}\int_0^{h_p} |\Pi_j^{n-1}(1 - hk_j)| dh dp \leq |\Omega^c_\delta| \leq \frac{C_1}{\delta^2}|\Omega|\epsilon.$$

This and $|\Pi_j^{n-1}(1 - hk_j)| \geq 1 - C_k\epsilon^\alpha$ ($C_k > 0$ a constant that depends on the curvatures of $\partial\Omega$) imply the existence of $C_2 > 0$ such that

$$\begin{aligned} &\mathcal{H}^{n-1}(\mathscr{S}^\alpha)\epsilon^\alpha(1 - C_k\epsilon^\alpha) \leq \frac{C_1}{\delta^2}|\Omega|\epsilon,\\ &\Rightarrow \mathcal{H}^{n-1}(\mathscr{S}^\alpha) \leq \frac{C_2}{\delta^2}\epsilon^{1-\alpha}\mathcal{H}^{n-1}(\partial\Omega). \end{aligned} \tag{3.3}$$

Next continuing with the canonical coordinates near $\partial\Omega$ (pag.354 in [16]) and utilizing that $|\nabla u_\epsilon| \geq |\frac{\partial u_\epsilon}{\partial d}|$, d the distance from $\partial\Omega$, and a standard estimate on the Jacobian, already employed above, via Lemma 2.3 we obtain from (3.3)

$$\begin{aligned} J^\epsilon_\Omega(u_\epsilon) &\geq (\sigma_z - \frac{1}{2}C_W\delta^2)\int_{\partial\Omega\setminus\mathscr{S}^\alpha} \min\{1, |\Pi_j^{n-1}(1 - \epsilon^\alpha k_j)|\} dp\\ &\geq (\sigma_z - \frac{1}{2}C_W\delta^2)\int_{\partial\Omega\setminus\mathscr{S}^\alpha} (1 - C_k\epsilon^\alpha) dp\\ &\geq (\sigma_z - \frac{1}{2}C_W\delta^2)\mathcal{H}^{n-1}(\partial\Omega)(1 - \frac{C_2}{\delta^2}\epsilon^{1-\alpha})(1 - C_k\epsilon^\alpha). \end{aligned}$$

It follows

$$J^\epsilon_\Omega(u_\epsilon) \geq \sigma_z\mathcal{H}^{n-1}(\partial\Omega)(1 - C_3(\delta^2 + \epsilon^\alpha + \frac{\epsilon^{1-\alpha}}{\delta^2})),$$

for some constant $C_3 > 0$. This implies (3.2) for $\alpha = \frac{1}{3}$, $\delta^2 = \epsilon^{\frac{1}{3}}$. □

Let $\delta_0 \in (0, \min_{i\neq j}\frac{1}{2}|a_i - a_j|)$ and $\delta \in (0, \delta_0)$ be fixed. A consequence of the upper bound, the lower bound and the density estimate (see for example Theorem 5.2 in [5]) is that a minimizer u remains near just one of the zeros of W throughout Ω.

3.1. **Proof of Theorem 1.2.**

Proof. 1. Assume $x_0 \in \Omega_\delta^c$. Then by Lemma 2.4 we have the gradient bound

$$|\nabla u_\epsilon| \leq \frac{M}{\epsilon}$$

which implies

$$\min_j |u_\epsilon(x) - a_j| > \frac{\delta}{2}, \quad x \in B_{\frac{\delta\epsilon}{2M}}(x_0). \tag{3.4}$$

Let $v : \frac{\Omega}{\epsilon} \to \mathbb{R}^m$ be defined by $v(\frac{x}{\epsilon}) = u_\epsilon(x)$. Then the minimality of u_ϵ implies that v minimizes $\int_{\frac{\Omega}{\epsilon}} (\frac{1}{2}|\nabla v|^2 + W(v))dy$ and (3.4) implies that v satisfies

$$\min_j |v(y) - a_j| > \frac{\delta}{2}, \quad y \in B_{\frac{\delta}{2M}}(y_0).$$

Then arguing as in the proof of Lemma 5.5 in [5] we obtain, via the density estimate, with $c'(\delta)$ a constant,

$$|B_r(y_0) \cap \{y : \min_j |v(y) - a_j| > \delta\}| \geq (\delta_0 - \delta)c'(\delta)r^{n-1}, \quad r \leq d(y_0, \partial(\frac{\Omega}{\epsilon})),$$

which, in terms of u_ϵ, becomes

$$\epsilon^{-n}|B_{\epsilon r}(x_0) \cap \Omega_\delta^c| \geq (\delta_0 - \delta)c'(\delta)r^{n-1}, \quad \epsilon r \leq d(x_0, \partial\Omega).$$

It follows

$$J^\epsilon_{B_{\epsilon r}(x_0)}(u_\epsilon) \geq w_\delta(\delta_0 - \delta)c'(\delta)(\epsilon r)^{n-1} = C_\delta(\epsilon r)^{n-1}, \quad \epsilon r \leq d(x_0, \partial\Omega), \tag{3.5}$$

where $w_\delta = \min_{x\in\Omega_\delta^c} W(u_\epsilon(x)) > 0$ and $C_\delta = w_\delta(\delta_0 - \delta)c'(\delta)$.

2. From the proof of Theorem 3.3 we see that the right hand side of (3.2) is an estimate of the energy contained in an $\epsilon^{\frac{1}{3}}$−neighborhood of $\partial\Omega$. Therefore we can add the energy $J_{B_{\epsilon r}(x_0)}$ and improve the lower bound given by (3.2) provided x_0 and r satisfy the condition

$$d(x_0, \partial\Omega) > \epsilon^{\frac{1}{3}} + \epsilon r.$$

In order to contradict the assumption that $x_0 \in \Omega_\delta^c$ by utilizing the upper bound given by Lemma 3.1 we choose r by imposing the condition

$$C_\delta(\epsilon r)^{n-1} = 2\sigma_z\mathcal{H}^{n-1}(\partial\Omega)C\epsilon^{\frac{1}{3}} \ \Leftrightarrow\ r = (\frac{2\sigma_z\mathcal{H}^{n-1}(\partial\Omega)C}{C_\delta})^{\frac{1}{n-1}}\epsilon^{-(1-\frac{1}{3(n-1)})},$$

where $C > 0$ is the constant in (3.2). With this choice of r we obtain that the existence of a point $x_0 \in \Omega_\delta^c$ that satisfies $d(x_0, \partial\Omega) > \epsilon^{\frac{1}{3}} + C'\epsilon^{\frac{1}{3(n-1)}}$ (here $C' = (\frac{2\sigma_z\mathcal{H}^{n-1}(\partial\Omega)C}{C_\delta})^{\frac{1}{n-1}}$) implies

$$J^\epsilon_\Omega(u_\epsilon) \geq \sigma_z\partial\Omega(1 + C\epsilon^{\frac{1}{3}})$$

which is in conflict with Lemma 3.1. This and the smoothness of the minimizer u imply that there is $a_0 \in A$ such that

$$d(x_0, \partial\Omega) \geq (2C' + 1)\epsilon^{\frac{1}{3(n-1)}} \ \Rightarrow\ |u_\epsilon(x) - a_0| < \delta. \tag{3.6}$$

3. Finally we show that $a_0 \in A_z$, $A_z \subset A$ the set of possible a_z's. If this is not the case, from Lemmas 2.3 and 3.1 we have

$$(\sigma^* - \frac{1}{2}C_W\delta^2)\mathcal{H}^{n-1}(\partial\Omega)(1 - C_K\epsilon^{\frac{1}{3(n-1)}}) \leq \sigma_z\mathcal{H}^{n-1}(\partial\Omega) + C_0\epsilon,$$

where $C_K > 0$ is a constant that depends on C' and from the curvature of $\partial\Omega$. Since $\sigma^* > \sigma_z$ this inequality cannot hold if δ and ϵ are sufficiently small. This proves the claim.

From (3.6) with $a_0 = a_z$ and via linear elliptic theory we obtain the exponential estimate. The proof is complete. □

4. Example 2

We consider the minimization problem

$$\min_{u\in\mathscr{A}} J^{\epsilon}_{\Omega}(u), \quad J^{\epsilon}_{\Omega}(u) = \int_{\Omega} \Big(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\Big)dx,$$
$$\mathscr{A} = \{u \in H^1(\Omega;\mathbb{R}^m) : u = g_\epsilon,\ x \in \partial\Omega\}. \tag{4.1}$$

under the hypothesis on the potential W and the Dirichlet data stated in the Introduction (below (1.1), in (1.2) and in (1.6))

4.1. $\frac{h}{l} > 1$ **The boundary layer case.** In this section we analyze in detail the structure of the minimizers of $J^{\epsilon}_{\Omega}(u)$ under the assumption

$$l < h. \tag{4.2}$$

We will establish Theorem 1.3 of the Introduction. The proof is based on the tight lower/upper energy bounds and on the vector analog of the Caffarelli-Cordoba density estimate (see [[5] Theorem 5.2]).

Set $J^{\epsilon}_{\mathbb{R}}(v) = \int_{\mathbb{R}}(\frac{\epsilon}{2}|v'|^2 + \frac{1}{\epsilon}W(v))ds$ and note that

$$\sigma = J^{\epsilon}_{\mathbb{R}}(\overline{u}(\frac{\cdot}{\epsilon})) = J_{\mathbb{R}}(\overline{u}).$$

4.1.1. *The Upper Bound.*

Proposition 4.1. *There is a constant* $C_1 > 0$ *such that, if* $u_\epsilon : \Omega \to \mathbb{R}^m$ *is a minimizer of the problem* (4.1) *then*

$$J^{\epsilon}_{\Omega}(u_\epsilon) \leq 2l\sigma + C_1\epsilon|\ln\epsilon|^3. \tag{4.3}$$

Remark 1. We don't believe that the logarithm can be removed.

Proof. We show that there exists a test map $\tilde{u} : \Omega \to \mathbb{R}^m$ that satisfies the bound (4.3). Let $\eta > 0$ a small number to be chosen later and set

$$L_1 = [C_0\epsilon, l - C_0\epsilon] \times [0, 2\eta],\ L_2 = [C_0\epsilon, l - C_0\epsilon] \times [h - 2\eta, h],$$
$$T^1 = \{(x,y) : 0 \leq y \leq \frac{2\eta}{C_0\epsilon}x,\ x \in [0, C_0\epsilon]\}.$$

We denote by T^2 the reflection of T^1 in the x axis, by T^3 the reflection of T^1 in the y axis, by T_4 the reflection of T^2 in the y axis and by T^j_τ, we denote T^j subjected to the translation $\tau \in \mathbb{R}^2$. Set

$$B = L_1 \cup L_2 \cup T^1 \cup T^2_{(0,h)} \cup T^3_{(l,0)} \cup T^4_{(l,h)}.$$

We set $\tilde{u}(x, y) = a_-$ for $(x, y) \in \Omega \setminus B$. To define $\tilde{u}$ in $L_1 \cup L_2$ we use the map $\overline{v} : [-\eta, \eta] \to \mathbb{R}^m$ given by

$$\overline{v}(s) = \begin{cases} \frac{1}{C_2\epsilon}\Big((-\eta + C_2\epsilon - s)a_- + (\eta + s)\overline{u}(C_2 - \frac{\eta}{\epsilon})\Big), & -\eta \leq s \leq -\eta + C_2\epsilon, \\ \overline{u}(\frac{s}{\epsilon}), \quad -\eta + C_2\epsilon \leq s \leq \eta - C_2\epsilon, \\ \frac{1}{C_2\epsilon}\Big((\eta - s)\overline{u}(\frac{\eta}{\epsilon} - C_2) + (s - \eta + C_2\epsilon)a_+\Big), & \eta - C_2\epsilon \leq s \leq \eta. \end{cases}$$

There are positive constants k and K such that (see Proposition 2.4 in [5])

$$\begin{aligned} &|\overline{u}(s) - a_-| \leq Ke^{ks}, \quad s \leq 0, \\ &|\overline{u}(s) - a_+| \leq Ke^{-ks}, \quad s \geq 0. \end{aligned} \tag{4.4}$$

We take $\eta = \frac{1}{2k}\epsilon|\ln\epsilon|$. Then (4.4) implies

$$\begin{aligned} &|\overline{u}(C_2 - \frac{\eta}{\epsilon}) - a_-| \leq Ke^{C_2}e^{\ln\epsilon^{\frac{1}{2}}} = C_3\epsilon^{\frac{1}{2}}, \\ &|\overline{u}(\frac{\eta}{\epsilon} - C_2) - a_+| \leq Ke^{C_2}e^{\ln\epsilon^{\frac{1}{2}}} = C_3\epsilon^{\frac{1}{2}}. \end{aligned}$$

This, (2.1), the definition of $\overline{v}$ and a standard computation yield

$$J^\epsilon_{(-\eta,-\eta+C_2\epsilon)}(\overline{v}), J^\epsilon_{(\eta-C_2\epsilon,\eta)}(\overline{v}) \leq \frac{1}{2}(\frac{C_3^2}{C_2} + C_W^2 C_3^2 C_2)\epsilon = C_4\epsilon. \tag{4.5}$$

This and the obvious inequality $J^\epsilon_{(-\eta+C_2\epsilon,\eta-C_2\epsilon)}(\overline{v}) \leq \sigma$ yield

$$J^\epsilon_{(-\eta,\eta)}(\overline{v}) \leq \sigma + 2C_4\epsilon. \tag{4.6}$$

We set

$$\tilde{u}(x, y) = \begin{cases} \overline{v}(y - h + \eta), \ (x, y) \in L_2, \\ \overline{v}(\eta - y), \ (x, y) \in L_1. \end{cases} \tag{4.7}$$

It remains to define $\tilde{u}$ on $T^1 \cup T^2_{(0,h)} \cup T^3_{(l,0)} \cup T^4_{(l,h)}$. We define $\tilde{u}$ on T^1. The definition on $T^2_{(0,h)}, T^3_{(l,0)}, T^4_{(l,h)}$ is similar. Let $(r, s) \in \mathbb{R}^2$ be coordinates with respect to a positively oriented system with origin in $(0, 0)$ and the s axis coinciding with the line $y = \frac{2\eta}{C_0\epsilon}x$. We define $\tilde{u}|_{T^1}$ on each line parallel to the s axis by linear interpolation between the values on the intersection of the line with the boundary of T^1. Set $\lambda = (C_0^2\epsilon^2 + 4\eta^2)^{\frac{1}{2}}$, $\alpha = \frac{C_0\epsilon}{\lambda}$ and $\beta = \frac{2\eta}{\lambda}$. With this definition it follows that

$$\tilde{u}(r, s) = \frac{\lambda - \frac{\beta}{\alpha}r - s}{\lambda - \frac{r}{\alpha\beta}} g(\frac{r}{\beta}, 0) + \frac{s - \frac{\alpha}{\beta}r}{\lambda - \frac{r}{\alpha\beta}} \overline{v}(-\eta + \frac{r}{\alpha}), \quad s \in [\frac{\alpha}{\beta}r, \lambda - \frac{\beta}{\alpha}r], \ r \in [0, \alpha\beta\lambda]. \tag{4.8}$$

Since $\tilde{u}$ is uniformly bounded for $\epsilon \to 0^+$ the same is true for $W(\tilde{u})$. It follows that

$$\frac{1}{\epsilon}\int_{T^1} W(\tilde{u}) \leq CC_0\eta = C\epsilon|\ln\epsilon|. \tag{4.9}$$

From (4.8) we have

$$\tilde{u}_s(r,s) = \frac{1}{\lambda - \frac{r}{\alpha\beta}}(\overline{v}(-\eta + \frac{r}{\alpha}) - g(\frac{r}{\beta},0)),$$

and therefore

$$\epsilon\int_{T^1}|\tilde{u}_s|^2 = \epsilon\int_0^{\alpha\beta\lambda}\int_{\frac{\alpha}{\beta}r}^{\lambda-\frac{\beta}{\alpha}r}|\tilde{u}_s|^2 ds dr = \epsilon\int_0^{\alpha\beta\lambda}\frac{\alpha\beta}{\alpha\beta\lambda - r}|\overline{v}(-\eta + \frac{r}{\alpha}) - g(\frac{r}{\beta},0)|^2 dr.$$

Observe that from $\beta\lambda = 2\eta$ and $\alpha\lambda = C_0\epsilon$ it follows that

$$r = \alpha\beta\lambda \quad \Rightarrow \quad \overline{v}(-\eta + \frac{r}{\alpha}) = \overline{v}(\eta) = g(\frac{r}{\beta},0)) = g(C_0\epsilon,0) = a_+.$$

This and the fact that $|\overline{v}'|, |g_x| \leq \frac{C}{\epsilon}$ imply

$$\begin{aligned}|\overline{v}(-\eta + \frac{r}{\alpha}) - g(\frac{r}{\beta},0)| &\leq |\overline{v}(-\eta + \frac{r}{\alpha}) - a_+| + |g(\frac{r}{\beta},0) - a_+| \\ &\leq \frac{C}{\epsilon}(|2\eta - \frac{r}{\alpha}| + |\frac{r}{\beta} - \alpha\lambda|) = \frac{C}{\epsilon}\frac{\alpha+\beta}{\alpha\beta}(\alpha\beta\lambda - r).\end{aligned} \tag{4.10}$$

With this estimate we obtain

$$\begin{aligned}\epsilon\int_{T^1}|\tilde{u}_s|^2 &\leq \epsilon\int_0^{\alpha\beta\lambda}\frac{C^2}{\epsilon^2}\frac{(\alpha+\beta)^2}{\alpha\beta}(\alpha\beta\lambda - r)dr \\ &= \frac{C^2}{2\epsilon}\alpha\beta\lambda^2(\alpha+\beta)^2 \leq \frac{C^2}{\epsilon}\alpha\beta\lambda^2 \leq C\epsilon|\ln\epsilon|.\end{aligned}$$

After some manipulation we obtain

$$\begin{aligned}\tilde{u}_r(r,s) &= \frac{1}{\alpha\beta}(\alpha^2\lambda - s)\frac{g(\frac{r}{\beta},0) - \overline{v}(-\eta + \frac{r}{\alpha})}{(\lambda - \frac{r}{\alpha\beta})^2} \\ &+ \frac{1}{\beta}\frac{\lambda - \frac{\beta}{\alpha}r - s}{\lambda - \frac{r}{\alpha\beta}}g_x(\frac{r}{\beta},0) + \frac{1}{\alpha}\frac{s - \frac{\alpha}{\beta}r}{\lambda - \frac{r}{\alpha\beta}}\overline{v}'(-\eta + \frac{r}{\alpha}).\end{aligned}$$

This and (4.10) imply

$$|\tilde{u}_r|^2 \leq \frac{C}{\epsilon^2}\frac{(\alpha^2\lambda - s)^2 + \alpha^2(\lambda - \frac{\beta}{\alpha}r - s)^2 + \beta^2(s - \frac{\alpha}{\beta}r)^2}{(\alpha\beta\lambda - r)^2}.$$

Observing that

$$\int_{\frac{\alpha}{\beta}r}^{\lambda-\frac{\beta}{\alpha}r}\Big((\alpha^2\lambda-s)^2+\alpha^2(\lambda-\frac{\beta}{\alpha}r-s)^2+\beta^2(s-\frac{\alpha}{\beta}r)^2\Big)ds=\frac{1+\alpha^6+\beta^6}{3\alpha^3\beta^3}(\alpha\beta\lambda-r)^3,$$

we finally obtain

$$\begin{aligned}\epsilon\int_{T^1}|\tilde{u}_r|^2&=\epsilon\int_0^{\alpha\beta\lambda}\int_{\frac{\alpha}{\beta}r}^{\lambda-\frac{\beta}{\alpha}r}|\tilde{u}_r|^2dsdr\\&\leq\frac{C}{\epsilon\alpha^3\beta^3}\int_0^{\alpha\beta\lambda}(\alpha\beta\lambda-r)dr=\frac{C\lambda^2}{2\epsilon\alpha\beta}\leq C\epsilon|\ln\epsilon|^3.\end{aligned}$$

The proof is complete. □

4.1.2. *The Lower Bound.* We now derive a lower bound for the energy of a minimizer $u_\epsilon:\Omega\to\mathbb{R}^m$ of problem (4.1) (see also Corollary 4.4 below).

Proposition 4.2. *There exist $\epsilon_0>0$ and a constant $\tilde{C}>0$ independent of $\epsilon\in(0,\epsilon_0]$ such that, if $u_\epsilon:\Omega\to\mathbb{R}^m$ is a minimizer of problem* (4.1)*, then*

$$J_\Omega^\epsilon(u_\epsilon)\geq\int_0^l\int_0^h\Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial y}|^2+\frac{1}{\epsilon}W(u_\epsilon)\Big)dydx\geq 2l\sigma-\tilde{C}\epsilon^{\frac{1}{2}}. \tag{4.11}$$

Proof. 1. Given $y\in(0,h)$, let $\Sigma_y=\{(x',y')\in\Omega:y'=y\}$ and let $Y\subset(0,h)$ be the set

$$Y=\{y\in(0,h):\Sigma_y\cap\{\min_{a\neq a_-}|u_\epsilon-a|\leq\delta\}=\emptyset\},$$

where $\delta=\delta_\epsilon$ is to be chosen later. That is the set of sections Σ_y on which u_ϵ is δ away from all $a\neq a_-$. To estimate the measure $\mathcal{H}^1(Y)$ from below we note that $y\in(0,h)\setminus Y$ implies the existence of $a\neq a_-$ and $(x(y),y)$ such that $|u_\epsilon(x(y),y)-a|<\delta$. From this, (4.3) and Lemma 2.3 we obtain

$$\begin{aligned}2l\sigma+C_1\epsilon|\ln\epsilon|^3&\geq\int_{(0,h)\setminus Y}\int_{\Sigma_y}(\frac{\epsilon}{2}|\frac{\partial}{\partial x}u_\epsilon(x,y)|^2+\frac{1}{\epsilon}W(u_\epsilon(x,y)))dxdy\\&\geq(2\sigma-C_W\delta^2)(h-\mathcal{H}^1(Y)).\end{aligned}$$

It follows, for $\delta>0$ small, that

$$\mathcal{H}^1(Y)\geq h-l-\mathrm{O}(\delta^2)\geq\frac{1}{2}(h-l).$$

2. Among the sections Σ_y considered in the previous step (a large set) consider those that contain a subset on which u_ϵ is also δ-away from a_-. We will show that

this subset has small measure for most of these sections. Set $\Sigma_y^* = \{(x,y) \in \Sigma_y : |u_\epsilon(x,y) - a_-| > \delta\}$. Then the definition of Y implies

$$\min_{a\in A} |u_\epsilon - a| > \delta, \quad (x,y) \in \Sigma_y^*, \ y \in Y,$$

and therefore, from Lemma 2.1 and Proposition 4.1 we get

$$\begin{aligned} &\frac{1}{2\epsilon} C_W^2 \delta^2 \int_Y \mathcal{H}^1(\Sigma_y^*) \leq 2l\sigma + C_1\epsilon|\ln\epsilon|^3, \\ &\Rightarrow \int_Y \mathcal{H}^1(\Sigma_y^*) \leq C^* \frac{\epsilon}{\delta^2}, \quad C^* = 8\frac{l\sigma}{C_W^2}. \end{aligned} \tag{4.12}$$

Let $Y^* = \{y \in Y : \mathcal{H}^1(\Sigma_y^*) \geq \frac{2}{\mathcal{H}^1(Y)} C^* \frac{\epsilon}{\delta^2}\}$. Then (4.12) yields $\mathcal{H}^1(Y^*) \leq \frac{1}{2}\mathcal{H}^1(Y)$, and in turn by step 1.

$$\mathcal{H}^1(Y \setminus Y^*) \geq \frac{1}{4}(h - l).$$

For each $y \in Y \setminus Y^*$ we have that $\mathcal{H}^1(\Sigma_y^*) < \frac{2}{\mathcal{H}^1(Y)} C^* \frac{\epsilon}{\delta^2}$. It follows that

$$y \in Y \setminus Y^* \ \Rightarrow \ \mathcal{H}^1(\{x \in (0,l) : |u_\epsilon(x,y) - a_-| \leq \delta\}) > l - \frac{2}{\mathcal{H}^1(Y} C^* \frac{\epsilon}{\delta^2}. \tag{4.13}$$

3. From step 2. there exist $y_1, y_2 \in (0,h)$, $y_2 - y_1 \geq \frac{1}{4}(h-l)$, that satisfy (4.13). The boundary condition implies $u_\epsilon(x,h) = a_+$ for $x \in (C_0\epsilon, l - C_0\epsilon)$. Hence from (4.13) we have

$$\mathcal{H}^1(\{x \in (C_0\epsilon, l - C_0\epsilon) : |u_\epsilon(x,y_2) - a_-| \leq \delta\}) > l - 2C_0\epsilon - \frac{2}{\mathcal{H}^1(Y)} C^* \frac{\epsilon}{\delta^2}.$$

This and Lemma 2.3 imply

$$\begin{aligned} J^\epsilon_{(C_0\epsilon, l-C_0\epsilon)\times(y_2,h)}(u_\epsilon) &\geq \int_{\{x\in(C_0\epsilon,l-C_0\epsilon):|u_\epsilon(x,y_2)-a_-|\leq\delta\}} J^\epsilon_{(y_2,h)}(u_\epsilon(x,\cdot))dx \\ &\geq (\sigma - \frac{1}{2}C_W\delta^2)(l - 2C_0\epsilon - \frac{2}{\mathcal{H}^1(Y)} C^* \frac{\epsilon}{\delta^2}). \end{aligned} \tag{4.14}$$

Since a similar estimate holds for $J^\epsilon_{(C_0\epsilon,l-C_0\epsilon)\times(0,y_1)}(u_\epsilon)$, the lower bound (4.11) follows from (4.14) with $\delta = \epsilon^{\frac{1}{4}}$ and from the fact that $J^\epsilon_{(0,y_1)}(u_\epsilon(x,\cdot))$ and $J^\epsilon_{(y_2,h)}(u_\epsilon(x,\cdot))$ only account for the derivative $\frac{\partial}{\partial y}u_\epsilon$. The proof is complete. □

Remark 2. The lower bound in Proposition 4.2 is an estimate from below of the energy of the minimizer u_ϵ in the set $D = [0,l] \times ([0,y_1] \cup [y_2,h])$ which is quite tight with respect the upper bound (4.3) on the whole domain. Hence we expect that, in $\Omega \setminus D$, the minimizer to be close to a_- uniformly.

Set $\delta_0 = \frac{1}{2}\min_{i\neq j}|a_i - a_j|$.

Lemma 4.3. *Fix $\delta \in (0, \delta_0]$ then there is a constant $C_\delta > 0$ such that*

$$z \in \Omega,\ d(z, D)) \geq C_\delta \epsilon^{\frac{1}{2}} \quad \Rightarrow\ |u_\epsilon(z) - a_-| \leq \delta. \tag{4.15}$$

Moreover there is a constant $k > 0$ such that

$$|u_\epsilon(z) - a_-| \leq \delta e^{-\frac{k}{\epsilon}(d(z,D)-C_\delta \epsilon^{\frac{1}{2}})^+},\ \ z \in \Omega \setminus D. \tag{4.16}$$

Proof. 1. We first establish (4.15). Suppose that $\min_{a\in A} |u_\epsilon(z_0) - a| > \delta$ for some $z_0 \in \Omega \setminus D$. From the gradient bound $|\nabla u_\epsilon| \leq \frac{M}{\epsilon}$ it follows that $\min_{a\in A} |u_\epsilon(z) - a| > \frac{\delta}{2}$ for $z \in B_{\frac{\delta\epsilon}{2M}}(z_0)$. Then by the Caffarelli-Cordoba (vector version) density estimate we obtain

$$J^\epsilon_{B_{\epsilon r}(z_0)}(u_\epsilon) \geq C'_\delta \epsilon r,\ \ \text{for } \epsilon r \leq d(z_0, \partial(\Omega \setminus D)), \tag{4.17}$$

for some $C'_\delta > 0$. We choose r by imposing

$$C'_\delta \epsilon r = 2\tilde{C}\epsilon^{\frac{1}{2}} \quad \Leftrightarrow \quad r = \frac{2\tilde{C}}{C'_\delta}\epsilon^{-\frac{1}{2}},$$

where $\tilde{C}$ is the constant in Proposition 4.2. With this choice of r we see that the existence of $z_0 \in \Omega \setminus D$, $d(z_0, \partial(\Omega \setminus D)) \geq \frac{2\tilde{C}}{C'_\delta}\epsilon^{\frac{1}{2}}$, implies by Proposition 4.2 and (4.17)

$$J^\epsilon_\Omega(u_\epsilon) \geq 2l\sigma + \tilde{C}\epsilon^{\frac{1}{2}}$$

that contradicts Proposition 4.1 for small ϵ. It follows that

$$z \in \Omega \setminus D,\ d(z, \partial(\Omega \setminus D)) \geq C^*_\delta \epsilon^{\frac{1}{2}} \quad \Rightarrow\ |u_\epsilon(z) - a_z| < \delta,\ \text{for some } a_z \in A, \tag{4.18}$$

where we have set $C^*_\delta = \frac{2\tilde{C}}{C'_\delta}$.

2. $a_z \equiv a_-$. The smoothness of $\partial\Omega$ implies the existence of $d > 0$ such that the points $z \in C_d$, $C_d = \{z \in \Omega : d(z, \partial\Omega) \leq d\}$ can be represented with coordinates (s, r) where $r = d(z, \partial\Omega)$ and s, in each connected component of $\partial\Omega$, is the arc-length of the orthogonal projection of z on $\partial\Omega$.

Given $z_0 = z(0, s_0) \in \partial\Omega$ and a small interval (s_1, s_2) that contains s_0 set

$$N_{s_0,(s_1,s_2)} = \{z = z(r, s) : r \in (0, C^*_\delta \epsilon^{\frac{1}{2}}),\ s \in (s_1, s_2)\}.$$

Assume that $d(N_{s_0,(s_1,s_2)}, D) \geq C^*_\delta \epsilon^{\frac{1}{2}}$ and observe that, since a_z is locally constant by the continuity of u_ϵ, $a_{z(C^*_\delta \epsilon^{\frac{1}{2}}, s_0)} \neq a_-$ implies via Lemma 2.3,

$$J^\epsilon_{N_{s_0,(s_1,s_2)}}(u_\epsilon) \geq (\sigma - C_W \delta^2)(s_2 - s_1)(1 - \frac{C^*_\delta \epsilon^{\frac{1}{2}}}{\rho}), \tag{4.19}$$

where $\rho > 0$ is a lower bound for the the radius of curvature of $\partial\Omega$. Since the proof of Proposition 4.2 implies

$$J^\epsilon_D(u_\epsilon) \geq 2l\sigma - \tilde{C}\epsilon^{\frac{1}{2}}, \tag{4.20}$$

(4.19) is in contradiction with the upper bound given by Proposition 4.1 and we conclude

$$a_{z(C^*_\delta \epsilon^{\frac{1}{2}}, s_0)} = a_-.$$

This, the fact that s_0 is arbitrary and the continuity of u_ϵ implies the claim and we have

$$z \in \Omega \setminus D,\ d(z, \partial(\Omega \setminus D)) \geq C_\delta^* \epsilon^{\frac{1}{2}} \quad \Rightarrow\ |u_\epsilon(x,y) - a_-| \leq \delta. \tag{4.21}$$

3. We now show that this can be upgraded to (4.15) thus eliminating $\partial\Omega^-$, that is the part of $\partial(\Omega \setminus D)$ near which we do not expect a boundary layer, and thus allowing z all the way to the boundary $\partial\Omega^-$. The tool for accomplishing this is the cut-off Lemma (Theorem 4.1 in [5]). We now describe a construction which is needed for closing the potential boundary layer. Let $C > 0$ be fixed and assume that, for some value s_0 of the curvilinear abscissa on $\partial\Omega$, it follows that

$$\begin{aligned} &d(z(r,s_0), D) \geq C\epsilon^{\frac{1}{2}},\quad r \in [0, C_\delta^* \epsilon^{\frac{1}{2}}],\\ &\max_{r\in[0,C_\delta^*\epsilon^{\frac{1}{2}}]} |u_\epsilon(z(r,s_0)) - a_-| \geq \delta. \end{aligned} \tag{4.22}$$

Then there exists $r_0 \in (0, C_\delta^* \epsilon^{\frac{1}{2}})$ such that

$$\begin{aligned} &|u_\epsilon(z(r,s_0)) - a_-| < \delta,\quad r \in [0, r_0),\\ &|u_\epsilon(z(r_0,s_0)) - a_-| = \delta. \end{aligned}$$

Since $\delta > 0$ is small Lemma 2.1 implies

$$W(u_\epsilon(z(r,s_0))) \geq \frac{c_W^2}{2}|u_\epsilon(z(r,s_0)) - a_-|^2 \ \ , r \in (0, r_0).$$

This and $v(r) = u_\epsilon(z(r,s_0)) - a_-$ imply

$$\begin{aligned} j(s_0) =: &\int_0^{r_0} \Big(\frac{\epsilon}{2}|\frac{\partial}{\partial r}u_\epsilon(z(r,s_0))|^2 + \frac{1}{\epsilon}W(u_\epsilon(z(r,s_0)))\Big)dr\\ \geq &\int_0^{r_0} \frac{1}{2}(\epsilon|v'|^2 + \frac{c_W^2}{\epsilon}|v|^2)dr \geq \int_0^{r_0} \frac{1}{2}(\epsilon|\hat{v}'|^2 + \frac{c_W^2}{\epsilon}|\hat{v}|^2)dr = \frac{\delta^2 c_W}{2}\frac{\cosh\frac{c_W}{\epsilon}r_0}{\sinh\frac{c_W}{\epsilon}r_0}, \end{aligned} \tag{4.23}$$

where the map $\hat{v}(r) = \delta\frac{\sinh\frac{c_W}{\epsilon}r}{\sinh\frac{c_W}{\epsilon}r_0}n$, $n = \frac{u(z_0)-a_-}{|u(z_0)-a_-|}$, minimizes $\int_0^{r_0}\frac{1}{2}(\epsilon|v'|^2 + \frac{c_W^2}{\epsilon}|v|^2)dr$ in the class of maps that satisfy $v(0) = 0$, $v(r_0) = \delta n$. It follows that

$$j(s_0) \geq \frac{c_W\delta^2}{2}. \tag{4.24}$$

Let $S = \{s_0 : (4.22)\ \text{holds}\}$. Then (4.24) implies

$$J^\epsilon_{\Omega\setminus D}(u_\epsilon) \geq \int_S j(s_0)(1 - \frac{C_\delta^*\epsilon^{\frac{1}{2}}}{\rho})ds_0 \geq \mathcal{H}^1(S)\frac{c_W\delta^2}{2}(1 - \frac{C_\delta^*\epsilon^{\frac{1}{2}}}{\rho}), \tag{4.25}$$

This, (4.25), (4.20) and (4.3) imply

$$\mathcal{H}^1(S)\frac{c_W\delta^2}{2}(1 - \frac{C_\delta^*\epsilon^{\frac{1}{2}}}{\rho}) + 2l\sigma - \tilde{C}\epsilon^{\frac{1}{2}} \leq 2l\sigma + C_1\epsilon|\ln\epsilon|^3$$

and we obtain

$$\mathcal{H}^1(S) \leq 2\frac{\tilde{C}\epsilon^{\frac{1}{2}} + C_1|\epsilon\ln\epsilon|^3}{c_W\delta^2(1-\frac{C_\delta^*\epsilon^{\frac{1}{2}}}{\rho})} \leq \hat{C}\epsilon^{\frac{1}{2}}. \tag{4.26}$$

From (4.26) it follows that, given $z(0,s_0) \in \partial\Omega^-$ with $d(z(0,s_0),D) \geq C\epsilon^{\frac{1}{2}}$ for some $C > 0$ sufficiently large, there are $s_1 < s_0 < s_2$ with $s_2 - s_1 \leq 2\hat{C}\epsilon^{\frac{1}{2}}$ such that (recalling also that we have $u_\epsilon = a_-$ on $\partial\Omega^-$)

$$|u_\epsilon(z) - a_-| \leq \delta, \ \ z \in \partial N_{s_0,(s_1,s_2)}.$$

This, since $\delta > 0$ is a small number, implies that we can invoke Theorem 4.1 in [5] and conclude that

$$|u_\epsilon(z) - a_-| \leq \delta, \ \ z \in N_{s_0,(s_1,s_2)}.$$

This and the fact that the only condition imposed to s_0 is $d(z(0,s_0),D) \geq C\epsilon^{\frac{1}{2}}$ imply (4.15) for some constant $C_\delta > 0$.

4. Finally we establish (4.16) by applying linear theory. Set $\Omega^\epsilon = \{\zeta : \epsilon\zeta \in \Omega\}$. The map $U^\epsilon : \Omega^\epsilon \to \mathbb{R}^m$, $U^\epsilon(\zeta) = u_\epsilon(\epsilon\zeta)$ is a minimizer of $\int_{\Omega^\epsilon}\Big(\frac{1}{2}|\nabla V|^2 + W(V)\Big)d\xi d\eta$. Fix $z \in \Omega$ with $d(z,D) \geq C\epsilon^{\frac{1}{2}}$. Now (4.15) implies that we can assume $C > 0$ sufficiently large to ensure

$$|u_\epsilon - a_-| \leq \delta, \quad \text{on} \quad B_R(z) \cap \Omega, \quad R = d(z,D) - C\epsilon^{\frac{1}{2}}.$$

Set $E = B_R(z) \cap \Omega$ and $E^\epsilon = \{\zeta : \epsilon\zeta \in E\}$. The minimizer U^ϵ satisfies

$$\begin{aligned} &|U^\epsilon - a_-| \leq \delta, \quad \text{on} \quad E^\epsilon, \\ &U^\epsilon = a_-, \quad \text{on} \quad \partial E^\epsilon \cap \partial\Omega^\epsilon. \end{aligned} \tag{4.27}$$

Arguing as in the proof of Lemma 4.4 (p.123 in [5]) with $A = E^\epsilon$ and $r = \delta$ we deduce

$$|U^\epsilon(\frac{z}{\epsilon}) - a_-|^2 \leq \delta^2\varphi(\frac{z}{\epsilon}),$$

where $\varphi : E^\epsilon \to \mathbb{R}$ is the solution of

$$\begin{aligned} &\Delta\varphi = c_W^2\varphi, \quad \text{on} \quad E^\epsilon, \\ &\varphi = \frac{1}{\delta^2}|U^\epsilon - a_-|^2 \leq 1, \quad \text{on} \quad \partial E^\epsilon. \end{aligned} \tag{4.28}$$

Since from (4.27) we have $\varphi = 0$ on $\partial E^\epsilon \cap \partial\Omega^\epsilon$, the restriction to E^ϵ of the solution ψ of the problem

$$\begin{aligned} &\Delta\psi = c_W^2\psi, \quad \text{on} \quad B_{\frac{R}{\epsilon}}(\frac{z}{\epsilon}), \\ &\varphi = 1, \quad \text{on} \quad \partial B_{\frac{R}{\epsilon}}(\frac{z}{\epsilon}), \end{aligned} \tag{4.29}$$

is a super-solution for problem (4.28) therefore, using also Lemma A.1 in [5], we have

$$|U^\epsilon(\frac{z}{\epsilon}) - a_-|^2 \leq \delta^2\psi(\frac{z}{\epsilon}) \leq \delta^2 e^{-k_0\frac{R}{\epsilon}}, \quad \text{for} \quad R \geq \epsilon\rho_0, \tag{4.30}$$

for appropriate $k_0 > 0$ and ρ_0. It follows that

$$|u_\epsilon(z) - a_-| \leq \delta e^{-\frac{k_0}{2\epsilon}(d(z,D)-C\epsilon^{\frac{1}{2}})}, \quad \text{for} \quad d(z,D) \geq C\epsilon^{\frac{1}{2}},$$

where, by adjusting the constant $C > 0$ we have absorbed the term $\epsilon\rho_0$ in $C\epsilon^{\frac{1}{2}}$. The proof is complete. □

Corollary 4.4. *The lower bound* (4.11) *can be upgraded to*

$$J^\epsilon_\Omega(u_\epsilon) \geq \int_0^l \int_0^h \Big(\frac{\epsilon}{2}|\frac{\partial u_\epsilon}{\partial y}|^2 + \frac{1}{\epsilon}W(u_\epsilon)\Big)dydx \geq 2\sigma l - \tilde{C}'\epsilon.$$

Proof. Consider y_1, y_2 in Step 3 in the proof of Proposition 4.2 above. Set $\hat{y}_1 = \frac{1}{2}(y_1 + y_2) - \frac{1}{4}(y_2 - y_1)$, $\hat{y}_2 = \frac{1}{2}(y_1 + y_2) + \frac{1}{4}(y_2 - y_1)$, then, from $y_2 - y_1 \geq \frac{1}{4}(h - l)$ and the exponential estimate in Lemma 4.3 it follows that

$$x \in (C_0\epsilon, l - C_0\epsilon), \ j = 1, 2 \quad \Rightarrow$$

$$|u_\epsilon(x, \hat{y}_j) - a_-| \leq \delta e^{-\frac{k}{16\epsilon}(h-l-C_\delta\epsilon^{\frac{1}{2}})} \leq \delta e^{-\frac{k}{32\epsilon}(h-l)}.$$

Then, by taking $\epsilon > 0$ sufficiently small we can assume

$$|u_\epsilon(x, \hat{y}_j) - a_-| \leq \epsilon^{\frac{1}{2}}, \ \ x \in (C_0\epsilon, l - C_0\epsilon), \ j = 1, 2.$$

Therefore, proceeding as for (4.14) with $\hat{y}_j$ in place of y_j and $\delta = \epsilon^{\frac{1}{2}}$, we obtain

$$J^\epsilon_{(0,\hat{y}_1)}(u_\epsilon(x,\cdot)), J^\epsilon_{(\hat{y}_2,h)}(u_\epsilon(x,\cdot)) \geq \sigma - C_W\epsilon, \ \ x \in (C_0\epsilon, l - C_0\epsilon), \tag{4.31}$$

and therefore the lower bound

$$J^\epsilon_\Omega(u_\epsilon) \geq \int_0^l J^\epsilon_{(0,h)}(u_\epsilon(x,\cdot))dx \geq 2l\sigma - \tilde{C}'\epsilon, \quad \tilde{C}' = 4\sigma C_0 + C_W l. \tag{4.32}$$

□

The estimate (4.32) is basic for deriving information on the structure of the minimizer u_ϵ in the set $[0,l]\times([0,y_1]\cup[y_2,h])$. We begin by studying the one-dimensional problem:

4.1.3. *The one-dimensional problem.* Consider the minimization problem

$$\begin{gathered}\min_{v\in\mathscr{V}} J^\epsilon_{(0,\lambda_v)}(v), \ \ J^\epsilon_{(0,\lambda_v)}(v) = \int_0^{\lambda_v} (\frac{\epsilon}{2}|v'|^2 + \frac{1}{\epsilon}W(v))ds, \\ \mathscr{V} = \{v \in H^1((0,\lambda_v);\mathbb{R}^m) : |v(0) - a_-| \leq \epsilon^{\frac{1}{2}}, \ |v(\lambda_v) - a_+| \leq \epsilon^{\frac{1}{2}}, \ \}.\end{gathered} \tag{4.33}$$

Note that in the definition of $\mathscr{V}$ we have assumed that the interval of definition of $v \in \mathscr{V}$ may change with v. Also observe that if $\lambda > 0$ is a fixed number it follows that

$$\min_{v\in\mathscr{V}} J^\epsilon_{(0,\lambda_v)}(v) \leq \min_{\{v\in\mathscr{V}:\lambda_v=\lambda\}} J^\epsilon_{(0,\lambda)}(v). \tag{4.34}$$

Lemma 4.5. *Assume v is a minimizer of* (4.33). *Then there exist* $0 < s^- < s^+ < \lambda_v$ *and a constant* $C^* > 0$ *such that*

$$\begin{aligned} & s^+ - s^- \leq C^* \epsilon^{\frac{1}{2}}, \quad (C^* = \frac{4\sigma}{c_W^2}) \\ & \textit{and} \\ & v(s) \in B_{\epsilon^{\frac{1}{4}}}(a_-), \quad s \in [0, s^-), \\ & v(s) \in \mathbb{R}^m \setminus \cup_{a \in A} B_{\epsilon^{\frac{1}{4}}}(a), \quad s \in [s^-, s^+], \\ & v(s) \in B_{\epsilon^{\frac{1}{4}}}(a_+), \quad s \in (s^+, \lambda_v). \end{aligned} \tag{4.35}$$

Proof. For the energy of a minimizer v we have the upper bound

$$J^{\epsilon}_{(0,\lambda_v)}(v) \leq \sigma. \tag{4.36}$$

This follows from Lemma 2.2 and the definition of σ. Set

$$\begin{aligned} s^- &= \sup\{s > 0 : v(s) \in B_{\epsilon^{\frac{1}{4}}}(a_-)\}, \\ s^+ &= \inf\{s < \lambda_v : v(s) \in B_{\epsilon^{\frac{1}{4}}}(a_+)\}. \end{aligned}$$

Lemma 2.4 and Lemma 2.5 in [5] imply that, for $\delta > 0$ small and $a \in A$, the existence of $s_1 < s_2 < s_3$ such that

$$v(s_i) \in B_\delta(a), \; i = 1, 3 \quad \text{and } v(s_2) \notin B_\delta(a)$$

is incompatible with the minimality of v. This property of minimizers implies that $(4.35)_2$ and $(4.35)_4$ hold and moreover that

$$v(s) \in \mathbb{R}^m \setminus (B_{\epsilon^{\frac{1}{4}}}(a_-) \cup B_{\epsilon^{\frac{1}{4}}}(a_+)), \quad s \in [s^-, s^+]. \tag{4.37}$$

From Lemma 2.2, $\sigma = J_{\mathbb{R}}(\overline{u})$, the minimality of $\overline{u}$ and the upper bound (4.36) we also have

$$v(s) \notin B_{\epsilon^{\frac{1}{4}}}(a), \quad a \in A \setminus \{a_-, a_+\}, \; s \in [s^-, s^+].$$

From this and (4.37) $(4.35)_3$ follows. From $(4.35)_3$ and Lemma 2.1 we have $W(v(s)) \geq \frac{1}{2} c_W^2 \epsilon^{\frac{1}{2}}$ and therefore from (4.36) we obtain

$$(s^+ - s^-) \frac{1}{2\epsilon} c_W^2 \epsilon^{\frac{1}{2}} \leq \sigma + \overline{C}\epsilon$$

and $(4.35)_1$ follows with $C^* = \frac{4\sigma}{c_W^2}$. The proof is complete. □

Now let $\lambda > 0$ be fixed and define $\mathscr{V}_a = \mathscr{V}_a^1 \cup \mathscr{V}_a^2$ where

$$\begin{aligned} & \mathscr{V}_a^1 = \{v \in \mathscr{V} : \lambda_v = \lambda, \; v(\bar{s}) \in B_{\epsilon^{\frac{1}{4}}}(a), \; \text{ for some } \; a \neq a_\pm, \text{ and } \bar{s} \in (0, \lambda)\}, \\ & \mathscr{V}_a^2 = \{v \in \mathscr{V} : \lambda_v = \lambda, \; v(\bar{s}_i) \in B_{\epsilon^{\frac{1}{4}}}(a_i), \; a_1 = a_-, \; a_2 = a_+, \; a_3 = a_- \\ & \text{for some } 0 \leq \bar{s}_1 < \bar{s}_2 < \bar{s}_3 < \lambda\}. \end{aligned} \tag{4.38}$$

Let $\mathscr{W} = \{v \in \mathscr{V} : \lambda_v = \lambda\} \setminus \mathscr{V}_a$. For $w \in \mathscr{W}$ we set

$$\begin{aligned}
s^{-,w} &= \sup\{s \in (0,\lambda) : w(s) \in B_{\epsilon^{\frac{1}{4}}}(a_-),\\
s^{+,w} &= \inf\{s \in (0,\lambda) : w(s) \in B_{\epsilon^{\frac{1}{4}}}(a_+),\\
S^w &= \{s \in (0,\lambda) : w(s) \in \mathbb{R}^m \setminus \cup_{a\in A} B_{\epsilon^{\frac{1}{4}}}(a)\}.
\end{aligned}$$

Let $K > 1$ a number to be chosen later and define

$$\begin{aligned}
\mathscr{W}^* = \Big\{ & w \in \mathscr{W} : |S^w| < 2C^*\epsilon^{\frac{1}{2}}, \text{ and}\\
& \left.\begin{array}{l} w(s) \in B_{K\epsilon^{\frac{1}{4}}}(a_-),\ s \in [0, s^{-,w}),\\ w(s) \in B_{K\epsilon^{\frac{1}{4}}}(a_+),\ s \in (s^{+,w}, \lambda] \end{array} .\right\}
\end{aligned} \tag{4.39}$$

Note that

$$s^{+,w} - s^{-,w} \leq 2C^*\epsilon^{\frac{1}{2}}, \quad w \in \mathscr{W}^*, \tag{4.40}$$

and that Lemma 2.3 implies

$$J^{\epsilon}_{(0,\lambda)}(w) \geq \sigma - C_W\epsilon^{\frac{1}{2}}, \quad w \in \mathscr{W}^*. \tag{4.41}$$

We divide $\mathscr{W}^c = \mathscr{W} \setminus \mathscr{W}^*$ in two disjoint parts $\hat{\mathscr{W}}^c$ and $\tilde{\mathscr{W}}^c$ defined as follows

$$\begin{aligned}
\hat{\mathscr{W}}^c &= \{w \in \mathscr{W}^c : |S^w| \geq 2C^*\epsilon^{\frac{1}{2}}\},\\
\tilde{\mathscr{W}}^c &= \{w \in \mathscr{W}^c : |S^w| < 2C^*\epsilon^{\frac{1}{2}}, \text{ and}\\
& w(\bar{s}) \notin B_{K\epsilon^{\frac{1}{4}}}(a_-) \cup B_{K\epsilon^{\frac{1}{4}}}(a_+), \text{ for some } \bar{s} \notin [s^{-,w}, s^{+,w}]\}.
\end{aligned} \tag{4.42}$$

We set $\mathscr{V}^c = \mathscr{V}_a \cup \hat{\mathscr{W}}^c \cup \tilde{\mathscr{W}}^c$ and observe that $\mathscr{V}^c = \{v \in \mathscr{V} : \lambda_v = \lambda\} \setminus \mathscr{W}^*$. From the definition of $\mathscr{V}^c$ we see that maps in $\mathscr{V}^c$ have a structure that differs substantially from that of minimizers of problem (4.33) described in Lemma 4.5. We need a quantitative estimate of the energy price paid by a map $w \in \mathscr{V}^c$.

Lemma 4.6. *Let $\mathscr{V}^c$ be as before. Then there is $K > 1$ such that*

$$w \in \mathscr{V}^c \quad \Rightarrow \quad J^{\epsilon}_{(0,\lambda)}(w) \geq \sigma + C_W\epsilon^{\frac{1}{2}}. \tag{4.43}$$

Proof. 1. If $w \in \mathscr{V}_a^1$ we have $J^{\epsilon}_{(0,\lambda)}(w) \geq \sigma + \eta$ for some $\eta > 0$. This follows from the minimality of $\bar{u}$. If $w \in \mathscr{V}_a^2$ Lemma 2.3 yields $J^{\epsilon}_{(0,\lambda)}(w) \geq 2\sigma - C_W\epsilon^{\frac{1}{2}}$.

2. If $w \in \hat{\mathscr{W}}^c$ we have

$$W(w(s)) \geq \frac{1}{2}c_W^2\epsilon^{\frac{1}{2}}, \quad s \in S^w.$$

It follows that

$$J^{\epsilon}_{(0,\lambda)}(w) \geq \frac{1}{\epsilon}W(w(s))|S^w| \geq 2c_W^2C^* = 4\sigma, \quad (C^* = \frac{2\sigma}{c_W^2}).$$

3. If $w \in \tilde{\mathscr{W}}^c$ we have

$$\begin{aligned}
w(s^{-,w}) &\in \overline{B}_{\epsilon^{\frac{1}{4}}}(a_-),\\
w(s^{+,w}) &\in \overline{B}_{\epsilon^{\frac{1}{4}}}(a_+).
\end{aligned}$$

This and Lemma 2.3 imply

$$J^{\epsilon}_{(s^{-,w},s^{+,w})}(w) \geq \sigma - C_W\epsilon^{\frac{1}{2}}. \tag{4.44}$$

By assumption there is $\bar{s} \notin (s^{-,w}, s^{+,w})$ that satisfies $w(\bar{s}) \in \mathbb{R}^m \setminus (B_{K\epsilon^{\frac{1}{4}}}(a_-) \cup B_{K\epsilon^{\frac{1}{4}}}(a_+))$. From this follows the existence of two intervals $I_j = [r_j, s_j] \subset (0,\lambda) \setminus (s^{-,w}, s^{+,w})$, $j = 1,2$ such that

$$W(w(s)) \geq \frac{1}{2}c_W^2\epsilon^{\frac{1}{2}}, \;\; s \in I_j,$$
$$||w(r_j)| - |w(s_j)|| \geq (K-1)\epsilon^{\frac{1}{4}}.$$

This and a standard computation imply

$$J^{\epsilon}_{I_j}(w) \geq \frac{1}{2}\Big(\epsilon^{\frac{3}{2}}\frac{(K-1)^2}{|I_j|} + \frac{1}{\epsilon^{\frac{1}{2}}}c_W^2|I_j|\Big) \geq c_W(K-1)\epsilon^{\frac{1}{2}},$$

that together with (4.44) yields

$$J^{\epsilon}_{(0,\lambda)}(w) \geq \sigma + (2c_W(K-1) - C_W)\epsilon^{\frac{1}{2}} \geq \sigma + C_W\epsilon^{\frac{1}{2}},$$

where we have chosen $K = 1 + \frac{C_W}{c_W}$. The proof is complete. □

4.1.4. *The structure inside the domain.* From (4.31) we have a good lower bound for the energy $J_{(0,\hat{y}_1)}(u_\epsilon(x,\cdot))$ or $J_{(\hat{y}_2,h)}(u_\epsilon(x,\cdot))$ of the restrictions of the minimizer $u_\epsilon : \Omega \to \mathbb{R}^m$ to each fiber $\{x\}\times(0,\hat{y}_1)$, or $\{x\}\times(\hat{y}_2,h)$ and therefore we expect that $u_\epsilon(x,\cdot)|_{(0,\hat{y}_1)}$ and $u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)}$ should be well approximated by a translation of $\bar{u}(-\cdot)$ and $\bar{u}$ respectively. We prove that this is indeed the case for most $x \in (C_0\epsilon, l - C_0\epsilon)$. To show this we apply the results in Lemma 4.5 and Lemma 4.6 to u_ϵ restricted to the fibers $\{x\}\times(0,\hat{y}_1)$ and $\{x\}\times(\hat{y}_2,h)$. This requires a natural reinterpretation of the notation. For instance, when dealing with the fiber $\{x\}\times(\hat{y}_2,h)$ the interval $(0,\lambda)$ in problem (4.33) should be replaced by the interval $(\hat{y}_2,h)$ and the intervals $[0,s^{-,w}]$ and $[s^{+,w},\lambda]$ in the definition (4.39) of $\mathscr{W}^*$ with the intervals $[\hat{y}_2, \hat{y}_2 + s^{-,w}]$ and $[\hat{y}_2 + s^{+,w}, h]$. For $w = u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)}$ we set

$$\eta(x) = h - \hat{y}_2 - s^{+,u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)}}.$$

Let $X = X_1 \cup X_2 \subset (C_0\epsilon, l - C_0\epsilon)$ be defined by

$$X_1 = \{x \in (C_0\epsilon, l - C_0\epsilon) : u_\epsilon(x,\cdot)|_{(0,\hat{y}_1)} \in \mathscr{V}^c\},$$
$$X_2 = \{x \in (C_0\epsilon, l - C_0\epsilon) : u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)} \in \mathscr{V}^c\}.$$

Lemma 4.7. *There exists a constant $C' > 0$ such that*

$$\mathcal{H}^1(X) \leq C'\epsilon^{\frac{1}{2}}|\ln\epsilon|^3. \tag{4.45}$$

Proof. From Lemma 4.6, (4.31) and Proposition 4.1 we have

$$\sum_{i\in\{1,2\}}\Big((\sigma - C_W\epsilon)(l - C_0\epsilon - \mathcal{H}^1(X_i)) + \mathcal{H}^1(X_i)(\sigma + C_W\epsilon^{\frac{1}{2}})\Big) \leq J^{\epsilon}_{\Omega}(u_\epsilon) \leq 2l\sigma + C_1\epsilon|\ln\epsilon|^3.$$

It follows that $\mathcal{H}^1(X_1) + \mathcal{H}^1(X_2) \leq C'\epsilon^{\frac{1}{2}}|\ln\epsilon|^3$ with $C' = \frac{C_1}{C_W}$. This and $\mathcal{H}^1(X_1 \cup X_2) \leq \mathcal{H}^1(X_1) + \mathcal{H}^1(X_2)$ conclude the proof. □

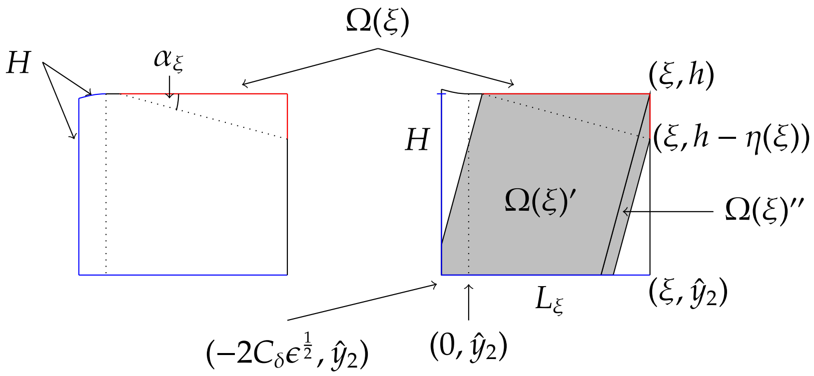

FIGURE 4. $\Omega(\xi), \Omega(\xi)', \Omega(\xi)'', L_\xi$ and H for two different Ω.

4.2. **The proof of Theorem 1.3.**

Proof. 1. For $\xi \in (0, \frac{l}{2}]$ we let $\Omega(\xi)$ be the connected component of $\Omega \cap (-2C_\delta\epsilon^{\frac{1}{2}}, \xi) \times (\hat{y}_2, +\infty)$ that contains $(0, \xi) \times (\hat{y}_2, h)$ (see Figure 4).

Set

$$L_\xi = [-2C_\delta\epsilon^{\frac{1}{2}}, \xi] \times \{\hat{y}_2\},$$
$$H = \partial\Omega(\xi) \cap (-\infty, \xi) \times (\hat{y}_2, h),$$

and observe that, for small $\epsilon > 0$, the estimate (4.16) in Lemma 4.3 implies

$$|u_\epsilon(z) - a_-| < \epsilon, \quad \text{on} \quad L_\xi \cup H. \tag{4.46}$$

Also note that, if $\xi \in [C_0\epsilon, \frac{l}{2}] \setminus X_2$ we have $u_\epsilon(x, \cdot)|_{(\hat{y}_2, h)} \in \mathscr{W}^*$ and therefore

$$|u_\epsilon(z) - a_+| < K\epsilon^{\frac{1}{4}}, \quad \text{on} \quad H_\eta, \tag{4.47}$$

where

$$H_\eta = \{\xi\} \times [h - \eta(\xi), h].$$

2. Assume that $\xi \in [C_0\epsilon, \frac{l}{2}] \setminus X_2$ and set

$$\sin\alpha_\xi = \frac{\eta(\xi)}{(\xi^2 + \eta^2(\xi))^{\frac{1}{2}}}, \qquad \cos\alpha_\xi = \frac{\xi}{(\xi^2 + \eta^2(\xi))^{\frac{1}{2}}}.$$

We regard $\Omega(\xi)$ as a union of fibers parallel to $(\sin\alpha_\xi, \cos\alpha_\xi)$ and let $\Omega(\xi)'$ be the union of the fibers that terminate on $[C_0\epsilon, \xi] \times \{h\}$ and $\Omega(\xi)''$ the union of the fibers that terminate on H_η. Since $u_\epsilon = a_+$ on $[C_0\epsilon, \xi] \times \{h\}$ and each fiber with the second extreme on $[C_0\epsilon, \xi] \times \{h\}$ has its first extreme on $L_\xi \cup H$, from (4.46) it follows that

$$J^\epsilon_{\Omega(\xi)'}(u_\epsilon) \geq (\sigma - C_W\epsilon)(\xi - C_0\epsilon - \mathcal{H}^1(X))\cos\alpha_\xi \geq \sigma\xi\cos\alpha_\xi - C\epsilon^{\frac{1}{2}}|\ln\epsilon|^3, \tag{4.48}$$

where we have used (4.45) and $C > 0$ denotes a generic constant independent of ϵ.

On the other hand (4.47) implies that the contribution $J_{\Omega(\xi)''}(u_\epsilon)$ of the fibers that intersect H_η satisfies

$$J^{\epsilon}_{\Omega(\xi)''}(u_\epsilon) \geq (\sigma - K^2 C_W \epsilon^{\frac{1}{2}}))\eta(\xi)\sin\alpha_\xi \geq \sigma\eta(\xi)\sin\alpha_\xi - C\epsilon^{\frac{1}{2}}.$$

This and (4.48) yield

$$J^{\epsilon}_{\Omega(\xi)}(u_\epsilon) \geq J^{\epsilon}_{\Omega(\xi)'}(u_\epsilon) + J^{\epsilon}_{\Omega(\xi)''}(u_\epsilon) \geq \sigma\eta(\xi)\sin\alpha_\xi + \sigma\xi\cos\alpha_\xi - C\epsilon^{\frac{1}{2}}|\ln\epsilon|^3. \tag{4.49}$$

From (4.31) with $R = (C_0\epsilon, l - C_0\epsilon) \times (0, \hat{y}_1)$ and $R(\xi) = (\xi, l - C_0\epsilon) \times (\hat{y}_2, h)$ we also have

$$\begin{aligned} J^{\epsilon}_{R}(u_\epsilon) &\geq (\sigma - C_W\epsilon)(l - 2C_0\epsilon) \geq \sigma l - C\epsilon, \\ J^{\epsilon}_{R(\xi)}(u_\epsilon) &\geq (\sigma - C_W\epsilon)(l - \xi - C_0\epsilon) \geq \sigma(l - \xi) - C\epsilon. \end{aligned} \tag{4.50}$$

The upper bound (4.3) and the estimate (4.49), (4.50) yield

$$\eta(\xi)\sin\alpha_\xi \leq \xi(1 - \cos\alpha_\xi) + C\epsilon^{\frac{1}{2}}|\ln\epsilon|^3. \tag{4.51}$$

The definition of $\sin\alpha_\xi$ and $\cos\alpha_\xi$ implies that (4.51) is equivalent to

$$\eta(\xi)^2 \leq 2C\xi\epsilon^{\frac{1}{2}}|\ln\epsilon|^3 + C^2\epsilon\ln\epsilon|^6.$$

Since we have $\xi \leq \frac{l}{2}$ and $\xi \notin X$ we conclude

$$\eta(\xi) \leq C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}, \quad \xi \in (C_0\epsilon, \frac{l}{2}] \setminus X. \tag{4.52}$$

for some constant $C^\sharp > 0$ independent of ϵ.

3. We can perform a similar analysis for estimating $\eta(\xi)$ for $\xi \in [\frac{l}{2}, l - C_0\epsilon)$ and for estimating the size of $s^{-,u(x,\cdot)|(0,\hat{y}_1)}$ for $\xi \in (C_0\epsilon, l - C_0\epsilon)$. Proceeding in this way and recalling the definition of $\eta(\xi)$ and the estimate (4.40) we finally obtain

$$\left.\begin{aligned} &h - \hat{y}_2 - s^{-,u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)}} \leq 2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}, \\ &s^{+,u_\epsilon(x,\cdot)|_{(0,\hat{y}_1)}} \leq 2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}, \end{aligned}\right. \qquad \text{for } x \in (C_0\epsilon, l - C_0\epsilon) \setminus X. \tag{4.53}$$

This implies that most of the energy of the minimizer u is concentrated near $\partial^+\Omega$. Set

$$D_\epsilon = \Big((C_0\epsilon, l - C_0\epsilon) \setminus X\Big) \times \Big((0, 2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}) \cup (h - 2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}, h)\Big).$$

From (4.53) and the fact that $u_\epsilon(x,\cdot)|_{(0,\hat{y}_1)}$ and $u_\epsilon(x,\cdot)|_{(\hat{y}_2,h)}$ belong to $\mathscr{W}^*$ it follows that

$$|u_\epsilon(z) - a_-| \leq K\epsilon^{\frac{1}{4}}, \quad z \in \Big((C_0\epsilon, l - C_0\epsilon) \setminus X\Big) \times (2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}, h - 2C^\sharp\epsilon^{\frac{1}{4}}|\ln\epsilon|^{\frac{3}{2}}), \tag{4.54}$$

and therefore Lemma 2.3 and (4.45) imply

$$J^{\epsilon}_{D_\epsilon}(u_\epsilon) \geq 2(\sigma - K^2C_W\epsilon^{\frac{1}{2}})(l - 2C_0\epsilon - C'\epsilon^{\frac{1}{2}}|\ln\epsilon|^3) \geq 2\sigma l - C\epsilon^{\frac{1}{2}}|\ln\epsilon|^3.$$

4. Conclusion. As in the proof of Lemma 4.3 we find that $\min_{a\in A}|u_\epsilon(z_0) - a| \geq \delta$ for some $z_0 \in \Omega \setminus D_\epsilon$ implies

$$J^{\epsilon}_{B_{\epsilon r}(z_0)}(u_\epsilon) \geq C'_\delta\epsilon r, \quad \text{for } \epsilon r \leq d(z_0, \partial(\Omega \setminus D_\epsilon)), \tag{4.55}$$

for some $C'_\delta > 0$. We choose r by imposing

$$C'_\delta \epsilon r = 3\sigma C' \epsilon^{\frac{1}{2}} |\ln \epsilon|^3 \quad \Leftrightarrow \quad r = \frac{3\sigma C'}{C'_\delta} \epsilon^{-\frac{1}{2}} |\ln \epsilon|^3.$$

With this choice of r we see that the existence of $z_0 \in \Omega \setminus D_\epsilon$, $d(z_0, \partial(\Omega \setminus D_\epsilon)) \geq \frac{3\sigma C'}{C'_\delta} \epsilon^{\frac{1}{2}} |\ln \epsilon|^3$, implies

$$J^\epsilon_{D_\epsilon}(u_\epsilon) + J^\epsilon_{B_{\epsilon r}(z_0)}(u_\epsilon) \geq 2l\sigma + \sigma C' \epsilon^{\frac{1}{2}} |\ln \epsilon|^3.$$

This estimate, if $\epsilon > 0$ is small, collides with the upper bound (4.3) and we conclude that

$$\min_{a \in A} |u_\epsilon(z) - a| < \delta, \quad z \in \Omega \setminus D_\epsilon, \ d(z, \partial(\Omega \setminus D_\epsilon)) \geq C \epsilon^{\frac{1}{2}} |\ln \epsilon|^3, \tag{4.56}$$

where $C = \frac{3\sigma C'}{C'_\delta}$. This, (4.54), (4.15) and the continuity of u imply that actually we have $a = a_-$ in (4.56). From the definition of D_ϵ it follows that for $C^1 > 0$ sufficiently large we have, for small $\epsilon > 0$,

$$\begin{aligned} & z \in \Omega, \ d(z, \partial^+\Omega) \geq C^1 \epsilon^{\frac{1}{4}} |\ln \epsilon|^{\frac{3}{2}} \quad \Rightarrow \\ & z \in \Omega \setminus D_\epsilon, \ d(z, \partial\Omega \setminus D_\epsilon) \geq C \epsilon^{\frac{1}{2}} |\ln \epsilon|^3, \end{aligned}$$

and from (4.56) with $a = a_-$ we conclude

$$\begin{aligned} & z \in \Omega, \ d(z, \partial^+\Omega) \geq C^1 \epsilon^{\frac{1}{4}} |\ln \epsilon|^{\frac{3}{2}}, \\ & \Rightarrow \quad |u_\epsilon(z) - a_-| < \delta. \end{aligned}$$

From this, proceeding as in the proof of (4.16) in the proof of Lemma 4.3, we derive the exponential estimate

$$|u_\epsilon(z) - a_-| \leq \delta e^{-\frac{k}{\epsilon}(d(z, \partial^+\Omega) - C^1 \epsilon^{\frac{1}{2}} |\ln \epsilon|^3)}, \quad z \in \Omega, \ d(z, \partial^+\Omega) \geq C^1 \epsilon^{\frac{1}{4}} |\ln \epsilon|^{\frac{3}{2}}.$$

The estimate (1.11) follows from this, (2.8) and a suitable choice of K. The proof is complete. □

Remark 3. Theorem 1.3 implies the existence of the pointwise limit

$$u_0 = \lim_{\epsilon \to 0^+} u_\epsilon,$$

where

$$u_0 = \begin{cases} a_-, & \text{on } \overline{\Omega} \setminus \partial^+\Omega, \\ a_+, & \text{on } \partial^+\Omega. \end{cases}$$

4.2.1. *On the thickness of the boundary layer.* Fix a point $(\hat{x}, \hat{y}) \in \overline{\Omega}$ and consider the rescaled map

$$U^\epsilon(\xi, \eta) = u_\epsilon(\epsilon\xi + \hat{x}, \epsilon\eta + \hat{y}), \quad (\epsilon\xi + \hat{x}, \epsilon\eta + \hat{y}) \in \overline{\Omega}. \tag{4.57}$$

The bound

$$|u_\epsilon| \leq M, \quad \Leftrightarrow \quad |U^\epsilon| \leq M,$$

the smoothness of W and of $\partial\Omega$ and elliptic theory imply

$$|U^\epsilon|_{C^{2,\alpha}(\overline{\Omega}^\epsilon;\mathbb{R}^m)} \leq C, \tag{4.58}$$

for some $\alpha \in (0,1)$ and some constant $C > 0$. Therefore, at least along a subsequence, there exists

$$U^0 = \lim_{\epsilon\to 0^+} U^\epsilon,$$

and the convergence is in the C^2 sense in compact sets. Clearly the limit function U^0 depends on the choice of the point $(\hat{x},\hat{y}) \in \overline{\Omega}$. From Theorem 1.3 we can easily characterize U^0 for various choices of $(\hat{x},\hat{y})$. For instance, if $(\hat{x},\hat{y}) \in \overline{\Omega} \setminus \overline{\partial^+\Omega}$ we have

$$U^0 = a_-, \quad \text{for} \quad (\xi,\eta) \in S,$$

where $S = \mathbb{R}^2$ if $(\hat{x},\hat{y}) \in \Omega$ and S is the half plane that contains Ω and is tangent to Ω at $(\hat{x},\hat{y})$ if $(\hat{x},\hat{y}) \in \partial\Omega \setminus \overline{\partial^+\Omega}$. If $(\hat{x}^\epsilon,\hat{y}) \in \partial\Omega$ has $\hat{x}^\epsilon \in (0, C\epsilon)$, $\hat{y} = 0$ and $g(\hat{x},0) = b \neq a^\pm$ we expect that $U^0 : S \to \mathbb{R}^m$ satisfies $\lim_{\xi\to\pm\infty} U^0(\xi,0) = a^\pm$ and that as $\eta \to +\infty$, $U^0(\cdot,\eta)$ converges exponentially to a translate of the heteroclinic connection $\overline{u}$, see [22] and Chapter 9 in [5]. It remains to consider the case where $(\hat{x},\hat{y}) \in \partial^+\Omega$. In this case, Theorem 1.3 suggests that

$$U^0 \equiv a_+, \tag{4.59}$$

on the half plane S. In spite of the fact that the estimate for the thickness of the boundary layer given by Theorem 1.3 may not be optimal as far as the power $\epsilon^{\frac{1}{2}}$ is concerned, it is correct in indicating that the layer is thicker than $O(\epsilon)$. In Theorem 4.8 below we establish (4.59) and also prove that the thickness is $\frac{\epsilon}{o(1)}$. This is compatible with the fact that there is no connecting orbit in the half plane.

The existence of the boundary layer is a higher dimensional effect and a new phenomenon. It is the result of a compromise between two competing minimization requirements: in the interior of $\partial^+\Omega$ to reduce energy the solution U^ϵ tries to behave like in the one dimensional case and push the layer in the interior while in a neighborhood of the extreme points of $\partial^+\Omega$ minimization requires the solution to remain near a_-.

Theorem 4.8. *Let $\hat{x} \in (0,l)$ be fixed. Then*

(i) For each $K > 0$ it results

$$\lim_{\epsilon\to 0^+} \sup_{y\leq K\epsilon} |u_\epsilon(\hat{x},y) - a_+| = 0,$$

(ii)

$$\liminf_{\epsilon\to 0^+} |u_\epsilon(\hat{x},y^\epsilon) - a_+| > 0 \quad \Rightarrow \quad \lim_{\epsilon\to 0^+} \frac{y^\epsilon}{\epsilon} = +\infty,$$

(iii)

$$\lim_{\epsilon\to 0^+} \epsilon \frac{\partial u_\epsilon}{\partial y}(\hat{x},0) = 0.$$

Proof. 1. We begin with the scalar case $m = 1$ that can be handled by simply combining known facts based on the Modica inequality which is not available in the vector case. Specifically we will make use of a result of Farina and Valdinoci [13, Theorem 1 (ii)] which in particular establishes the validity of the Modica inequality on the half space (as opposed to the whole $\mathbb{R}^n$).

The map U^ϵ defined in (4.57) for fixed $\hat{x} \in (0, l)$ and $\hat{y} = 0$ satisfies

$$\Delta U^\epsilon = W'(U^\epsilon), \ \ (\epsilon\xi + \hat{x}, \epsilon\eta) \in \Omega; \ \ \xi = \frac{x - \hat{x}}{\epsilon}, \eta = \frac{y}{\epsilon}.$$

Since we are in the scalar case we can assume $a^\pm \in \mathbb{R}$, $a_- < a_+$. Then we have $a_- \le U^\epsilon \le a_+$ which follows from the boundary conditions on $\partial\Omega$ via the maximum principle. By linear elliptic theory

$$U^\epsilon \in C^{2,\alpha}(\overline{\Omega}^\epsilon), \ \ \Omega^\epsilon = \{(\xi, \eta) : (\epsilon\xi + \hat{x}, \epsilon\eta) \in \Omega\},$$

and so, along a subsequence,

$$U^{\epsilon_j} \overset{C^2}{\to} U^0, \ \ \text{on compacts in } \ S = \mathbb{R} \times [0 + \infty).$$

The limit function U^0 satisfies

$$\begin{cases} \Delta U^0 = W'(U^0), \ \ \text{in}\, S = \mathbb{R} \times (0, +\infty), \\ U^0(\xi, 0) = a_+, \ \ \xi \in \mathbb{R}. \end{cases} \tag{4.60}$$

Setting $u = -U^0 + a_+$, $-F'(u) = W(a_+ - u)$ we can apply Theorem 1 (ii) in [13] and conclude, via the positivity of W and the specific bounds $U^0 \in [a_-, a_+]$, the Modica estimate

$$\frac{1}{2}|\nabla U^0|^2 \le W(U^0), \ \ \text{in } S. \tag{4.61}$$

As we mention below, classical estimates from linear elliptic theory imply

$$U^0 \in C^{2,\alpha}(\overline{S}), \tag{4.62}$$

which we accept for the moment This extends the validity of (4.61) up to the boundary of S. Arguing now as in [19, Theorem I] we set

$$\phi(t) = U^0((\xi, 0) + tn) + a_+, \ \ n \in \mathbb{S}^1, \ t \in \mathbb{R}, \ (\xi, 0) + tn \in S,$$

and obtain via (4.61) that there is $\delta > 0$ such that

$$|\phi'(t)|^2 \le C|\phi(t)|^2, \ \ \phi(0) = 0, \ |t| \le \delta,$$

and conclude that $\phi(t) = 0$ for $|t| < \delta$. This shows that the set $\{(\xi, \eta) \in \overline{S} : U^0(\xi, \eta) = a_+\}$ which is nonempty and closed in $\overline{S}$ is also open and therefore coincides with $\overline{S}$, hence

$$U^0 \equiv a_+, \ \ \text{on}\, \mathbb{R} \times [0, +\infty). \tag{4.63}$$

From this Theorem 4.8 follows. To avoid repetition, we give details concerning the various statements of the theorem, when dealing with the vector case.

To prove (4.62) we note that the boundedness of U^0 and Theorem 8.29 in [16] imply a Hölder estimate for U^0 up to the boundary of S. This and the smoothness of

W yields $W'(U^0) \in C^\alpha(\overline{S})$, for some $\alpha \in (0,1)$. Next with the global Schauder estimate we have

$$\|U^0\|_{C^{2,\alpha}(\overline{S})} \leq C_1\|W'(U^0)\|_{C^\alpha(\overline{S})} + C_2\|U^0\|_{L^\infty(S)},$$

and (4.62) follows.

Remark 4. The previous discussion of the scalar case can be slightly generalized. Indeed the argument developed to derive (4.63) goes through in the same way also if we allow the point $\hat{x} \in (C\epsilon, l - C\epsilon)$ in the definition of U^ϵ to depend on ϵ.

2. Now we move to the vector case $m > 1$. The objective is to establish (4.63) and the difficulty is due to the absence of the Modica estimate. We will utilize the upper bound (4.3) in Proposition 4.1, the refined lower bound (4.32), which is sharp as far as the power of ϵ is concerned as well as the zero order term, Gui's Hamiltonian identities (e.g. [5], 3.4) and Theorem 1.3 above.

Lemma 4.9.

$$\int_0^l\int_0^h |\frac{\partial}{\partial x}u_\epsilon|^2 dydx \leq C|\ln\epsilon|^3, \ \textit{some constant } C > 0,$$

$$\int_{-\frac{\hat{x}}{\epsilon}}^{\frac{l-\hat{x}}{\epsilon}}\int_0^{\frac{h}{\epsilon}} |U^\epsilon_\xi|^2 d\eta d\xi \leq C|\ln\epsilon|^3.$$

Proof. From (4.3) and (4.32) we have

$$\frac{\epsilon}{2}\int_0^l\int_0^h |\frac{\partial}{\partial x}u_\epsilon|^2 dydx \leq 2l\sigma + C_1\epsilon|\ln\epsilon|^3 - \int_0^l J^\epsilon_{(0,h)}(u_\epsilon(x,\cdot))dx \leq C_1\epsilon|\ln\epsilon|^3 + \tilde{C}'\epsilon,$$

and the first inequality is established. The second inequality follows from the first by changing variables. □

From the boundary conditions we have $U^\epsilon(\xi, 0) = a_+$ for $\xi \in (-\frac{\hat{x}}{\epsilon} + C, \frac{l-\hat{x}}{\epsilon} - C)$, so by linear elliptic theory (e.g. Theorem 8.29 in [16])

$$|U^\epsilon(\xi,\eta) - U^\epsilon(\xi,0)| \leq \tilde{C}|\eta|^\alpha, \tag{4.64}$$

for some constant $\tilde{C} > 0$ and $\alpha \in (0,1)$. Thus we obtain in this range of η's the estimate

$$|U^\epsilon(\xi,\eta) - a_+| \leq \tilde{C}|\eta|^\alpha, \tag{4.65}$$

Lemma 4.10. *There exist $\xi^{\pm}(\epsilon)$ such that*

$$\begin{aligned} &\xi^{-}(\epsilon) \in (-\frac{\hat{x}}{\epsilon} + C, -\frac{\hat{x}}{\epsilon} + C + \epsilon^{-\frac{3}{4}}), \\ &\xi^{+}(\epsilon) \in (\frac{l-\hat{x}}{\epsilon} - C - \epsilon^{-\frac{3}{4}}, \frac{l-\hat{x}}{\epsilon} - C), \\ &\int_0^{\frac{h}{\epsilon}} |U^{\epsilon}_{\xi}(\xi^{\pm}(\epsilon), \eta)|^2 d\eta \le C\epsilon^{\frac{3}{4}} |\ln \epsilon|^3. \end{aligned} \tag{4.66}$$

Proof. Set $f(\xi) = \int_0^{\frac{h}{\epsilon}} |U^{\epsilon}_{\xi}(\xi, \eta)|^2 d\eta$, $\xi \in (-\frac{\hat{x}}{\epsilon} + C, \frac{l-\hat{x}}{\epsilon} - C)$. Then Lemma 4.9 and the Mean Value Theorem imply, for each $p \in (-\frac{\hat{x}}{\epsilon} + C, \frac{l-\hat{x}}{\epsilon} - C - \epsilon^{-\frac{3}{4}})$,

$$\epsilon^{-\frac{3}{4}} f(\xi(\epsilon)) = \int_p^{p+\epsilon^{-\frac{3}{4}}} f(\xi) d\xi \le C |\ln \epsilon|^3, \text{ some } \xi(\epsilon) \in (p, p + \epsilon^{-\frac{3}{4}}),$$

□

We now utilize the Hamiltonian Identity in the rectangle $[\xi^{-}(\epsilon), \xi^{+}(\epsilon)] \times [0, \frac{h}{2\epsilon}]$ arguing as in the proof of Theorem 3.2 in [5]. Taking into account the boundary condition $U^{\epsilon}(\xi, 0) = a_{+}$, we have

$$\begin{aligned} &\int_{\xi^{-}(\epsilon)}^{\xi^{+}(\epsilon)} [\frac{1}{2}(|U^{\epsilon}_{\xi}|^2 - |U^{\epsilon}_{\eta}|^2) + W(U^{\epsilon})]|_{\eta = \frac{h}{2\epsilon}} d\xi + \frac{1}{2} \int_{\xi^{-}(\epsilon)}^{\xi^{+}(\epsilon)} |U^{\epsilon}_{\eta}(\xi, 0)|^2 d\xi \\ &= \int_0^{\frac{h}{2\epsilon}} U^{\epsilon}_{\xi} \cdot U^{\epsilon}_{\eta}|_{\xi = \xi^{-}(\epsilon)} d\eta - \int_0^{\frac{h}{2\epsilon}} U^{\epsilon}_{\xi} \cdot U^{\epsilon}_{\eta}|_{\xi = \xi^{+}(\epsilon)} d\eta. \end{aligned} \tag{4.67}$$

Lemma 4.11.

$$\begin{aligned} &\lim_{\epsilon \to 0^+} \int_{\xi^{-}(\epsilon)}^{\xi^{+}(\epsilon)} [\frac{1}{2}(|U^{\epsilon}_{\xi}|^2 - |U^{\epsilon}_{\eta}|^2) + W(U^{\epsilon})]|_{\eta = \frac{h}{2\epsilon}} d\xi = 0, \\ &\lim_{\epsilon \to 0^+} \int_0^{\frac{h}{2\epsilon}} |U^{\epsilon}_{\xi} \cdot U^{\epsilon}_{\eta}| \big|_{\xi = \xi^{\pm}(\epsilon)} d\eta = 0. \end{aligned} \tag{4.68}$$

Proof. From $\Delta U^{\epsilon} = W_u(U^{\epsilon}) = W_u(U^{\epsilon}) - W_u(a_{-})$ and Theorem 1.3, via a local L^p-estimate, we obtain [e changed to exp below – too small fonts]

$$|\nabla U^{\epsilon}| \le C \exp\left(-k\left(\min\{\eta, \frac{h}{\epsilon} - \eta\} - C_1 \frac{|\ln \epsilon|^3}{\epsilon^{\frac{1}{2}}}\right)\right), \text{ in } [-\frac{\hat{x}}{\epsilon}, \frac{l-\hat{x}}{\epsilon}] \times [0, \frac{h}{\epsilon}], \tag{4.69}$$

where $C > 0$ here and below is a constant possibly different from line to line. This with $\eta = \frac{h}{2\epsilon}$ and the smallness of $W(U^\epsilon(\xi, \frac{h}{2\epsilon}))$, $\xi \in (-\frac{\hat{x}}{\epsilon}, \frac{l-\hat{x}}{\epsilon})$ that follows by (1.11), proves $(4.68)_1$.

To complete the proof we write

$$\int_0^{\frac{h}{2\epsilon}} |U_\xi^\epsilon \cdot U_\eta^\epsilon| \big|_{\xi=\xi^\pm(\epsilon)} \, d\eta = \int_0^{C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}} |U_\xi^\epsilon \cdot U_\eta^\epsilon| \big|_{\xi=\xi^\pm(\epsilon)} \, d\eta + \int_{C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}}^{\frac{h}{2\epsilon}} |U_\xi^\epsilon \cdot U_\eta^\epsilon| \big|_{\xi=\xi^\pm(\epsilon)} \, d\eta = I + II.$$

We estimate each term separately. From (4.58) and $(4.66)_2$ and (4.69) we have

$$I \leq C \left(\int_0^{C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}} |U_\xi^\epsilon(\xi^\pm(\epsilon), \eta)|^2 d\eta \right)^{\frac{1}{2}} \left(\frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}\right)^{\frac{1}{2}} \leq C(\epsilon^{\frac{3}{4}} |\ln\epsilon|^3)^{\frac{1}{2}} \left(\frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}\right)^{\frac{1}{2}} \leq C\epsilon^{\frac{1}{8}} |\ln\epsilon|^3, \quad (4.70)$$

$$\begin{aligned} II &\leq \left(\int_0^{\frac{h}{\epsilon}} |U_\xi^\epsilon(\xi^\pm(\epsilon), \eta)|^2 d\eta \right)^{\frac{1}{2}} \left(\int_{C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}}^{\frac{h}{2\epsilon}} |U_\eta^\epsilon(\xi^\pm(\epsilon), \eta)|^2 d\eta \right)^{\frac{1}{2}} \\ &\leq C(\epsilon^{\frac{3}{4}} |\ln\epsilon|^3)^{\frac{1}{2}} \left(\int_{C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}}}^{\frac{h}{2\epsilon}} e^{-2k(\eta - C_1 \frac{|\ln\epsilon|^3}{\epsilon^{\frac{1}{2}}})} d\eta \right)^{\frac{1}{2}} \leq C(\epsilon^{\frac{3}{4}} |\ln\epsilon|^3)^{\frac{1}{2}}. \end{aligned} \quad (4.71)$$

□

From Lemma 4.11 and (4.67) we obtain

$$\lim_{\epsilon \to 0^+} \int_{\xi^-(\epsilon)}^{\xi^+(\epsilon)} |U_\eta^\epsilon(\xi, 0)|^2 d\xi = 0, \quad (4.72)$$

and since, along a sequence $\{\epsilon_j\}$, U^{ϵ_j} converges locally in $\overline{S} = \mathbb{R} \times [0, +\infty)$, to U^0 in the C^2 sense, we conclude

$$\int_{\mathbb{R}} |U_\eta^0(\xi, 0)|^2 d\xi = 0, \quad (4.73)$$

hence

$$U_\eta^0(\xi, 0) = 0, \quad \xi \in \mathbb{R}. \quad (4.74)$$

Passing to the limit as $\epsilon \to 0^+$ in (4.65) we also have

$$U^0(\xi, 0) = 0, \quad \xi \in \mathbb{R}. \quad (4.75)$$

A classical argument based on (4.73) and (4.75) shows that the map $\tilde{U} : \mathbb{R}^2 \to \mathbb{R}^m$

$$\tilde{U} = \begin{cases} a_+, & \text{on } \mathbb{R} \times (-\infty, 0), \\ U^0, & \text{on } \mathbb{R} \times [0, +\infty), \end{cases}$$

is a $W^{1,2}$ solution of $\Delta U = W_u(U)$. Obviously the same is true for the map identically equal to a_+. This and a unique continuation theorem in [15] imply $\tilde{U} \equiv a_+$ and therefore that (4.63) holds also in the vector case.

We are now in the position to conclude the proof. Assume that (i) does not hold. Then there exists $\overline{K} > 0$, $\delta > 0$ and sequences $\{\epsilon_j\}$, $\epsilon_j \to 0^+$ as $j \to +\infty$, $\{y_j\}$ such that

$$|u_{\epsilon_j}(\hat{x}, y_j) - a_+| \geq \delta, \quad y_j \leq \overline{K}\epsilon_j, \quad j = 1, 2, \ldots$$

This is equivalent to

$$|U^{\epsilon_j}(0, \eta_j) - a_+| \geq \delta, \quad \eta_j \leq \overline{K}, \quad j = 1, 2, \ldots$$

which, since U^ϵ converges to U^0 uniformly in compacts, contradicts (4.63). This contradiction proves (i). Statement (ii) is a straightforward consequence of (i). Finally if (iii) does not hold we have

$$\epsilon_j \frac{\partial}{\partial y} u_\epsilon(\hat{x}, 0) \geq \delta, \quad j = 1, 2, \ldots$$

for some sequence $\{\epsilon_j\}$, $\epsilon_j \to 0^+$ as $j \to +\infty$ and therefore

$$U_\eta^{\epsilon_j}(0, 0) \geq \delta, \quad j = 1, 2, \ldots$$

This again contradicts (4.63) since, along a subsequence, $U_\eta^{\epsilon_j}$ converges in the C^2 sense, uniformly in compacts, to U^0. The proof of Theorem 4.8 is complete.

□

4.3. $\frac{h}{l} < 1$**, The internal layer case.** In this section we analyze in detail the structure of the minimizers of problem (4.1) under the assumption

$$l > h. \tag{4.76}$$

We will establish Theorem 1.4 of the Introduction. The proof is based on the tight lower/upper energy bounds, on the vector analogue of the Caffarelli-Cordoba density estimate (see [5] Theorem 5.2), and on the study of the one-dimensional problem in Section 4.1.3

4.3.1. *The upper bound.*

Proposition 4.12. *There exists* $C > 0$ *independent of* ϵ *such that, if* u *is a minimizer of* (4.1)*, then*

$$J_\Omega^\epsilon(u) \leq 2\sigma h + C\epsilon.$$

Proof. 1. From (1.9) it follows that there is a number $\delta > 0$ and smooth functions $p, q : [-\delta, l + \delta] \to \mathbb{R}$ such that

$$\Omega \cap [-\delta, l + \delta] \times [-\delta, h + \delta] = S_\delta,$$
$$S_\delta = \{(x, y) : y \in (p(x), q(x)),\ x \in [-\delta, l + \delta]\},$$

$$
\begin{aligned}
&p(x) = 0, \quad x \in [0,l]; \; q(x) = h, \quad x \in [0,l],\\
&p'(0) = p'(l) = q'(0) = q'(l) = 0,\\
&p''(0) = p''(l) = q''(0) = q''(l) = 0.
\end{aligned}
\tag{4.77}
$$

We define a test function u_{test} in several steps. We let $\delta_\epsilon > 0$ be a small number to be chosen later and set

$$
\begin{aligned}
&u_{\text{test}}(z) = a_-, \quad z \in \overline{\Omega \setminus S_{\delta_\epsilon}},\\
&u_{\text{test}}(z) = a_+, \quad z \in [\delta_\epsilon, l - \delta_\epsilon] \times [0,h].
\end{aligned}
\tag{4.78}
$$

2. To complete the definition of u_{test} we observe that S_{δ_ϵ} is mapped onto the rectangle $R_{\delta_\epsilon} = [-\delta_\epsilon, l + \delta_\epsilon] \times [0,h]$ via the map $z = (x,y) \to \zeta = (\xi, \eta)$ defined by

$$
\begin{aligned}
&\xi = x, \quad x \in (-\delta_\epsilon, l + \delta_\epsilon),\\
&\eta = \frac{h(y - p(x))}{q(x) - p(x)}, \quad y \in (p(x), q(x)),\\
&\text{with inverse}\\
&x = \xi, \quad \xi \in (-\delta_\epsilon, l + \delta_\epsilon),\\
&y = \frac{\eta(q(\xi) - p(\xi))}{h} + p(\xi), \quad \eta \in (0,h).
\end{aligned}
\tag{4.79}
$$

From (4.77), if δ_ϵ is chosen sufficiently small, say $\delta_\epsilon \leq \epsilon^{\frac{1}{2}}$, we have that $p, q - h$, p', q' are $\mathrm{O}(\epsilon)$ for $x \in [-\delta_\epsilon, l + \delta_\epsilon]$. It follows that

$$
\begin{aligned}
&\frac{\partial \zeta}{\partial z}(z) = I + \mathrm{O}(\epsilon),\\
&\det \frac{\partial z}{\partial \zeta}(\zeta) = 1 + \mathrm{O}(\epsilon),
\end{aligned}
\tag{4.80}
$$

where $\frac{\partial \zeta}{\partial z}$ is the Jacobian matrix of ζ, $\det M$ the determinant of the matrix M and $z : S_{\delta_\epsilon} \to R_{\delta_\epsilon}$ the inverse of ζ. From (4.80) it follows that, if $v : R_{\delta_\epsilon} \to \mathbb{R}^m$ is a smooth bounded map and if we set

$$
u_{\text{test}}(z) = v(\zeta(z)), \quad z \in S_{\delta_\epsilon},
$$

then it follows that

$$\begin{aligned}
&\int_{S_{\delta_\epsilon}} |\nabla u_{\text{test}}(z)|^2 dz \leq \int_{R_{\delta_\epsilon}} |\frac{\partial \zeta}{\partial z}(z(\zeta))|^2 |\nabla v(\zeta)|^2 |\det \frac{\partial z}{\partial \zeta}(\zeta)| d\zeta, \\
&\leq (1 + \mathrm{o}(\epsilon)) \int_{R_{\delta_\epsilon}} |\nabla v(\zeta)|^2 d\zeta, \\
&\int_{S_{\delta_\epsilon}} W(u_{\text{test}}(z)) dz = \int_{R_{\delta_\epsilon}} W(v(\zeta)) |\det \frac{\partial z}{\partial \zeta}(\zeta)| d\zeta, \\
&\leq (1 + \mathrm{o}(\epsilon)) \int_{R_{\delta_\epsilon}} W(v(\zeta)) d\zeta.
\end{aligned} \tag{4.81}$$

3. The estimates (4.81) show that we can work on R_{δ_ϵ}. Obviously we need to impose $v = a_+$ on $[\delta_\epsilon, l - \delta_\epsilon] \times [0, h]$ and, to ensure that u_{test} satisfies the boundary conditions in the minimization problem (4.1), we must require

$$v(\zeta) = \tilde{g}_\epsilon(\zeta) =: g_\epsilon(z(\zeta)), \quad \zeta \in [-\delta_\epsilon, l + \delta_\epsilon] \times \{0, h\},$$

where g_ϵ is as in (1.10). More explicitly we have

$$\begin{aligned}
&\tilde{g} = a_-, \quad (\xi, \eta) \in [-\delta_\epsilon, 0] \cup [l, l + \delta_\epsilon] \times \{0, h\}, \\
&\tilde{g} = g_\epsilon, \quad (\xi, \eta) \in [0, C_0\epsilon] \cup [l - C_0\epsilon, l] \times \{0, h\}, \\
&\tilde{g} = a_+, \quad (\xi, \eta) \in [C_0\epsilon, \delta_\epsilon] \cup [l - \delta_\epsilon, l - C_0\epsilon] \times \{0, h\},
\end{aligned} \tag{4.82}$$

with g_ϵ and C_0 as in (1.10).

4. We complete the definition of v. We divide $[-\delta_\epsilon, \delta_\epsilon] \cup [l - \delta_\epsilon, l + \delta_\epsilon] \times [0, h]$ in parts denoted A, B, C, D as sketched in Fig. 5. The definition of v in regions labeled with the same letter is similar. Therefore we only define v on one (marked by the superscript $*$) of the regions denoted with the same letter. The extension to the entire domain is then straightforward. We set $\delta_\epsilon = \epsilon + \epsilon\lambda_\epsilon$, with $\lambda_\epsilon = |\ln \epsilon^n|$ and $n \geq 1$ a sufficiently large number, and define

$$\begin{aligned}
&v(\xi, \eta) = \bar{u}(\frac{\xi}{\epsilon}), \quad (\xi, \eta) \in A^* = [-\epsilon\lambda_\epsilon, \epsilon\lambda_\epsilon] \times [\epsilon, h - \epsilon], \\
&v(\xi, \eta) = \frac{\eta}{\epsilon}(\bar{u}(\frac{\xi}{\epsilon}) - \tilde{g}(\xi, 0)) + \tilde{g}(\xi, 0), \quad (\xi, \eta) \in B^* = [-\epsilon\lambda_\epsilon, \epsilon\lambda_\epsilon] \times [0, \epsilon], \\
&v(\xi, \eta) = (1 - (\frac{\xi}{\epsilon} - \lambda_\epsilon))(\bar{u}(\lambda_\epsilon) - a_+) + a_+, \\
&\quad (\xi, \eta) \in C^* = [\epsilon\lambda_\epsilon, \epsilon\lambda_\epsilon + \epsilon] \times [\epsilon, h - \epsilon], \\
&v(\xi, \eta) = (1 - (\frac{\xi}{\epsilon} - \lambda_\epsilon))\frac{\eta}{\epsilon}(\bar{u}(\lambda_\epsilon) - a_+) + a_+, \\
&\quad (\xi, \eta) \in D^* = [\epsilon\lambda_\epsilon, \epsilon\lambda_\epsilon + \epsilon] \times [0, \epsilon].
\end{aligned} \tag{4.83}$$

with these definitions we immediately have

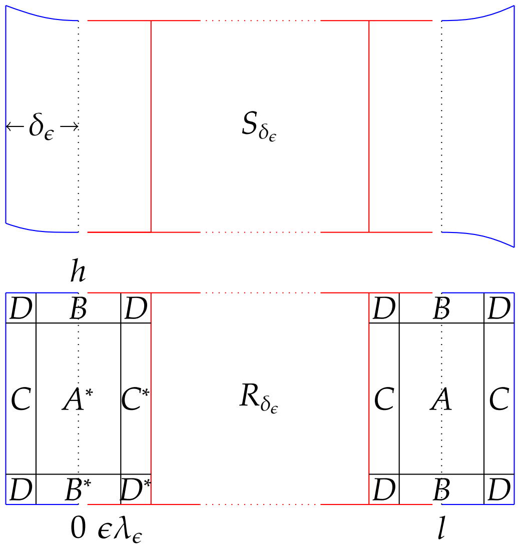

FIGURE 5. R_{δ_ϵ}, S_{δ_ϵ} and the sets A, B, C, D.

$$J^\epsilon_{A^*}(v) \leq \sigma h. \tag{4.84}$$

To proceed with the estimates of $J_{B^*}(v)$ etc, we recall (4.4) and that similar estimates are valid for the derivative of $\overline{u}$. Moreover the fact that v is bounded and the assumption that $a^\pm$ are nondegenerate imply that there is a constant $C > 0$ such that

$$W(v) \leq C|v - a^\pm|^2. \tag{4.85}$$

To estimate $J^\epsilon_{B^*}(v)$ we divide B^* as $B^* = B^- \cup B' \cup B^+$ where $B^- = [-\epsilon\lambda_\epsilon, 0] \times [0, \epsilon]$, $B' = [0, C_0\epsilon] \times [0, \epsilon]$ and $B^+ = [C_0\epsilon, \epsilon\lambda_\epsilon] \times [0, \epsilon]$. From the properties of g_ϵ in (1.10) and the definition of v it follows that $(\xi, \eta) \in B'$ implies $|\nabla v| \leq \frac{C}{\epsilon}$ and $W(v) \leq C$. This and $|B'| \leq C\epsilon^2$ imply

$$J^\epsilon_{B'}(v) \leq C\epsilon. \tag{4.86}$$

Next we estimate $J^\epsilon_{B^+}(v)$. Since, for $\xi \in [C_0\epsilon, \epsilon\lambda_\epsilon]$, it follows that $\tilde{g}(\xi, 0) = a_+$ and therefore $v(\xi, \eta) = \frac{\eta}{\epsilon}(\overline{u}(\frac{\xi}{\epsilon}) - a_+) + a_+$, from (4.4) and (4.85) we obtain

$$J^\epsilon_{B^+}(v) \leq \frac{C}{\epsilon} \int_0^\epsilon \int_{C_0\epsilon}^{\epsilon\lambda_\epsilon} \Big(|\overline{u}(\frac{\xi}{\epsilon}) - a_+|^2 + \frac{\eta^2}{\epsilon^2}|\overline{u}'(\frac{\xi}{\epsilon})|^2\Big) d\xi d\eta$$

$$\leq \frac{C}{\epsilon} \int_0^\epsilon \int_{C_0\epsilon}^{\epsilon\lambda_\epsilon} e^{-2k\frac{\xi}{\epsilon}} d\xi d\eta \leq C\epsilon.$$

In a similar way we also obtain $J^\epsilon_{B^-}(v) \leq C\epsilon$. From this and (4.86) we conclude

$$J^\epsilon_{B^*}(v) \leq C\epsilon. \tag{4.87}$$

On C^* we have $|\frac{\partial v}{\partial \xi}| \leq |\overline{u}(\lambda_\epsilon) - a_+|$, $\frac{\partial v}{\partial \eta} = 0$ and, from (4.85), $W(v) \leq C|\overline{u}(\lambda_\epsilon) - a_+|^2$. This and (4.4) imply

$$\begin{aligned} J^\epsilon_{C^*}(v) &\leq \frac{C}{\epsilon}|\overline{u}(\lambda_\epsilon) - a_+|^2 \int_{\epsilon\lambda_\epsilon}^{\epsilon\lambda_\epsilon+\epsilon} d\xi \\ &\leq Ce^{\ln \epsilon^{2nk}} \leq C\epsilon, \end{aligned} \tag{4.88}$$

provided n is chosen sufficiently large. Finally, using as before (4.4), (4.85) and $|\overline{u}(\lambda_\epsilon) - a_+|^2 \leq Ce^{\ln \epsilon^{2nk}} \leq C\epsilon$, it is seen that $J^\epsilon_{D^*}(v)$ is of higher order

$$J^\epsilon_{D^*}(v) \leq C\epsilon^2. \tag{4.89}$$

From (4.84), (4.87), (4.88) and (4.84) and the similar estimates valid for $J_A(v)$ etc. we obtain

$$J^\epsilon_{R_{\delta_\epsilon}}(v) \leq 2\sigma h + C\epsilon.$$

This concludes the proof. □

4.3.2. *The lower bound.* We now derive a lower bound for the energy of a minimizer of problem (4.1) (see also Corollary 4.14 below).

Proposition 4.13. *There exist $C_4 > 0$ and $\epsilon_0 > 0$ such that, if u is a minimizer of* (4.1)*, then*

$$J^\epsilon_\Omega(u) \geq \int_\Omega \Big(\frac{\epsilon}{2}|\frac{\partial u}{\partial x}|^2 + \frac{1}{\epsilon}W(u)\Big)dxdy \geq 2\sigma h - C_4\epsilon^{\frac{1}{2}}, \quad \epsilon \in (0, \epsilon_0]. \tag{4.90}$$

Remark 5. By Corollary 4.14 below

$$\int_\Omega \Big(\frac{\epsilon}{2}|\frac{\partial u}{\partial x}|^2 + \frac{1}{\epsilon}W(u)\Big)dxdy \geq 2\sigma h - \tilde{C}\epsilon.$$

As a consequence to this and the upper bound above, one obtains the following key estimate

$$\int_\Omega |\frac{\partial u}{\partial y}|^2 dxdy \leq C.$$

Proof. 1. Let $\delta = \delta(\epsilon) > 0$ a number to be chosen later that satisfies $\frac{\epsilon}{\delta^2} = \mathrm{o}(1)$, as $\epsilon \to 0$. Set $\Omega_\delta = \{z \in \Omega : \min_{a \in A} |u(z) - a| < \delta\}$ and let Ω^c_δ be the complement of Ω_δ in Ω. Then (2.2) and Proposition 4.12 imply

$$\begin{aligned} &\frac{c_W^2\delta^2}{2\epsilon}|\Omega^c_\delta| \leq \frac{1}{\epsilon}\int_\Omega W(u)dz \leq J^\epsilon_\Omega(u) \leq 2\sigma h + C\epsilon, \\ &\Rightarrow \ |\Omega^c_\delta| \leq C_5\frac{\epsilon}{\delta^2}. \end{aligned} \tag{4.91}$$

2. For $x \in (0, l)$ set $\Sigma_x = \{x\} \times (0, h)$ and define

$$X = \{x \in (C_0\epsilon, l - C_0\epsilon) : \mathcal{H}^1(\Sigma_x \cap \Omega_\delta) > h - \eta\frac{\epsilon}{\delta^2}\},$$
$$X^c = \{x \in (C_0\epsilon, l - C_0\epsilon) : \mathcal{H}^1(\Sigma_x \cap \Omega_\delta^c) \geq \eta\frac{\epsilon}{\delta^2}\},$$

where $\eta > 0$ is a constant to be selected later. From (4.91) we have

$$\eta\frac{\epsilon}{\delta^2}\mathcal{H}^1(X^c) \leq \int_{X^c} \mathcal{H}^1(\Sigma_x \cap \Omega_\delta^c)dx \leq |\Omega_\delta^c| \leq C_5\frac{\epsilon}{\delta^2}.$$

It follows that

$$\mathcal{H}^1(X^c) \leq \frac{C_5}{\eta},$$

and therefore

$$\mathcal{H}^1(X) \geq l - 2C_0\epsilon - \frac{C_5}{\eta}.$$

3. We divide the sections in X in two parts X_- and $X_+ = X \setminus X_-$ where

$$X_- = \{x \in X : \exists\, a \in A \setminus \{a_+\} \text{ and } z \in \Sigma_x \text{ such that } |u(z) - a| < \delta\},$$
$$X_+ = \{x \in X : \Sigma_x \cap \Omega_\delta = \{z \in \Sigma_x : |u(z) - a_+| < \delta\}\}.$$

From the boundary condition at $x \in (0, l)$, $y = 0, h$, Lemma 2.3 and Proposition 4.12 it follows that

$$(2\sigma - C_W\delta^2)\mathcal{H}^1(X_-) \leq 2\sigma h + C\epsilon,$$
$$\Rightarrow\ \mathcal{H}^1(X_-) \leq \frac{h + \frac{C}{2\sigma}\epsilon}{1 - \frac{C_W}{2\sigma}\delta^2} \leq h + \frac{C}{\sigma}\epsilon + \frac{C_W h}{\sigma}\delta^2,$$

and, in turn by Step 2 above,

$$\mathcal{H}^1(X_+) \geq l - h - (2C_0 + \frac{C}{\sigma})\epsilon - \frac{C_5}{\eta} - \frac{C_W h}{\sigma}\delta^2.$$

Since $l > h$ this estimate shows that if $\epsilon > 0$ and $\delta > 0$ are sufficiently small and if $\eta > 0$ is sufficiently large we have $\mathcal{H}^1(X_+) \geq \frac{1}{2}(l - h)$. Then from the characterization of X_+ and X we have

$$x \in X_+ \ \Rightarrow\ \mathcal{H}^1(\{z \in \Sigma_x : |u(z) - a_+| < \delta\}) \geq h - \eta\frac{\epsilon}{\delta^2}.$$

This and Lemma 2.3 imply

$$\int_\Omega \Big(\frac{\epsilon}{2}|\frac{\partial u}{\partial x}|^2 + \frac{1}{\epsilon}W(u)\Big)dxdy \geq (2\sigma - C_W\delta^2)(h - \eta\frac{\epsilon}{\delta^2}) \geq 2\sigma h - (C_W h + 2\sigma\eta)\epsilon^{\frac{1}{2}},$$

where we have set $\delta = \epsilon^{\frac{1}{4}}$. The proof is complete. □

For $z_1, z_2 \in \mathbb{R}^2$ we denote $\mathrm{sg}(z_1, z_2)$ the open segment with end points z_1 and z_2. We use brackets for closed or half closed segments. For $y \in (0, h)$ let $\mathrm{sg}[z_y^1, z_y^2]$ the connected component of $\Sigma_y = \Omega \cap ((-\infty, +\infty) \times \{y\})$ that contains $(0, l) \times \{y\}$.

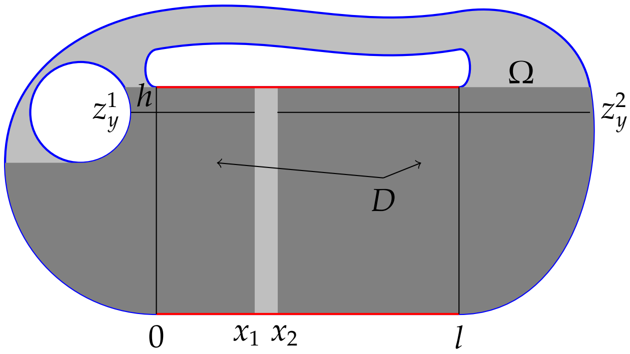

FIGURE 6. The set D and the segments $\mathrm{sg}(z_y^1, (x_1, y))$ and $\mathrm{sg}((x_2, y), z_y^2)$.

Remark 6. From the proof of Proposition 4.13 there exist $0 < x_1 < x_2 < l$ that satisfy $x_2 - x_1 \geq \frac{1}{2}(l - h)$ and are such that in the lower bound in Proposition 4.13, Ω can be replaced by the subset (see Figure 6)

$$D = \bigcup_{y \in (0,h)} \mathrm{sg}(z_y^1, (x_1, y)) \cup \mathrm{sg}((x_2, y), z_y^2).$$

Here D plays the same role as the set denoted again D in the boundary layer case (see Remark 2)

Arguing as in the proof of Theorem 1.2, we show that, in $\Omega \setminus D$, u remains in a neighborhood of A. Actually, by adapting to the case at hand the argument in Steps 1. and 2. in the proof of Theorem 1.2 we see that the existence of a point $z \in \Omega \setminus D$ that satisfies

$$d(z, \partial(\Omega \setminus D)) \geq C\epsilon^{\frac{1}{2}}, \quad \text{and} \quad \min_{a \in A} |u(z) - a| \geq \delta,$$

for $C > 0$ is sufficiently large, contradicts Proposition 4.12. It follows that

$$\begin{aligned} &z \in \Omega \setminus D, \quad d(z, \partial(\Omega \setminus D)) \geq C\epsilon^{\frac{1}{2}} \\ &\Rightarrow \quad |u(z) - a| \leq \delta, \quad \text{for some } a \in A. \end{aligned} \tag{4.92}$$

Set $Q = (x_1, x_2) \times (0, h)$ and $\Omega^* = \Omega \setminus (\overline{D \cup Q})$. From (4.92), by means of the approach developed in Steps 2. and 3. in the proof of Lemma 4.3, we obtain via $u = g_\epsilon$ on $\partial\Omega$ and the continuity of u up to the boundary

$$\begin{aligned} &z \in Q, \quad \min_i |x - x_i| \geq C\epsilon^{\frac{1}{2}} \\ &\Rightarrow \quad |u(z) - a_+| \leq \delta, \end{aligned} \tag{4.93}$$

and, if Ω^* is nonempty,

$$\begin{aligned} &z \in \Omega^*, \quad d(z, D) \geq C\epsilon^{\frac{1}{2}} \\ &\Rightarrow \quad |u(z) - a_-| \leq \delta, \end{aligned} \tag{4.94}$$

and then derive the exponential estimates

$$\begin{aligned} &|u(z)-a_+| \leq Ke^{-\frac{k}{\epsilon}(\min_i |x-x_i| - C\epsilon^{\frac{1}{2}})^+}, \quad z \in Q, \\ &|u(z)-a_-| \leq Ke^{-\frac{k}{\epsilon}(d(z,D) - C\epsilon^{\frac{1}{2}})^+}, \quad z \in \Omega^*. \end{aligned} \tag{4.95}$$

The estimate $(4.95)_1$ implies the following

Corollary 4.14. *The lower bound* (4.90) *above can be upgraded to*

$$\int_\Omega \Big(\frac{\epsilon}{2}|\frac{\partial u}{\partial x}|^2 + \frac{1}{\epsilon}W(u)\Big)dxdy \geq 2\sigma h - \tilde{C}\epsilon. \tag{4.96}$$

Proof. Since $x_2 - x_1 \geq \frac{1}{2}(l-h)$, for $x = x_m = \frac{1}{2}(x_1+x_2)$ we have $\min_i |x_m - x_i| \geq \frac{1}{4}(l-h)$. This and $(4.95)_2$ yield, for $\epsilon > 0$ sufficiently small,

$$|u(x_m,y) - a_+| \leq Ke^{-\frac{k}{\epsilon}(\frac{1}{4}(l-h) - C\epsilon^{\frac{1}{2}})} \leq \epsilon^{\frac{1}{2}}. \tag{4.97}$$

Therefore Lemma 2.3 implies

$$J^\epsilon_{(x_y^1,x_m)}(u(\cdot,y)),\ J^\epsilon_{(x_m,x_y^2)}(u(\cdot,y)) > \sigma - \frac{1}{2}C_W\epsilon, \quad y \in (0,h), \tag{4.98}$$

where x_y^i is defined by $(x_y^i, y) = z_y^i$, $i = 1,2$, and (4.96) follows and the proof is complete. □

4.3.3. *The proof of Theorem 1.4.* In this Section we parallel the reasoning developed in Section 4.1.4 to analyze the fine structure of a minimizer u inside the set D and on the basis of this information we prove Theorem 1.4. The estimates in (4.98) suggest that $u(\cdot,y)|_{(x_y^1,x_m)}$ should be a perturbation of a translation of the heteroclinic $\bar{u}$ and similarly that $u(\cdot,y)|_{(x_m,x_y^2)}$ should be a perturbation of the map $s \to \bar{u}(-s)$ that connects a_+ to a_-. By applying Lemma 4.5 and Lemma 4.6 to the restrictions of u to the segments $\mathrm{sg}(z_y^1,(x_m,y))$ and $\mathrm{sg}((x_m,y),z_y^2)$ we show that this is true for most $y \in (0,h)$. When dealing with $\mathrm{sg}(z_y^1,(x_m,y))$ the interval $(0,\lambda)$ in (4.33) should be replaced by the interval (x_y^1,x_m) where x_y^1 is defined by $(x_y^1,y) = z_y^1$, and the intervals $[0,s^{-,w}]$ and $[s^{+,w},\lambda]$ with the intervals $[x_y^1, x_y^1 + s^{-,w}]$ and $[x_y^1 + s^{+,w}, x_m]$.

Let $Y = Y_1 \cup Y_2 \subset (0,h)$ be defined by

$$\begin{aligned} Y_1 &= \{y \in (0,h) : u(\cdot,y)|_{(x_y^1,x_m)} \in \mathscr{V}^c\}, \\ Y_2 &= \{y \in (0,h) : u(\cdot,y)|_{(x_m,x_y^2)} \in \mathscr{V}^c\}. \end{aligned}$$

Lemma 4.15. *There exists* $C > 0$ *such that*

$$\mathcal{H}^1(Y) \leq C\epsilon^{\frac{1}{2}}. \tag{4.99}$$

Proof. From Lemma 4.6, (4.98) and Proposition 4.12 we obtain

$$\sum_{i\in\{1,2\}} \Big((\sigma - \frac{1}{2}C_W\epsilon)(h - \mathcal{H}^1(Y_i)) + \mathcal{H}^1(Y_i)(\sigma + C_W\epsilon^{\frac{1}{2}})\Big) \leq 2\sigma h + C\epsilon.$$

It follows that $\mathcal{H}^1(Y) \leq \mathcal{H}^1(Y_1) + \mathcal{H}^1(Y_2) \leq C\epsilon^{\frac{1}{2}}$. □

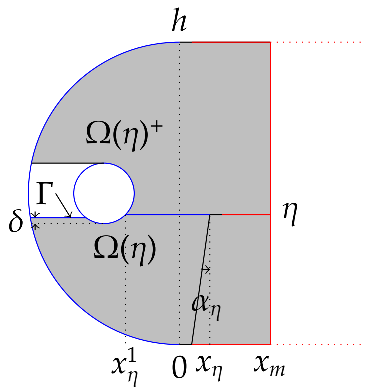

FIGURE 7. $\Omega(\eta)$, $\Omega(\eta)^+$ and Γ.

For $\eta \in (0, \frac{h}{2}]$ we define the set $\Omega(\eta)$, the analogue of the set $\Omega(\xi)$ in the proof of Theorem 1.3. A set Ω that satisfies (1.9) may have a rather complex structure and the same is true for the set D, see Figure 6. Therefore some care is needed for the definition of $\Omega(\eta)$. We start by setting

$$\tilde{\Omega}(\eta) = \cup_{y\in(0,\eta)}\mathrm{sg}(z_y^1, (x_m, y)),$$
$$\Omega(\eta)^+ = \cup_{y\in[\eta,h)}\mathrm{sg}(z_y^1, (x_m, y)).$$

Set $\Pi(\eta) = \Omega^c \cup \Omega(\eta)^+ \cup \{z : x \geq x_m\}$ and define

$$\Omega(\eta) = \tilde{\Omega}_\delta(\eta) \setminus \Pi(\eta),$$

where Ω^c is the complement of Ω in $\mathbb{R}^2$, $\tilde{\Omega}_\delta(\eta) = \bigcup_{z\in\tilde{\Omega}(\eta)} B_\delta(z)$ and $\delta > 0$ is a small number. The plan is now to obtain a precise estimate of $J^\epsilon_{\Omega(\eta)}(u)$. To do this we collect information of the values of the minimizer u on $\partial\Omega(\eta)$. The boundary of $\Omega(\eta)$ includes $\mathrm{sg}(z_\eta^1, (x_m, \eta))$ and $\partial\Omega(\eta) \cap \partial\Omega$. We observe that if

$$\Gamma = \partial\Omega(\eta) \setminus \Big(\mathrm{sg}(z_\eta^1, (x_m, \eta)) \cup (\partial\Omega(\eta) \cap \partial\Omega)\Big) \neq \emptyset,$$

then, for each $z \in \Gamma$ we have $d(z, D) \geq \delta$ (cfr. Figure 7).

This, $(4.95)_1$ and $u = a_-$ on $\partial\Omega^-$, imply that, for $\epsilon > 0$ sufficiently small we have

$$|u(z) - a_-| < \epsilon, \quad z \in \Gamma \cup (\partial\Omega(\eta) \cap \partial\Omega^-).$$

Assume now that $\eta \notin Y$ then we have $u(\cdot, \eta)|_{(x_\eta^1, x_m)} \in \mathscr{W}^*$. This, (4.39) and (4.40) imply that, if we set $x_\eta = s^{-, u(\cdot,\eta)|_{(x_\eta^1, x_m)}} - x_\eta^1$, we have

$$|u(x, \eta) - a_-| \leq K\epsilon^{\frac{1}{4}}, \quad x \in (x_\eta^1, x_\eta),$$
$$|u(x, \eta) - a_+| \leq K\epsilon^{\frac{1}{4}}, \quad x \in (x_\eta + 2C^*\epsilon^{\frac{1}{2}}, x_m).$$

To complete the description of the boundary values of u on $\partial\Omega(\eta)$ we recall (4.97) which implies

$$|u(x_m, y) - a_+| \leq \epsilon^{\frac{1}{2}}, \quad y \in (0, \eta),$$

and $u = g_\epsilon$ on $\partial\Omega$ and in particular

$$u(x, 0) = a_+, \quad x \in (C_0\epsilon, x_m).$$

We proceed to estimate $J^\epsilon_{\Omega(\eta)}(u)$. We give the details for the case $\eta \in (0, \frac{h}{2})$ and $x_\eta > 0$, since the other cases can be discussed in the same way with obvious modifications. We regard $\Omega(\eta)$ as the union of fibers orthogonal to the segment $\mathrm{sg}((C_0\epsilon, 0), (x_\eta, \eta))$. We let $\Omega(\eta)'$ the union of the fibers that have one of their extreme points on the $\mathrm{sg}((C_0\epsilon, 0), (x_m, 0))$ and $\Omega(\eta)''$ the union of the fibers that have one of their extreme points on the $\mathrm{sg}((x_m, 0), (x_m, \eta))$. From the above discussion on the boundary values of u on $\partial\Omega(\eta)$ and Lemma 2.3 we obtain

$$\begin{aligned}
&J^\epsilon_{\Omega(\eta)'}(u) \geq (\sigma - C_W K^2 \epsilon^{\frac{1}{2}})(x_m - C_0\epsilon)\sin\alpha_\eta, \\
&J^\epsilon_{\Omega(\eta)''}(u) \geq (\sigma - C_W K^2 \epsilon^{\frac{1}{2}})\Big(\frac{\eta}{\cos\alpha_\eta} - (x_m - C_0\epsilon)\sin\alpha_\eta\Big), \\
&\Rightarrow \\
&J^\epsilon_{\Omega(\eta)}(u) \geq (\sigma - C_W K^2 \epsilon\epsilon^{\frac{1}{2}})\frac{\eta}{\cos\alpha_\eta},
\end{aligned} \tag{4.100}$$

where

$$\sin\alpha_\eta = \frac{x_\eta - C_0\epsilon}{\sqrt{(x_\eta - C_0\epsilon)^2 + \eta^2}}, \quad \cos\alpha_\eta = \frac{\eta}{\sqrt{(x_\eta - C_0\epsilon)^2 + \eta^2}}.$$

On the other hand, since by construction $\Omega(\eta) \cap \Omega(\eta)^+ = \emptyset$, from (4.98) we obtain

$$\begin{aligned}
&J^\epsilon_\Omega(u) \geq J^\epsilon_{\Omega(\eta)}(u) + J^\epsilon_{D\setminus\Omega(\eta)}(u), \\
&J^\epsilon_{D\setminus\Omega(\eta)}(u) \geq (2h - \eta)(\sigma - C\epsilon).
\end{aligned}$$

This, (4.100) and Lemma 4.12 yield

$$(\sigma - C_W K^2 \epsilon^{\frac{1}{2}})\frac{\eta}{\cos\alpha_\eta} + (2h - \eta)(\sigma - C\epsilon) \leq 2h\sigma + C\epsilon,$$

which, via the expression of $\cos\alpha_\eta$, implies

$$\begin{aligned}
&\sqrt{(x_\eta - C_0\epsilon)^2 + \eta^2} - \eta \leq C\epsilon^{\frac{1}{2}}, \\
&\Rightarrow \quad x_\eta \leq C\epsilon^{\frac{1}{4}}.
\end{aligned} \tag{4.101}$$

Set

$$D_\epsilon = \Big((-C\epsilon^{\frac{1}{4}}, C\epsilon^{\frac{1}{4}}) \cup (l - C\epsilon^{\frac{1}{4}}, l + C\epsilon^{\frac{1}{4}})\Big) \times \Big((0, h) \setminus Y\Big).$$

Then (4.41), Lemma 4.15 and (4.101) imply

$$J^{\epsilon}_{D_\epsilon}(u) \geq 2(\sigma - C_W \epsilon^{\frac{1}{2}})(h - \mathcal{H}^1(Y)),$$
$$\Rightarrow \quad J^{\epsilon}_{D_\epsilon}(u) \geq 2\sigma h - C\epsilon^{\frac{1}{2}}.$$

This shows that most of the energy of u is concentrated in D_ϵ and as a consequence applying the arguments developed for (4.95) we establish the exponential estimates in Theorem 1.4. The proof of Theorem 1.4 is complete.

References

[1] Alama S., Bronsard L. and Gui C. Stationary layered solutions in $\mathbb{R}^2$ for an Allen–Cahn system with multiple well potential. *Calc. Var.* **5** No. 4 (1997), pp. 359—390.

[2] Alikakos N.D. A new proof for the existence of an equivariant entire solution connecting the minima of the potential for the system $\Delta u - W_u(u) = 0$. *Commun. Partial Diff. Equ.*, **37** No. 12 (2012), pp. 2093—2115.

[3] Alikakos N.D. and Fusco G. Entire solutions to equivariant elliptic systems with variational structure. *Arch. Rat. Mech. Anal.*, **202** No. 2 (2011), pp. 567—597.

[4] Alikakos N.D. and Fusco, G. Density estimates for vector minimizers and application. *Discr. Contin. Dynam. Syst.*, **35** No. 12 (2015), pp. 5631—5663.

[5] Alikakos N.D., Fusco G. and Smyrnelis P. *Elliptic systems of phase transition type.* Progress in Nonlinear Differential Equations and their applications n.91, Birkhäuser, (2018).

[6] Ansini N., Braides A. and Piat V.C. Gradient theory of phase transitions for composite media. *Proc. Roy. Soc. Edin.* A, **133** No. 2 (2003), pp. 265—296.

[7] Baldo S. Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. *Ann. Inst. Henri Poincaré* **7** No. 2 (1990), pp. 67—90.

[8] Bates P.W., Fusco G. and Smyrnelis P. Multi-phase solutions to the vector Allen-Cahn equation: crystalline and other complex symmetric structures. *Arch. Ration. Mech. Anal.* **225** No. 2 (2017), pp. 685—715

[9] Braides A. *A handbook of* Γ*-convergence* in *Stationary Partial Differential Equations*. *Volume* **3** (M. Chipot and R. Quiffner Eds.) (2006), pp. 67—90.

[10] Bronsard L., Gui C. and Schatzman M. A three-layered minimizer in $\mathbb{R}^2$ for a variational problem with a symmetric three-well potential. *Comm. Pure. Appl. Math.* **49** No. 7 (1996), pp. 677–715.

[11] Caffarelli L. and Cordoba A. Uniform convergence of a singular perturbation problem. *Commun. Pure Appl. Math.* **48** (1995), pp. 1—12.

[12] del Pino M., Kowalczyk M. and Wei J. On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. Math.* **174**, 1485–1569 (2011)

[13] Farina A. and Valdinoci E. A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature. *Adv. Math.* **225**, (2010), pp. 2808–2827.

[14] Fusco G. Equivariant entire solutions to the elliptic system $\Delta u - W_u(u) = 0$ for general G-invariant potentials. Calc. Var. Part. Diff. Equ. **49** No. 3 (2014), pp. 936–985.

[15] Garofalo N. and Lin F. H. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.* **35** (1986), 245–267.

[16] Gilbarg D. and Trudinger N. S. *Elliptic Partial Equations of Second Order.* Springer-Verlag Berlin Heidelberg New York Tokyo, 1998 edition.

[17] Gui, C. Hamiltonian identities for partial differential equations. *J. Funct. Anal.* **254** No. 4 (2008), pp. 904— 933.

[18] Gui, C. and Schatzman M. Symmetric quadruple phase transitions. *Ind. Univ. Math. J.* **57** No. 2, 781–836 (2008)

[19] Modica L. A gradient bound and a Liouville theorem for nonlinear Poisson equations. *Comm. Pure Appl. Math* **38** No. 5 (1985), pp. 679–684.
[20] Monteil A. and Santambrogio F. Metric methods for heteroclinic connections. *Math. Methods Appl. Sci.* **41** (3) (2016), pp. 1019-1024.
[21] Savin, O. Regularity of flat level sets in phase transitions. *Ann. of Math.* **169**, 41–78 (2009)
[22] Schatzman M. Asymmetric heteroclinic double layers. *Control Optim. Calc. Var.* **8** (A tribute to J. L. Lions), 965–1005 (electronic) (2002)
[23] Smyrnelis P. Gradient estimates for semilinear elliptic systems and other related results. *Proc. R. Soc. Edinb.* Sect. A **145** No. 6, (2015), pp. 1313—1330.
[24] Smyrnelis P. Connecting orbits in Hilbert spaces and application to PDE. *Commun. Pure Appl. Anal.* **19** n.5, (2019), pp. 2797—2818.
[25] Wei J. Geometrization program of semilinear elliptic equations. AMS/IP **51**, 831–857 (2012)

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