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THE MAXIMUM AND ITS POSITION IN DIFFUSIVELY RESCALED SYMMETRIC CONDENSING ZERO-RANGE PROCESSES

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Abstract. Condensing zero range processes (ZRPs) are stochastic interacting particle systems that exhibit phase separation with the emergence of a condensate. Standard approaches for deriving a hydrodynamic limit of the density fail in these models, and an effective macroscopic description has not been rigorously established. In this article we focus on the macroscopic description of the condensate and study the evolution of its size and its position.

1. Introduction

Zero-range processes (ZRPs) are stochastic interacting particle systems on lattices. Particles leave from a site they occupy at an exponential rate $\mathfrak{g}(k)$ which depends only on the number k of particles on that same site through a non-negative function $\mathfrak{g}$. Thus each particle interacts only with the particles on the same site hence the term zero-range. The stochastic dynamics pose no restriction on the number of particles per site.

ZRPs were introduced in 1970 by Spitzer [13]. Since then, they have attracted a lot of attention, especially after it was observed that for suitable eventually non-increasing jump rates they exhibit phase separation [6] above a finite critical density $\rho_c > 0$. This phase transition is manifested in the thermodynamic limit: as was proved in [10, 2] for particular jump rate functions $\mathfrak{g}$, the finite dimensional projections of the canonical equilibrium states on the communication classes $\mathbb{M}_{N,K}$ consisting of configurations of K particles on a lattice with N sites converge weakly, as the density K/N of particles tends to $\rho > \rho_c$, to the grand canonical equilibrium measure of mean density ρ_c. The excess $\rho - \rho_c$ macroscopic mass is concentrated on a randomly located site.

A main open problem in the theory of condensing ZRPs is obtaining the hydrodynamic equation describing their macroscopic evolution in the appropriate scaling limit. The difficulties stem from both the microscopic and the macroscopic level. On the microscopic level difficulties arise from the slow ergodic averaging that such processes exhibit and the lack of an upper bound on the number of particles at each site of the lattice. Due to the rich dynamical behavior of condensing ZRPs the particular initial states from which the process starts are expected to play an important role in the hydrodynamic description. An intriguing choice of initial data is to start the ZRP from a supercritical profile $\rho_0(u) > \rho_c$ for u in some non-trivial domain. In this case, large clusters are initially formed which then exchange particles during a coarsening phase. The coarsening regime has been heuristically studied extensively in dimension $d = 1$ [9, 7, 10].

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Rigorous results when the number of particles tends to infinity on a fixed finite lattice have been obtained in [3]. The simplest case of initial data in the study of the hydrodynamic limit is to start from a subcritical initial profile. This has been rigorously covered in [14].

The natural next step is to consider the case of an initial profile which is subcritical outside a single well formed condensate, since in this case the coarsening phase of the evolution should not enter the picture. This is the motivation for the current article. We prove that if a symmetric ZRP starts from such an initial profile in dimensions $d \geq 3$, then the (rescaled) size of the condensate and its position do not change in the diffusive hydrodynamic timescale, where time is rescaled by N^2 when the lattice spacing is $\frac{1}{N}$. In the course of the proof we also obtain that the second largest coordinate does not change in size in the diffusive timescale. This behavior in dimensions $d \geq 3$ is different from the conjectured behavior in dimension $d = 1$ as heuristic arguments and Monte Carlo simulations [8, 12] suggest, where out of equilibrium initial conditions are expected to lead to mass transfer between the bulk and the condensate in the hydrodynamic scale. The behavior of the ZRP in dimension $d \geq 3$ is also different to the one obtained in [3] regarding the coarsening phase in a fixed finite lattice in dimension $d = 1$ on the diffusive timescale, where it is proved that clusters will exchange mass and die out until a single condensate remains. In dimension $d = 2$ both behaviors seem plausible to us, depending on the density of the bulk around the condensate.

In general, solutions to the hydrodynamic equation, which for symmetric ZRPs is expected to be a non-linear saturated diffusion, must be considered to be measure-valued in order to account for the existence of the condensate. A rigorous derivation of the hydrodynamic equation has not yet been achieved. The results in this article indicate that in dimensions $d \geq 3$ and for initial data with a unique condensate one can expect the measure-valued solution to consist of a Dirac mass at the initial position of the condensate and an absolutely continuous part. This also opens up the possibility for a local replacement lemma away from the immobile condensate.

2. Preliminaries and Statement of Results

Let $\mathbb{T}_N^d := \{0, 1, \dots, N-1\}^d$, $N \in \mathbb{N}$, be the discrete torus with periodic boundary conditions and let $\mathbb{M}_N^d$ be the space of configurations of particles on $\mathbb{T}_N^d$, i.e. the set of all functions $\eta\colon \mathbb{T}_N^d \to \mathbb{Z}_+ := \{0, 1, 2, \dots\}$. Thus, for $\eta \in \mathbb{M}_N^d$ and $x \in \mathbb{T}_N^d$, $\eta(x)$ is the number of particles at x. Given a configuration $\eta \in \mathbb{M}_N^d$ with $\eta(x) \neq 0$ we will denote by $\eta^{x,y}$ the configuration resulting from η by moving a particle from site x to site y. If there are no particles at site x we set $\eta^{x,y} = \eta$. The symmetric ZRP on $\mathbb{M}_N^d$ with elementary transition probabilities p on $\mathbb{T}_N^d$ and jump rate $\mathfrak{g}\colon \mathbb{Z}_+ \to \mathbb{R}_+$ is the Markov jump process with generator

$$L_N f(\eta) = \sum_{x,y \in \mathbb{T}_N^d} \{f(\eta^{x,y}) - f(\eta)\}\mathfrak{g}(\eta(x))p(x,y),$$

acting on functions $f\colon \mathbb{M}_N^d \to \mathbb{R}$. The transition probabilities p are assumed to be symmetric and irreducible. We also assume that they are translation invariant, so that, with a slight abuse of notation, $p(x, y) = p(y - x)$ for some probability measure p on $\mathbb{T}_N^d$. The jump rate $\mathfrak{g}$ is assumed to satisfy the non-degeneracy condition $\mathfrak{g}(k) = 0 \iff k = 0$. All examples of condensing ZRPs in the literature that exhibit phase separation have bounded jump rate $\mathfrak{g}$. So we also assume $\mathfrak{g}$ to be bounded. Given an initial probability distribution μ_0^N on $\mathbb{M}_N^d$ we will denote by $P^N \equiv P^{\mu_0^N}$ the law of the *diffusively rescaled* (i.e. sped up by N^2) ZRP on the Skorokhod space of càdlàg (right continuous with left hand limits) paths, $D(\mathbb{R}_+; \mathbb{M}_N^d)$, starting from μ_0^N. The reader is referred to [5, Chapter 3] for the theory of Skorokhod spaces.

For $\eta \in \mathbb{M}_N^d$ we write $|\eta|_1 := \sum_{x \in \mathbb{T}_N^d} \eta(x)$ for the total number of particles in the configuration η. Since p is irreducible, the hyperplanes $\mathbb{M}_{N,k}^d := \{\eta \in \mathbb{M}_N^d \mid |\eta|_1 = k\}$, $k \in \mathbb{Z}_+$, are finite irreducible communication classes and, as such, they have a unique equilibrium state $\nu_{N,k}$. The collection $\{\nu_{N,k}\}_{(N,k)}$ is the canonical ensemble. The grand canonical ensemble consists of the product and translation invariant equilibrium states on $\mathbb{M}_N^d$. Setting $\mathfrak{g}!(k) = \mathfrak{g}(1) \cdot \ldots \cdot \mathfrak{g}(k)$ for $k \in \mathbb{N}$ and $\mathfrak{g}!(0) = 1$ these have one site marginal of the form $\bar{\nu}_\varphi(k) = c\varphi^k/\mathfrak{g}!(k)$ for some normalizing constant $c = c(\varphi)$ and a parameter $\varphi \geq 0$ such that the series $Z(\varphi) := \sum_{k=0}^{\infty} \varphi^k/\mathfrak{g}!(k)$ converges. The normalizing constant is $c(\varphi) = 1/Z(\varphi)$. The density function is defined by $R(\varphi) := \int k \, \mathrm{d}\bar{\nu}_\varphi(k)$. It is strictly increasing on its proper domain $\mathcal{D}_R := \{\varphi \geq 0 \mid R(\varphi) < +\infty\}$ and its inverse $\Phi \colon R(\mathcal{D}_R) \to \mathcal{D}_R$ is well defined. The critical density is $\rho_c := \sup_{\varphi \in \mathcal{D}_R} R(\varphi)$ and whenever $\rho_c < +\infty$, Z converges at its radius of convergence φ_c, i.e. $Z(\varphi_c) < +\infty$ (see [11, Section 2.2.3]). The grand canonical ensemble $\bar{\nu}_\varphi^N := \bar{\nu}_\varphi^{\otimes N}$ is parametrized by the density by setting $\nu_\rho^N := \bar{\nu}_{\Phi(\rho)}^N$ for all $\rho \in [0, \rho_c] \cap \mathbb{R}_+$. We write $\nu_\rho := \bar{\nu}_{\Phi(\rho)}$ for the 1-site marginal of ν_ρ^N.

A class of jump rate functions $\mathfrak{g}$ for which the ZRP exhibits phase transition in the thermodynamic limit are those whose factorial $\mathfrak{g}!(k)$ satisfies the asymptotic relation

$$\lim_{k \to +\infty} \frac{\mathfrak{g}!(k)}{k^b} = C(b) \in (0, +\infty)$$

for $b > 2$. Particular examples are the Evans jump rate given by $\mathfrak{g}(k) = 1 + \frac{b}{k}$, $k \geq 1$, with the asymptotic behavior $\mathfrak{g}!(k) \sim \Gamma(b+1)^{-1} k^b$ ([10]) and the Beltrán-Landim jump rate defined by $\mathfrak{g}(1) = 1$ and $\mathfrak{g}(k) = (\frac{k}{k-1})^b$, whose factorial is exactly $\mathfrak{g}!(k) = k^b$.

In this article we study the evolution of the maximum of the ZRP in the diffusive timescale. Let $M_N \colon \mathbb{M}_N^d \to \mathbb{Z}_+$ and $K_N \colon \mathbb{M}_N^d \to 2^{\mathbb{T}_N^d}$ be the functions

$$M_N(\eta) = \max_{x \in \mathbb{T}_N^d} \eta(x), \quad K_N(\eta) := \{x \in \mathbb{T}_N^d \mid \eta(x) = M_N(\eta)\}.$$

We consider the process $M^N \colon \mathbb{R}_+ \times D(\mathbb{R}_+; \mathbb{M}_N^d) \to \mathbb{R}_+$ and $K^N \colon \mathbb{R}_+ \times D(\mathbb{R}_+, \mathbb{M}_N^d) \to 2^{\mathbb{T}_N^d}$ defined by $M_t^N(\eta) = M_N(\eta_t)$ and $K_t^N(\eta) = K_N(\eta_t)$. We will also consider the strict second maximum $M_{2,N} \colon \mathbb{M}_N^d \to \mathbb{R}_+$ defined by $M_{2,N}(\eta) = \max_{x \in \mathbb{T}_N^d \setminus K_N} \eta(x)$ and the induced process $M_t^{2,N}(\eta) = M_{2,N}(\eta_t)$ for $\eta \in D(\mathbb{R}_+; \mathbb{M}_N^d)$, $t \geq 0$. The processes (M_t^N), (K_t^N) and $(M_t^{2,N})$ are functions of the ZRP and thus their analysis can be based on their associated martingales

$$\Big(F(\eta_t) - F(\eta_0) - \int_0^t L_N F(\eta_s) \, \mathrm{d}s\Big)_{t \geq 0} \quad \text{for } F = M_N,\ K_N \text{ and } M_{2,N}.$$

In all the results we assume that the initial states μ_0^N have finite first order moments i.e. $\int \eta(x) \, \mathrm{d}\mu_0^N < +\infty$ for all $x \in \mathbb{T}_N^d$ and total mass asymptotically bounded from above in probability, i.e. that there exists $\bar{\rho} < +\infty$ such that

$$\lim_{N \to +\infty} \mu_0^N \Big\{ \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) > \bar{\rho} \Big\} = 0. \tag{1}$$

Any additional assumptions required on the initial data will be stated in the results.

We will make use of the following notation. Given a measurable map $f \colon X \to Y$ between measurable spaces and a measure μ on the σ-algebra of X we will denote by $f_\sharp \mu$ the push forward measure defined by $f_\sharp\mu(B) = \mu \circ f^{-1}(B)$ for all $B \subseteq Y$ in the σ-algebra of the space Y. Any topological space will be considered as a measurable space with its Borel σ-algebra. Also for any topological space X and $T > 0$ we will denote by $\mathfrak{c} \equiv \mathfrak{c}_X : X \to C(0, T; X)$ the map that assigns to each $x \in X$ the constant path $\mathfrak{c}(x)(t) = x$, $0 \le t \le T$. When necessary we will use the subindex X in the map $\mathfrak{c}_X$ to clarify the target space X of the constant maps.

Theorem 2.1. *Let $\{\mu_0^N\}_{N\in\mathbb{N}}$ be a sequence of initial distributions on $\mathbb{M}_N^d$ with finite first order moments and total mass bounded above in the sense of* (1)*. Let P^N be the law on $D(0, T; \mathbb{M}_N^d)$ of a diffusively rescaled symmetric ZRP on $\mathbb{T}_N^d$ with bounded jump rate $\mathfrak{g}$ starting from μ_0^N. If $d \ge 3$, then the rescaled maximum is asymptotically constant in the hydrodynamic scale as $N \to +\infty$, i.e.*

$$\lim_{N\to+\infty} P^N\Big\{ \sup_{0\le s,t\le T} \Big| \frac{M_t^N - M_s^N}{N^d} \Big| > \varepsilon \Big\} = 0, \quad \forall\, \varepsilon, T > 0, \tag{2}$$

the sequence $\{Q^N\} := \{\big((N^{-d}M_t^N)_{t\ge0}\big)_\sharp P^N\}$ of the laws of the rescaled maximum on $D(0, T; \mathbb{R}_+)$ is relatively compact in the weak topology and any limit point Q^* on $D(0, T; \mathbb{R}_+)$ of $\{Q^N\}$ is concentrated on constant paths. Furthermore, if the initial laws of the rescaled initial maximum $\{(N^{-d}M_N)_\sharp\mu_0^N\}_{N\in\mathbb{N}}$ converge weakly to a measure m_0 on $\mathbb{R}_+$ then the whole sequence $\{Q^N\}_{N\in\mathbb{N}}$ converges weakly to the measure $Q_{m_0} := (\mathfrak{c}_{\mathbb{R}_+})_\sharp m_0$ which is supported by constant paths and assigns probabilities according to m_0.

Of course this implies that if the initial distributions $\{\mu_0^N\}$ have initially exactly $\gamma > 0$ condensing mass in probability, in the sense that $\lim_{N\to+\infty} \mu_0^N\{|N^{-d}M_N - \gamma| > \varepsilon\} = 0$ for all $\varepsilon > 0$, then so does the ZRP for all times, $m_0 = \delta_\gamma$ and the measure Q_{m_0} is the Dirac measure concentrated on the constant path equal to γ.

Remark 2.1. *In the particular case of the Beltrán-Landim model with $b > 3$, under stronger assumptions on the sequence $\{\mu_0^N\}$ of the initial distributions, we can obtain that the laws $\{Q^N\}$ of the rescaled maximum are relatively compact also in dimension $d = 2$ and that all the limit points of $\{Q^N\}$ are concentrated on continuous trajectories. The stronger assumptions are that $\mu_0^N\{|\cdot|_1 = k_N\} = 1$ for some sequence $\{k_N\}_{N=1}^\infty$ such that*

$$\lim_{N\to+\infty} k_N/N^d = \rho > \rho_c \tag{3}$$

and that $\{\mu_0^N\}$ is close to the canonical state on the hyperplane with k_N particles, in the sense that for some pair of conjugate exponents $p \in [1, +\infty)$, $q \in (1, +\infty]$

$$\lim_{N\to+\infty} \frac{1}{N^{\frac{2(b-2)}{p}-2}} \Big\| \frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_{N,k_N}} \Big\|_{L^q(\nu_{N,k_N})} = 0. \tag{4}$$

In the next section we will see examples of states μ_0^N that satisfy assumption (4). We have stated Remark 2.1 only for the Beltrán-Landim model with $b > 3$, and not for all jump rates $\mathfrak{g}$ whose factorial scales as $C(b)k^b$ when k tends to infinity, because their proofs rely on results in [1, Section 9] which are proved there only for the Beltrán-Landim jump rate with $b > 3$. Since Remark 2.1 concerns only partial results in dimension $d = 2$, which is not the main focus of this article, we do not attempt to extend the results of [1, Section 9] to more general jump rates. However these results are expected to hold for all non-increasing jump rates whose factorial scales as $\mathfrak{g}!(k) \sim C(b)k^b$ with $b > 2$.

In order to obtain information on the position of the maximally occupied coordinate we also study the strict second largest coordinate $(M_t^{2,N})_{t\ge0}$.

Theorem 2.2. *Let μ_0^N, $N \in \mathbb{N}$, be an initial distribution on $\mathbb{M}_N^d$, $d \geq 3$, with finite first order moments and let P^N be the law on $D(0,T;\mathbb{M}_N^d)$ of a diffusively rescaled symmetric ZRP with bounded jump rate $\mathfrak{g}$ starting from μ_0^N. Assume that the sequence $\{\mu_0^N\}$ of the initial distributions has total mass bounded above in the sense of* (1)*, that the maximum is achieved on a singleton, i.e.*

$$\lim_{N\to+\infty} \mu_0^N\big\{\sharp K_N \geq 2\big\} = 0 \tag{5}$$

and that the rescaled strict second maximum is initially asymptotically distinct from the rescaled maximum, i.e. for some $\delta_0 > 0$

$$\lim_{N\to+\infty} \mu_0^N\Big\{\frac{M_N - M_{2,N}}{N^d} < \delta_0\Big\} = 0. \tag{6}$$

Then the maximizing set is a singleton and the rescaled strict second maximum is asymptotically constant in the hydrodynamic scaling in any finite time horizon, i.e.

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq t\leq T} \sharp K_t^N \geq 2\Big\} = 0, \quad \forall T > 0 \tag{7}$$

and

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq s,t\leq T} \Big|\frac{M_t^{2,N} - M_s^{2,N}}{N^d}\Big| > \varepsilon\Big\} = 0, \quad \forall \varepsilon, T > 0. \tag{8}$$

It follows that the strict second maximum remains asymptotically distinct from the first maximum at all times, namely for all $\delta \in (0,\delta_0)$

$$\lim_{N\to+\infty} P^N\Big\{\inf_{0\leq t\leq T} \frac{M_t^N - M_t^{2,N}}{N^d} < \delta\Big\} = 0, \quad \forall T > 0. \tag{9}$$

As a corollary we obtain that the position of the maximum does not change in the diffusive hydrodynamic scale in any finite time horizon $T > 0$. There are various approaches to defining the position of the maximum x_N. One is to order the set K_N in some complete order, for example the lexicographic order, and define x_N as the minimal element of the set K_N in this order. Another is to add an isolated point $\mathfrak{d}$ to the microscopic toruses, $\bar{\mathbb{T}}_N^d := \mathbb{T}_N^d \cup \{\mathfrak{d}\}$ and the macroscopic torus, $\bar{\mathbb{T}}^d := \mathbb{T}^d \cup \{\mathfrak{d}\}$ and then define the function $x_N \colon \mathbb{M}_N^d \to \bar{\mathbb{T}}_N^d$ by the formula

$$x_N = \begin{cases} y & \text{if } K_N = \{y\},\ y \in \mathbb{T}_N^d \\ \mathfrak{d} & \text{if } \sharp K_N \geq 2 \end{cases}.$$

Since $\mathfrak{d}$ is an isolated point, $\bar{\mathbb{T}}^d$ consists by definition of the two connected components $\mathbb{T}^d$ and $\{\mathfrak{d}\}$. The macroscopic position of the condensate is then defined as $u_N = x_N/N$ if $x_N \neq \mathfrak{d}$ and $u_N = \mathfrak{d}$ otherwise. As usual we define the induced processes by $x_t^N(\eta) = x_N(\eta_t)$ and $u_t^N(\eta) = u_N(\eta_t)$ for $\eta \in D(0,T;\mathbb{M}_N^d)$. We can extend the metric d of the torus $\mathbb{T}^d$ to a metric $\bar{d}$ on $\bar{\mathbb{T}}^d$ by choosing a constant $R \geq \frac{1}{2} = \operatorname{diam}\mathbb{T}^d/2$ and defining $\bar{d}(u,\mathfrak{d}) = R$ for all $u \in \mathbb{T}^d$, so that $(\bar{\mathbb{T}}^d, \bar{d})$ becomes a compact metric space.

Corollary 2.1. *Under the assumptions of Theorem 2.2, let $U^N := ((u_t^N)_{0\leq t\leq T})_\sharp P^N$ be the law on $D(0,T;\bar{\mathbb{T}}^d)$ of the macroscopic position of the maximal coordinate of the diffusively rescaled ZRP. Then $\{U^N\}_{N\in\mathbb{N}}$ is relatively compact in the weak topology and all of its limit points are supported by the constant paths in $\mathbb{T}^d$. If we assume in addition that the macroscopic position of the condensate has initially a well defined profile λ_0, in the sense that the laws*

$\lambda_N := u_{N\sharp}\mu_0^N$ converge weakly to a measure λ_0 on $\bar{\mathbb{T}}^d$, then the process U^N converges to the law $U_{\lambda_0} := (\mathfrak{c}_{\bar{\mathbb{T}}^d})_\sharp\lambda_0$.

Related Results. The dynamics of the location of the condensate in dimension $d = 1$ have been studied extensively in the literature. Some of the first results [8, 12] focused on the Evans jump rate, relied on heuristic arguments and simulations and conjectured that the condensate moves at the timescale N^{1+b}. This was proved rigorously in [4] for the particular case of the Beltrán-Landim jump rate on a fixed finite lattice as the number of particles converges to infinity and in [1] for the Beltrán-Landim jump rate with $b > 20$, when the lattice size converges to infinity together with the number of particles. Although our main results focus only on dimensions $d \geq 3$ and at the faster diffusive time scale N^2, they apply to all bounded jump rates for quite general initial profiles and also describe the evolution of the size of the condensate. They are thus expected to be useful in a rigorous derivation of the expected hydrodynamic equation of the ZRP.

3. Examples of admissible initial states

Before proceeding with the proofs of the main results, we give in this section concrete examples of initial states that satisfy our assumptions.

Example 3.1. *A simple example of initial states $\{\mu_0^N\}_{N\in\mathbb{N}}$ that satisfy the assumptions of Theorems 2.1 and 2.2 and of Corollary 2.1 is the family of product measures with slowly varying profile $\rho_0\colon \mathbb{T}^d \to [0, \rho_c]$ and a Dirac mass $\gamma > 0$ at a point $v \in \mathbb{T}^d$. More precisely, given sequences $\{\gamma_N\} \subseteq \mathbb{R}_+$ and $\{y_N \in \mathbb{T}_N^d\}_{N\in\mathbb{N}}$ such that $\lim_{N\to+\infty} \gamma_N/N^d = \gamma > 0$ and $\lim_{N\to+\infty} y_N/N = v \in \mathbb{T}^d$, we define a probability measure μ_0^N on $\mathbb{M}_N^d$ by*

$$\mu_0^N := (\gamma_N\delta_{y_N}) \otimes \Big(\bigotimes_{x\neq y_N} \nu_{\rho_0(x/N)}\Big).$$

The family $\{\mu_0^N\}$ satisfies assumptions (1), (5) *and* (6), *the law $(N^{-d}M_N)_\sharp\mu_0^N$ of its maximum converges weakly to the Dirac measure δ_γ on $\mathbb{R}_+$ and the law $u_{N\sharp}\mu_0^N$ of the position of the condensate converges weakly to the Dirac measure δ_v on $\mathbb{T}^d$.*

Proof. The family $\{\mu_0^N\}$ is associated to the macroscopic profile $\mu_0 := \rho_0\,\mathrm{d}\mathcal{L}_{\mathbb{T}^d} + \gamma\delta_v$ in the sense that

$$\mu_0^N\Big\{\Big|\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} G\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}^d} G(u)\,\mathrm{d}\mu_0(u)\Big| > \varepsilon\Big\} = 0$$

for all $G \in C(\mathbb{T}^d)$ and all $\varepsilon > 0$. This is a consequence of the law of large numbers, [11, Section 3]. In particular, if $\bar\rho > \mu_0(\mathbb{T}^d)$ then $\{\mu_0^N\}$ has total mass bounded above by $\bar\rho$ and thus $\{\mu_0^N\}$ satisfies assumption (1). To check that assumption (5) is also satisfied we note that since $M_N \geq \eta(y_N) \geq \gamma_N$ μ_0^N-a.s.,

$$\mu_0^N\{\sharp K_N \geq 2\} \leq \sum_{x\neq y_N} \mu_0^N\{\eta(x) \geq \gamma_N\} = \sum_{x\neq y_N}\int_{\mathbb{Z}_+} \mathbb{1}_{[\gamma_N,+\infty)}(k)\,\mathrm{d}\nu_{\rho_0(x/N)}(k).$$

Thus, since the family $\{\nu_\rho\}_{\rho\in[0,\rho_c]}$ is stochastically non-decreasing [11, Lemma 2.3.5] and the function $\mathbb{1}_{[\gamma_N,+\infty)}$ is non-decreasing

$$\mu_0^N\{\sharp K_N \geq 2\} \leq (N^d - 1)\nu_{\rho_c}([\gamma_N, +\infty)) \leq \frac{N^d-1}{\gamma_N}\int k\mathbb{1}_{[\gamma_N,+\infty)}(k)\nu_{\rho_c}(k) \overset{N\to+\infty}{\longrightarrow} 0,$$

where the last limit holds since $\int k \, \mathrm{d}\nu_{\rho_c}(k) = \rho_c < +\infty$ and $\lim_{N\to+\infty} \frac{\gamma_N}{N^d} = \gamma > 0$. Likewise, if $\delta_0 < \gamma$ then

$$\begin{aligned}\mu_0^N\Big\{\frac{M_N - M_{2,N}}{N^d} < \delta_0\Big\} &\leq \sum_{x\neq y_N} \mu_0^N\{\eta(x) > \gamma_N - \delta_0 N^d\} \\ &\leq (N^d - 1)\nu_{\rho_c}(\gamma_N - \delta_0 N^d, +\infty) \overset{N\to+\infty}{\longrightarrow} 0,\end{aligned}$$

which shows that the initial states $\{\mu_0^N\}$ also satisfy assumption (6). Since $M_N \geq \gamma_N$,

$$\mu_0^N\{M_N \neq \gamma_N\} = \mu_0^N\{M_N > \gamma_N\} \leq \sum_{x\neq y_N} \mu_0^N\{\eta(x) > \gamma_N\} \overset{N\to+\infty}{\longrightarrow} 0.$$

This shows that $\lim_{N\to+\infty}(N^{-d}M_N)_\sharp\mu_0^N = \delta_\gamma$. Similarly, $\lim_{N\to+\infty} \mu_0^N\{x_N \neq y_N\} = 0$ which shows that $\lim_{N\to+\infty} u_{N\sharp}\mu_0^N = \delta_v$. □

Example 3.2. *In the context of the Beltrán-Landim jump rate with $b > 3$, a simple example of initial states out of (but close to) equilibrium for which the assumptions of Remark 2.1 are satisfied is given by fixing $\rho > \rho_c$ and conditioning the canonical state $\nu_{N,[\rho N^d]}$ so that the occupation variables obtain their maximum at a particular site. To be precise, given $\rho > \rho_c$ and a point $v \in \mathbb{T}^d$ in the macroscopic torus we define a sequence $\{\nu_{\rho;v}^N\}_{N\in\mathbb{N}}$ of initial distributions by*

$$\nu_{\rho;v}^N(\cdot) := \nu_{N,[\rho N^d]}\big(\,\cdot\,|M_N = \eta([Nv])\big).$$

Then the sequence $\{\nu_{\rho;v}^N\}$ has deterministic total mass equal to $[\rho N^d]$, has the macroscopic profile $\mu := \rho_c \, \mathrm{d}\mathcal{L}_{\mathbb{T}^d} + (\rho - \rho_c)\delta_v$ and if $b > 3$ then for any pair of conjugate exponents p, q with $p \in [1, +\infty)$

$$\Big\|\frac{\mathrm{d}\nu_{\rho;v}^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_{N,[\rho N^d]})} \backsim N^{\frac{d}{p}} \quad \textit{as } N \to +\infty. \tag{10}$$

Consequently, if $b > 3 + p$ then the assumption (4) is satisfied in dimension $d = 2$.

Proof. A simple computation shows that $\|\frac{\mathrm{d}\nu_{\rho;v}^N}{\mathrm{d}\nu_N}\|_{L^q(\nu_N)} = \nu_{N,[\rho N^d]}\big\{M_N = \eta([Nv])\big\}^{-\frac{1}{p}}$. Thus in order to show (10) it suffices to show that

$$\lim_{N\to\infty} N^d \nu_{N,[\rho N^d]}\big\{M_N = \eta([Nv])\big\} = 1.$$

Since the function $\sharp K_N$ is translation invariant and the events $\{\sharp K_N = 1,\ M_N = \eta(x)\}$ and $\{\sharp K_N = 1,\ M_N = \eta(y)\}$ are disjoint when $x \neq y$, we have by the translation invariance of the law $\nu_{N,[\rho N^d]}$ that

$$\nu_{N,[\rho N^d]}\{\sharp K_N = 1,\ M_N = \eta([Nv])\} = \frac{1}{N^d}\nu_{N,[\rho N^d]}\{\sharp K_N = 1\}.$$

Similarly, by translation invariance, $\nu_{N,[\rho N^d]}\{M_N = \eta([Nv])\} \geq \frac{1}{N^d}$ and consequently we have by Lemma 4.1 that

$$\begin{aligned}1 \leq N^d \nu_{N,[\rho N^d]}\big\{M_N = \eta([Nv])\big\} &\leq \nu_{N,[\rho N^d]}\{\sharp K_N = 1\} + N^d \nu_{N,[\rho N^d]}\{\sharp K_N \geq 2\} \\ &\leq 1 + CN^{-(b-3)d}.\end{aligned}$$

Since $b > 3$, the claim follows by taking the limit as $N \to +\infty$. □

4. Proofs

In Section 4.1 we prove Theorem 2.1 and the remark following it and in Section 4.2 we prove Theorem 2.2 and Corollary 2.1.

4.1. **On the size of the maximum.** We break down the proof of Theorem 2.1 in smaller subsections. First, we introduce the martingale associated to the maximum process via the martingale problem and prove its asymptotic negligibility. Secondly, we prove that the maximum remains asymptotically constant in time in the hydrodynamic scale. We then use this fact to prove the relative compactness of the rescaled maximum process in the weak topology of probability measures and also give the proof of Remark 2.1. Finally, we complete the proof of Theorem 2.1 by proving the convergence of the laws of the rescaled maximum.

The martingale associated to the maximum.

Proposition 4.1. *Assume that the ZRP starts from an initial state μ_0^N on $\mathbb{M}_N^d$ with finite first moments. Then the process*

$$A_t^N := \frac{M_t^N}{N^d} - \frac{M_0^N}{N^d} - \frac{1}{N^{d-2}}\int_0^t L_N M_N(\eta_s)\,\mathrm{d}s, \quad t \geq 0, \tag{11}$$

is a square integrable martingale with respect to the natural filtration $(\mathcal{F}_t^N)_{t>0}$ of the ZRP and its quadratic variation $\langle A^N\rangle$ (in the sense that the process $V^N := (A^N)^2 - \langle A^N\rangle$ is a martingale) is given by

$$\langle A^N\rangle_t = \frac{1}{N^{2d-2}}\int_0^t \sum_{x,y\in\mathbb{T}_N^d}\big(M_N(\eta_s^{x,y}) - M_N(\eta_s)\big)^2\mathfrak{g}(\eta_s(x))p(y-x)\,\mathrm{d}s. \tag{12}$$

Proof. The jump rate of the ZRP is assumed bounded. The claim would thus follow immediately from standard results on Markov jump processes (e.g. [11, Lemma A1.5.1]) if the map M_N was bounded. Here we apply [11, Lemma A1.5.1] to the bounded map $M_{N,\varepsilon}\colon \mathbb{M}_N^d \to \mathbb{R}_+$ given by $M_{N,\varepsilon} = \frac{M_N}{1+\varepsilon|\cdot|_1}$. Then, setting $M_t^{N,\varepsilon}(\eta) = M_{N,\varepsilon}(\eta_t)$ for $\eta \in D(0,T;\mathbb{M}_N^d)$, we have by [11, Lemma A1.5.1] that for each fixed $\varepsilon > 0$ the process

$$A_t^{N,\varepsilon} := \frac{M_t^{N,\varepsilon}}{N^d} - \frac{M_0^{N,\varepsilon}}{N^d} - \frac{1}{N^{d-2}}\int_0^t L_N M_{N,\varepsilon}(\eta_s)\,\mathrm{d}s$$

is a bounded, and thus an L^2-martingale, with quadratic variation

$$\langle A^{N,\varepsilon}\rangle_t = \frac{1}{N^{2d-2}}\int_0^t \sum_{x,y\in\mathbb{T}_N^d}\big(M_{N,\varepsilon}(\eta_s^{x,y}) - M_{N,\varepsilon}(\eta_s)\big)^2\mathfrak{g}(\eta_s(x))p(y-x)$$

and the process $V_t^{N,\varepsilon} := (A^{N,\varepsilon})^2 - \langle A^{N,\varepsilon}\rangle$ is a martingale. Since $|\eta^{x,y}|_1 = |\eta|_1$ for all $\eta \in \mathbb{M}_N^d$ we have that $L_N M_{N,\varepsilon}(\eta) = \frac{1}{(1+\varepsilon|\eta|_1)}L_N M_N(\eta)$ and thus, since $|\eta_t|_1 = |\eta_0|_1$ for P^N-almost all paths, we also have that $A_t^{N,\varepsilon} = \frac{1}{(1+\varepsilon|\eta_0|_1)}A_t^N$ P^N-a.s. Since A_t^N is in L^1 by the assumption that μ_0^N has finite first order moments, we can apply the conditional dominated convergence theorem to conclude that A^N is a martingale.

Next we note that $\langle A^{N,\varepsilon}\rangle_t = \frac{1}{(1+\varepsilon|\eta_0|_1)^2}\langle A^N\rangle_t \leq \langle A^N\rangle_t \in L^1$ and therefore by Fatou's lemma

$$\mathbb{E}^N(A_t^N)^2 \leq \liminf_{\varepsilon\downarrow 0} \mathbb{E}^N(A_t^{N,\varepsilon})^2 = \lim_{\varepsilon\downarrow 0} \mathbb{E}^N\langle A^{N,\varepsilon}\rangle_t = \mathbb{E}^N\langle A^N\rangle_t < +\infty. \tag{13}$$

In particular A^N is square integrable and since for each $t > 0$ the random variable $V_t^{N,\varepsilon} = (A_t^{N,\varepsilon})^2 - \langle A^{N,\varepsilon}\rangle_t$ is dominated by the integrable random variable $(A_t^N)^2 + \langle A^N\rangle_t$ we can apply the conditional dominated convergence theorem to conclude the proof. □

In order to use the martingale A_t^N to study how the maximum M_t^N evolves in the diffusive timescale, we compute $L_N M_N$ explicitly. First, any jump outside the set K_N does not affect the maximum. On the other hand, any jump from any site $x \in \mathbb{T}_N^d$ to a site $y \in K_N$ will increase the maximum by one. Finally, jumps from $x \in K_N$ to $y \in K_N^c := \mathbb{T}_N^d \setminus K_N$ decrease the maximum by one if $\sharp K_N = 1$ and $\eta(y) \leq M_N - 2$ and leave it unaffected otherwise. Therefore

$$M_N(\eta^{x,y}) - M_N(\eta) = \begin{cases} +1 & \text{if } y \in K_N \\ -1 & \text{if } x \in K_N,\ \eta(y) \leq M_N(\eta) - 2 \text{ and } \sharp K_N = 1 \end{cases} \tag{14}$$

and $L_N M_N$ is equal to

$$L_N M_N = \sum_{\substack{x\in\mathbb{T}_N^d \\ y\in K_N}} \mathfrak{g}\big(\eta(x)\big)p(y-x) - \mathfrak{g}(M_N)\mathbb{1}_{\{\sharp K_N=1\}} \sum_{y:\eta(y)\leq M_N-2} p(y-x_N). \tag{15}$$

It is obvious by (15) that $L_N M_N$ satisfies the trivial bound

$$-\|\mathfrak{g}\|_\infty \leq -\|\mathfrak{g}\|_\infty \mathbb{1}_{\{\sharp K_N=1\}} \leq L_N M_N \leq \|\mathfrak{g}\|_\infty \sharp K_N. \tag{16}$$

Similarly, by (12) and (14) we see that the quadratic variation $\langle A^N\rangle$ satisfies the bound

$$\langle A^N\rangle_t - \langle A^N\rangle_s \leq \frac{2\|\mathfrak{g}\|_\infty}{N^{2d-2}} \int_s^t \mathbb{1}_{\{\sharp K_r^N=1\}}\, \mathrm{d}r + \frac{\|\mathfrak{g}\|_\infty}{N^{2d-2}} \int_s^t \sharp K_r^N \mathbb{1}_{\{\sharp K_r^N\geq 2\}}\, \mathrm{d}r. \tag{17}$$

We examine now the asymptotic negligibility of the martingale A^N.

Proposition 4.2. *In dimensions $d \geq 3$ the martingale $(A_t^N)_{t\geq 0}$ defined in* (11) *is asymptotically negligible as $N \to +\infty$, i.e. for all $\varepsilon > 0$,*

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq t\leq T} |A_t^N| > \varepsilon\Big\} = 0.$$

Proof. The quadratic variation of A^N is given by (12) and by the Chebyshev-Doob inequality

$$P^N\Big\{\sup_{0\leq t\leq T} |A_t^N| > \varepsilon\Big\} \leq \frac{1}{\varepsilon^2}\mathbb{E}^N(A_T^N)^2 = \frac{1}{\varepsilon^2}\mathbb{E}^N\langle A^N\rangle_T.$$

Therefore it suffices to show that $\lim_{N\to+\infty} \mathbb{E}^N\langle A^N\rangle_T = 0$. By (17) and the crudest possible estimate, $\sharp K_N \leq N^d$,

$$\langle A^N\rangle_t - \langle A^N\rangle_s \leq \frac{2\|\mathfrak{g}\|_\infty}{N^{2d-2}} \int_s^t \sharp K_r^N\, \mathrm{d}r \leq \frac{2\|\mathfrak{g}\|_\infty(t-s)}{N^{d-2}}. \tag{18}$$

Therefore $\mathbb{E}^N\langle A^N\rangle_T \leq \frac{2\|\mathfrak{g}\|_\infty T}{N^{d-2}} \xrightarrow{N\to+\infty} 0$ since $d \geq 3$. □

Remark 4.1. *In dimension $d = 2$, in the particular case of the Beltrán-Landim model with $b > 3$, we can show that when the ZRP starts from initial states that satisfy*

$$\lim_{N\to+\infty} \frac{1}{N^{\frac{b-2}{p}\cdot 2}} \Big\| \frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_{N,k_N}} \Big\|_{L^q(\nu_{N,k_N})} = 0 \tag{19}$$

for a pair of conjugate exponents $p \in [1, +\infty)$, $q \in (1, +\infty]$, then the martingale A^N is asymptotically negligible. According to (10) *the initial states of Example 3.2 satisfy the assumption* (19) *for all $b > 3$ in dimension $d = 2$.*

In the proof of Remarks 2.1 and 4.1 we will use the following two lemmas.

Lemma 4.1. *Suppose that the ZRP has the Beltrán-Landim jump rate with $b > 3$. Let $\rho > \rho_c$ and let $\{k_N\}_{N\in\mathbb{N}}$ be a sequence satisfying* (3)*. There exists then a constant $C \geq 0$ independent of N such that for all large enough $N \in \mathbb{N}$*

$$\nu_{N,k_N}\{\sharp K_N \geq 2\} \leq CN^{-(b-2)d}.$$

Proof. This follows by Propositions 9.3 and 9.4 in [1]. They are stated there in the case of one spatial dimension, but since the elementary transition probability p of the ZRP does not appear in the formulas of the canonical and grand canonical equilibrium states, they are valid in all dimensions. To simplify the notation we set $\nu_N := \nu_{N,k_N}$. By (3), given $\theta \in (0, 1)$, there exists $N_\theta \in \mathbb{N}$ such that $k_N \geq \rho_c N^d + \theta(\rho - \rho_c)N^d$ for all $N \geq N_\theta$. Let $a_N := \frac{1}{2}(k_N - [\rho_c N^d])$. Then $a_N \geq \frac{\theta}{2}(\rho - \rho_c)N^d$ for all $N \geq N_\theta$ and therefore by applying [1, Proposition 9.3] we obtain that there exists a constant $C < +\infty$ such that

$$\nu_N\Big\{M_N \leq \frac{k_N - [\rho_c N^d]}{2}\Big\} \leq CN^d a_N^{1-b} \leq \frac{2^{b-1}C}{\theta^{b-1}(\rho - \rho_c)^{b-1}} \frac{1}{N^{(b-2)d}} \tag{20}$$

for all large $N \in \mathbb{N}$. Next, we let x_N be the minimal element of K_N in the lexicographic order of $\mathbb{T}_N^d \cong \{0, 1, \ldots, N\}^d \subseteq \mathbb{Z}^d$, and consider the second largest coordinate $\widetilde{M}_{N,2} := \max_{x\neq x_N} \eta(x)$. By applying [1, Proposition 9.4] with $\beta_N = a_N = \frac{1}{2}(k_N - [\rho_c N^d])$ we obtain a constant $C < +\infty$ such that for all large enough N

$$\begin{aligned}\nu_N\{\widetilde{M}_{N,2} > \beta_N\} &= \nu_N\{M_N > k_N - [\rho_c N^d] - a_N, \widetilde{M}_{N,2} > \beta_N\} + \nu_N\{M_N \leq a_N\} \\ &\leq CN^d(a_N^{1-b} + \beta_N^{1-b}) \leq \frac{2^b C}{\theta^{b-1}(\rho - \rho_c)^{b-1}} \frac{1}{N^{(b-2)d}}.\end{aligned} \tag{21}$$

According to the definition of $\widetilde{M}_{N,2}$ we have $\{\sharp K_N \geq 2\} = \{M_N = \widetilde{M}_{N,2}\}$ and therefore we can estimate the probability of the event $\{\sharp K_N \geq 2\}$ by

$$\begin{aligned}\nu_{N,k_N}\{\sharp K_N \geq 2\} &\leq \nu_{N,k_N}\{M_N = \widetilde{M}_{N,2} \leq a_N\} + \nu_{N,k_N}\{M_N = \widetilde{M}_{N,2} > a_N\} \\ &\leq \nu_{N,k_N}\{M_N \leq a_N\} + \nu_{N,k_N}\{\widetilde{M}_{N,2} > a_N\}.\end{aligned}$$

Thus the claim follows by (20) and (21). □

The following lemma is the reason we need to impose stronger conditions on the sequence of the initial distribution of the ZRP in Remarks 2.1 and 4.1.

Lemma 4.2. *Suppose that the ZRP has the Beltrán-Landim jump rate with $b > 3$, let $\{k_N\} \subseteq \mathbb{N}$ be a sequence satisfying* (3) *for some $\rho > \rho_c$, and let $\nu_N := \nu_{N,k_N}$ be the canonical equilibrium state of the ZRP on $\mathbb{M}^d_{N,k_N}$. If $\{\mu_0^N\}$ is a sequence of initial distributions with deterministic* *particle number k_N, i.e. $\mu_0^N\{|\cdot|_1 = k_N\} = 1$, then there exists a constant $C < +\infty$ independent of N such that for any pair of conjugate exponents $p \in [1, +\infty)$, $q \in (1, +\infty]$*

$$\mathbb{E}^N\Big(\int_0^T \mathbb{1}_{\{\sharp K_t^N \geq 2\}}\,\mathrm{d}t\Big) \leq \frac{C^{\frac{1}{p}}T}{N^{\frac{b-2}{p}d}}\Big\|\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_N)}. \tag{22}$$

Proof. Let μ be a probability measure on the measurable space $M = (M, \mathcal{F})$ and let $1 \leq p \leq \infty$. Any Markov kernel $P\colon M \times \mathcal{F} \to [0,1]$ on M induces a linear contraction $P\colon L^p(\mu P) \to L^p(\mu)$ by the formula $Pf(x) = \int_M f(y)\,\mathrm{d}P^x(y)$. In particular if μ is invariant for the kernel P, i.e. $\mu P = \mu$, then P induces a linear contraction on $L^p(\mu)$. In our particular case, since ν_N is an equilibrium distribution, the semigroup $(S_t^N)_{t\geq 0}$ of the diffusively rescaled ZRP on $\mathbb{M}_N^d$ induces a semigroup of linear contractions $S_t^N\colon L^p(\nu_N) \to L^p(\nu_N)$. Thus $\|S_t^N f^N\|_{L^p(\nu_N)} \leq \|f^N\|_{L^p(\nu_N)}$ for any map $f^N \in L^p(\nu_N)$. We apply this to the ZRP starting from μ_0^N for the function $f^N = \mathbb{1}_{\{\sharp K_N \geq 2\}}$ and use Hölder's inequality to obtain

$$\begin{aligned}\mathbb{E}^N\Big(\int_0^T \mathbb{1}_{\{\sharp K_t^N \geq 2\}}\,\mathrm{d}t\Big) &= \int_0^T \int_{\mathbb{M}_N^d} \big(S_t^N \mathbb{1}_{\{\sharp K_N \geq 2\}}\big)\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\,\mathrm{d}\nu_N\,\mathrm{d}t \\ &\leq \int_0^T \big\|S_t^N \mathbb{1}_{\{\sharp K_N \geq 2\}}\big\|_{L^p(\nu_N)}\Big\|\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_N)}\,\mathrm{d}t \\ &\leq T\cdot \nu_N(\sharp K_N \geq 2)^{\frac{1}{p}}\Big\|\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_N)}.\end{aligned}$$

By Lemma 4.1 this yields (22). □

As in the proof of Proposition 4.2, in order to prove Remark 4.1 regarding the asymptotic negligibility of the martingale A^N in dimension $d = 2$, it suffices to show that $\lim_{N\to+\infty}\mathbb{E}^N\langle A^N\rangle_T = 0$. For this we use (17) and Lemma 4.2 to estimate that for some constant $C < +\infty$ independent of N, for all large N

$$\begin{aligned}\mathbb{E}^N\langle A^N\rangle_T &\leq \frac{2\|\mathfrak{g}\|_\infty}{N^{2d-2}}\mathbb{E}^N\Big(\int_0^T \mathbb{1}_{\{\sharp K_N = 1\}}\,\mathrm{d}t\Big) + \frac{\|\mathfrak{g}\|_\infty}{N^{d-2}}\mathbb{E}^N\Big(\int_0^T \mathbb{1}_{\{\sharp K_t^N \geq 2\}}\,\mathrm{d}t\Big) \\ &\leq \frac{2\|\mathfrak{g}\|_\infty T}{N^2} + \frac{C^{\frac{1}{p}}T\|\mathfrak{g}\|_\infty}{N^{\frac{b-2}{p}\cdot 2}}\Big\|\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_N)}.\end{aligned}$$

This upper bound converges to zero by assumption (19).

Evolution of the Maximum. In this subsection we will prove that in dimensions $d \geq 3$ the maximum does not change in time. We will show first that the maximum can not decrease and then use this fact to show that whenever the initial states have at least some positive condensing mass it can not increase either. Finally, we use the strong Markov property to remove the assumption that the initial states have some positive condensing mass. As always, $\{\mu_0^N\}$ is a sequence of initial distributions on $\mathbb{M}_N^d$ with finite first order moments and total mass bounded above in probability by $\bar{\rho} < +\infty$ and P^N is the law of a diffusively rescaled symmetric ZRP with bounded jump rate, starting from μ_0^N.

Lemma 4.3. *If $d \geq 3$ then the rescaled maximum process $(N^{-d}M_t^N)_{t\geq 0}$ is asymptotically non-decreasing in the hydrodynamic scale, i.e.*

$$\lim_{N\to+\infty} P^N\Big\{\inf_{0\leq s\leq t\leq T}\frac{M_t^N - M_s^N}{N^d} < -\varepsilon\Big\} = 0 \quad \textit{for all } \varepsilon, T > 0. \tag{23}$$

Proof. By the expression (11) of the martingale A^N

$$P^N\Big\{\inf_{0\leq s\leq t\leq T}\frac{M_t^N - M_s^N}{N^d} < -\varepsilon\Big\} \leq P^N\Big\{\inf_{0\leq s\leq t\leq T}(A_t^N - A_s^N) < -\frac{\varepsilon}{2}\Big\}$$
$$+ P^N\Big\{\inf_{0\leq s\leq t\leq T}\frac{1}{N^{d-2}}\int_s^t L_N M_N(\eta_r)\,\mathrm{d}r < -\frac{\varepsilon}{2}\Big\}.$$

For $d \geq 3$ the first term in the summand above vanishes as $N \to +\infty$ by the asymptotic negligibility of the martingale A^N on any finite time horizon, while the event appearing in the second summand is empty for large enough $N \in \mathbb{N}$, since by (16) we have that $\int_s^t L_N M_N(\eta_r)dr \geq -\|\mathfrak{g}\|_\infty(t-s)$ for all $s, t \in \mathbb{R}_+$. $\square$

As a consequence, if the initial states $\{\mu_0^N\}$ have at least γ condensing mass asymptotically as $N \to +\infty$, in the sense that $\lim_{N\to+\infty}\mu_0^N\big\{\frac{M_N}{N^d} < \gamma - \varepsilon\big\} = 0$ for all $\varepsilon > 0$, then the same is also true in any finite time horizon in the hydrodynamic scale, i.e.

$$\lim_{N\to+\infty} P^N\Big\{\inf_{0\leq t\leq T}\Big(\frac{M_t^N}{N^d} - \gamma\Big) < -\varepsilon\Big\} = 0 \quad \text{for all } \varepsilon, T > 0. \tag{24}$$

Taking into account this fact one can use the elementary inequality $M_N \cdot \sharp K_N \leq |\cdot|_1$ to see that if the ZRP starts from a sequence of initial distributions $\{\mu_0^N\}$ with total mass bounded from above and at least some condensing mass in the maximal coordinate, then the set K_t^N is a.s. finite in probability as $N \to +\infty$. This can in turn be used to show that the rescaled maximum is in fact asymptotically constant in the hydrodynamic scale.

Lemma 4.4. *Let $d \geq 3$. Suppose that the initial states $\{\mu_0^N\}$ have at least $\gamma > 0$ condensing mass in the maximal coordinate and total mass bounded above by $\bar\rho < +\infty$. Then*

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq t\leq T}\sharp K_t^N > \frac{\bar\rho}{\gamma} + \delta\Big\} = 0 \quad \textit{for each } \delta, T > 0. \tag{25}$$

Proof. By the inequality $M_N \cdot \sharp K_N \leq |\cdot|_1$, for every $\delta, T > 0$,

$$B_\delta^N := \Big\{\sup_{0\leq t\leq T}\sharp K_t^N > \frac{\bar\rho}{\gamma} + \delta\Big\} \subseteq \Big\{\sup_{0\leq t\leq T}\frac{|\eta_t|_1}{M_t^N} > \frac{\bar\rho}{\gamma} + \delta\Big\}.$$

We choose $\varepsilon \in (0,\gamma)$, decompose the event B_δ^N according to whether the rescaled maximum has dropped by ε and use the conservation of the total number of particles to obtain

$$P^N(B_\delta^N) \leq P^N\Big\{\sup_{0\leq t\leq T}\frac{|\eta_t|_1}{M_t^N} > \frac{\bar\rho}{\gamma} + \delta,\ \inf_{0\leq t\leq T}\Big(\frac{M_t^N}{N^d} - \gamma\Big) \geq -\varepsilon\Big\}$$
$$+ P^N\Big\{\inf_{0\leq t\leq T}\Big(\frac{M_t^N}{N^d} - \gamma\Big) < -\varepsilon\Big\}$$
$$\leq P^N\Big\{\frac{|\eta_0|_1}{N^d} > \Big(\frac{\bar\rho}{\gamma} + \delta\Big)(\gamma - \varepsilon)\Big\} + P^N\Big\{\inf_{0\leq t\leq T}\Big(\frac{M_t^N}{N^d} - \gamma\Big) < -\varepsilon\Big\}.$$

The second term converges to 0 as $N \to +\infty$ by (24) for any $\varepsilon > 0$, while if we choose $\varepsilon > 0$ small enough then the first term also goes to zero by the assumption that the sequence of initial states has at most $\bar{\rho}$ total mass in probability. □

Using Lemma 4.4 it easy to prove that, if the ZRP has at least $\gamma > 0$ condensing mass in the hydrodynamic scale, then any limit point of the sequence of the laws of the rescaled maximum $(N^{-d}M_t^N)_{t\geq 0}$ is concentrated on constant curves.

Lemma 4.5. *Suppose that the initial distributions $\{\mu_0^N\}$ have at least $\gamma > 0$ condensing mass and $d \geq 3$. Then the rescaled maximum is asymptotically non-increasing in the hydrodynamic scale, that is*

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq s\leq t\leq T} \frac{M_t^N - M_s^N}{N^d} > \varepsilon\Big\} = 0 \quad \textit{for all } \varepsilon, T > 0.$$

Proof. Let $\varepsilon, T > 0$ be given and let F_N, $E_{N,\varepsilon}$ be the events

$$E_{N,\varepsilon} := \Big\{\sup_{0\leq s\leq t\leq T} \frac{M_t^N - M_s^N}{N^d} > \varepsilon\Big\}, \quad F_N := \Big\{\sup_{0\leq t\leq T} \sharp K_t^N \leq \frac{\bar{\rho}+1}{\gamma}\Big\}.$$

Then $P^N(E_{N,\varepsilon}) \leq P^N(E_{N,\varepsilon} \cap F_N) + P^N(F_N^c)$ and the term $P^N(F_N^c)$ converges to zero as $N \to +\infty$ by Lemma 4.4. For the other term, by (11) we have that

$$P^N(E_{N,\varepsilon} \cap F_N) \leq P^N\Big\{\sup_{0\leq s\leq t\leq T} (A_t^N - A_s^N) > \frac{\varepsilon}{2}\Big\}$$
$$+ P^N\Big(F_N \bigcap \Big\{\sup_{0\leq s\leq t\leq T} \frac{1}{N^{d-2}} \int_s^t L_N M_M(\eta_r)\,\mathrm{d}r > \frac{\varepsilon}{2}\Big\}\Big).$$

The first term above converges to 0 by the asymptotic negligibility of the martingale A^N. For the second term, in view of (16) and Lemma 4.4, we note that on the event F_N

$$\frac{1}{N^{d-2}} \int_s^t L_N M_M(\eta_r)\,\mathrm{d}r \leq \frac{\|\mathfrak{g}\|_\infty}{N^{d-2}} \int_s^t \sharp K_r^N dr \leq \frac{\|\mathfrak{g}\|_\infty (t-s)}{N^{d-2}} \frac{\bar{\rho}+1}{\gamma}$$

for all $0 \leq s \leq t \leq T$. This upper bound converges to 0 as $N \to +\infty$, which completes the proof. □

Proposition 4.3. *Suppose that the sequence of initial distributions of the ZRP has at least $\gamma \geq 0$ condensing mass. Then*

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq s,t\leq T} \Big|\frac{M_t^N - M_s^N}{N^d}\Big| > \varepsilon\Big\} = 0 \quad \textit{for all } \varepsilon, T > 0.$$

Proof. Immediate by combining Lemmas 4.3 and 4.5 □

In particular when the maximum has exactly $\gamma > 0$ condensing mass at time zero, then so it does in any finite time horizon in the hydrodynamic scale, i.e.

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\leq t\leq T} \Big|\frac{M_t^N}{N^d} - \gamma\Big| > \varepsilon\Big\} = 0, \quad \forall \varepsilon, T > 0. \tag{26}$$

We remove now the assumption that the initial states have some condensing mass $\gamma > 0$. We will see first that if the sequence of initial distributions has no condensing mass, in the sense that $\lim_{N\to+\infty}\mu_0^N\big\{\frac{M_N}{N^d}>\varepsilon\big\}=0$ for all $\varepsilon>0$, then the same holds at all times in the diffusive time-scale. More generally we will show that if the ZRP has initially at most $\gamma\geq 0$ condensing mass in the sense that

$$\lim_{N\to+\infty}\mu_0^N\Big\{\frac{M_N}{N^d}-\gamma>\varepsilon\Big\}=0,\quad\forall\varepsilon>0 \tag{27}$$

then so it does at all later times. Note that, according to the definitions, the sequence $\{\mu_0^N\}$ has no condensing mass iff it has 0 condensing mass iff it has at most γ condensing mass for all $\gamma>0$.

Proposition 4.4. *Suppose that the sequence of initial distributions of the ZRP has at most $\gamma\geq 0$ condensing mass at the maximal coordinate. Then*

$$\lim_{N\to+\infty}P^N\Big\{\sup_{0\leq t\leq T}\Big(\frac{M_t^N}{N^d}-\gamma\Big)>\varepsilon\Big\}=0,\quad\forall\varepsilon,T>0. \tag{28}$$

Proof. The heuristic idea of the proof is the following. For each $\varepsilon>0$, consider the stopping time $\tau_N^\varepsilon\colon D(\mathbb{R}_+;\mathbb{M}_N^d)\to[0,+\infty]$ given by

$$\tau_N^\varepsilon:=\inf\big\{t\geq 0\,\big|\,M_t^N\geq[(\gamma+\varepsilon)N^d]\big\}.$$

On the stochastic interval $[0,\tau_N^\varepsilon)$, the maximum satisfies $M_t^N<[(\gamma+\varepsilon)N^d]$ while by the right continuity of the paths we have that $M_{\tau_N^\varepsilon}^N=[(\gamma+\varepsilon)N^d]$. But by the strong Markov property from the time τ_N^ε and onwards the maximum behaves as the maximum of a ZRP that starts from a sequence of initial laws that has total mass bounded above by $\bar\rho>0$ and $\gamma+\varepsilon>0$ condensing mass, namely the sequence $\mu_0^{N,\varepsilon}:=(\eta_{\tau_N^\varepsilon})_\sharp P^N$, $N\in\mathbb{N}$. Thus, by (26) the rescaled maximum will have exactly $\gamma+\varepsilon$ condensing mass for all times in the stochastic interval $[\tau_N^\varepsilon,+\infty)$. So in general, at all times it will have at most $\gamma+\varepsilon>0$ condensing mass. Since $\varepsilon>0$ is arbitrary this proves the claim.

We proceed now with the rigorous proof. The rescaled maximum $(N^{-d}M_t^N)_{t\geq 0}$ is asymptotically non-decreasing by Lemma 4.3 and thus in order to prove (28) it suffices to show that

$$\lim_{N\to+\infty}P^N\Big\{\frac{M_T^N}{N^d}>\gamma+\varepsilon\Big\}=0,\quad\forall\varepsilon,T>0. \tag{29}$$

For each $r,T>0$ we set $a_N(r,T):=P^N\{M_T^N/N^d>\gamma+r\}$. Then, in order to prove (29) we can equivalently show that for all $\varepsilon,\delta>0$ any subsequence of $\{a_N(\varepsilon+\delta,T)\}_{N\in\mathbb{N}}$ has a further subsequence which converges to zero. The reason for considering an extra parameter δ is that we allow for the case that $\gamma=0$, in which case we want the maximum to have positive condensing mass equal to ε at the time τ_N^ε, since our aim is to reduce the claim to the limit (26). So let $\{a_N\}_{N\in\mathbb{N}}$ be an arbitrary subsequence (which we continue to denote by the same index) of $\{a_N(\varepsilon+\delta,T)\}_{N\in\mathbb{N}}$. Let $\Omega^{N,\varepsilon}:=(\Omega^{N,\varepsilon},(\mathcal{F}_t^{N,\varepsilon})_{t\geq 0},P^{N,\varepsilon})$ be the filtered probability space of the ZRP conditioned on the event $\{\tau_N^\varepsilon<+\infty\}$, provided that this event is of positive probability. Then, the evaluation $\eta_{\tau_N^\varepsilon}$ of the ZRP at the stopping time τ_N^ε is a well-defined random variable on $\Omega^{N,\varepsilon}$, with law $\mu_0^{N,\varepsilon}:=\eta_{\tau_N^\varepsilon\sharp}P^{N,\varepsilon}$ on $\mathbb{M}_N^d$. By the strong Markov property the process $\eta^{N,\varepsilon}:=(\eta_{\tau_N^\varepsilon+t})_{t\geq 0}$ is a ZRP on $\Omega^{N,\varepsilon}$ starting from $\mu_0^{N,\varepsilon}$. If $\liminf_{N\to+\infty}P^N\{\tau_N^\varepsilon<+\infty\}=0$ then there exists a subsequence of $\{a_{k_N}\}$ of $\{a_N\}$ along which the limit inferior is achieved. Since $M_t^N<[(\gamma+\varepsilon)N^d]$ for all $t\geq 0$ on the event $\{\tau_N^\varepsilon=+\infty\}$, we then have that

$$a_{k_N}=P^{k_N}\Big\{\frac{M_T^{k_N}}{k_N^d}>\gamma+\varepsilon+\delta\Big\}\leq P^{k_N}\{\tau_{k_N}^\varepsilon<+\infty\}\overset{N\to+\infty}{\longrightarrow}0,$$

and thus in this case $\{a_{k_N}\}$ is the required further subsequence that converges to zero. If on the other hand $\liminf_{N\to+\infty} P^N\{\tau_N^\varepsilon < +\infty\} > 0$, then the conditioned probability space $\Omega^{N,\varepsilon}$ is well-defined for large enough N. Since $M_T^N < [(\gamma+\varepsilon)N^d]$ on the event $\{\tau_N^\varepsilon > T\}$, if we set here $M^{N,\varepsilon} := (M^N_{\tau_N^\varepsilon+t})_{t\geq 0}$, then for all N large enough

$$\begin{aligned} a_N &\leq P^{N,\varepsilon}\Big(\Big\{\frac{M^N_{\tau_N^\varepsilon+(T-\tau_N^\varepsilon)}}{N^d} > \gamma+\varepsilon+\delta\Big\} \cap \{\tau_N^\varepsilon \leq T\}\Big) \\ &\leq P^{N,\varepsilon}\Big\{\sup_{0\leq t\leq T}\frac{M_t^{N,\varepsilon}}{N^d} > \gamma+\varepsilon+\delta\Big\}. \end{aligned}$$

Therefore, it suffices to show that

$$\lim_{N\to+\infty} P^{N,\varepsilon}\Big\{\sup_{0\leq t\leq T}\Big|\frac{M_t^{N,\varepsilon}}{N^d} - (\gamma+\varepsilon)\Big| > \delta\Big\} = 0. \tag{30}$$

The process $M^{N,\varepsilon}$ is the maximum process of the ZRP $\eta^{N,\varepsilon}$. The sequence $\{\mu_0^{N,\varepsilon}\}_{N\in\mathbb{N}}$ of its initial distributions has total mass bounded by $\bar\rho > 0$ and finite first order moments, as is immediate from the conservation of the number of particles and the fact that the sequence $\{\mu_0^N\}$ of the initial states of the original ZRP satisfies these properties. Thus, if we can show that $\{\mu_0^{N,\varepsilon}\}$ has exactly $\gamma+\varepsilon > 0$ condensing mass then the limit in (30) follows by (26). But this follows from the definition of the stopping time τ_N^ε and the assumption that the initial state $\{\mu_0^N\}$ has at most γ condensing mass. Indeed, $M^N_{\tau_N^\varepsilon} = [(\gamma+\varepsilon)N^d]$ on the event $\{0 < \tau_N^\varepsilon < +\infty\}$, and therefore for any $\delta > 0$ the event

$$\Big\{\Big|\frac{M^N_{\tau_N^\varepsilon}}{N^d} - (\gamma+\varepsilon)\Big| > \delta\Big\} \cap \{0 < \tau_N^\varepsilon < +\infty\}$$

is empty for all sufficiently large $N \in \mathbb{N}$. Therefore, for large N

$$\mu_0^{N,\varepsilon}\Big\{\Big|\frac{M_N}{N^d} - (\gamma+\varepsilon)\Big| > \delta\Big\} = P^{N,\varepsilon}\Big\{\Big|\frac{M^N_{\tau_N^\varepsilon}}{N^d} - (\gamma+\varepsilon)\Big| > \delta\Big\} \leq P^{N,\varepsilon}\{\tau_N^\varepsilon = 0\}.$$

But by the right continuity of the paths of the ZRP, $\{\tau_N^\varepsilon = 0\} \subseteq \{M_0^N \geq [(\gamma+\varepsilon)N^d]\}$ and therefore the assumption that there is at most γ condensing mass at the initial state implies that $\lim_{N\to+\infty} P^N\{\tau_N^\varepsilon = 0\} = 0$ for all $\varepsilon > 0$. This proves that the sequence of distributions $\{\mu_0^{N,\varepsilon}\}_{N\in\mathbb{N}}$ has exactly $\gamma+\varepsilon > 0$ condensing mass. □

We complete now the proof of (2) by removing all additional assumptions on the initial condensing mass. We fix arbitrary $\varepsilon, T > 0$ and set

$$B^N_{\varepsilon,T} := \Big\{\sup_{0\leq s,t\leq T}\Big|\frac{M_t^N - M_s^N}{N^d}\Big| > \varepsilon\Big\}, \quad N \in \mathbb{N}.$$

It suffices to show that any subsequence of $\{P^N(B^N_{\varepsilon,T})\}_{N\in\mathbb{N}}$, which we continue to denote by the same subscript, has a further subsequence that converges to zero. For each $\delta > 0$ we define the event $I_\delta^N := \{M_N \leq [\delta N^d]\}$. If there exists $\delta > 0$ such that $\liminf_{N\to+\infty}\mu_0^N(I_\delta^N) = 0$, then we can choose a subsequence $\{\mu_0^{k_N}(I_\delta^{k_N})\}$ of $\{\mu_0^N(I_\delta^N)\}$ (which may depend on $\delta > 0$) that achieves the limit inferior. Along this subsequence

$$\mu_0^{k_N}\Big\{\frac{M_{k_N}}{k_N^d} < \delta\Big\} \leq \mu_0^{k_N}\{M_{k_N} < [\delta k_N^d] + 1\} = \mu_0^{k_N}(I_\delta^{k_N}) \overset{N\to+\infty}{\longrightarrow} 0$$

which shows that $\{\mu_0^{k_N}\}_{N\in\mathbb{N}}$ has at least $\delta > 0$ condensing mass. Therefore $\{a_{k_N}(\varepsilon, T)\}$ is the required subsequence by Proposition 4.3. So in what follows we can assume that $p_\delta := \liminf_{N\to+\infty} \mu_0^N(I_\delta^N) > 0$ for all $\delta > 0$. Now, if there exist numbers $\delta > 0$ arbitrarily close to 0 such that $p_\delta = 1$, then the sequence $\{\mu_0^N\}$ has no condensing mass and so the claim also holds in this case by Proposition 4.4 applied with $\gamma = 0$. Thus we can assume that there exists $\delta_1 > 0$ such that $p_\delta \in (0,1)$ for all $\delta \in (0, \delta_1)$. Then for any $0 < \delta < \delta_1 \wedge \frac{\varepsilon}{2}$ we choose a subsequence $\{\mu_0^{k_N}(I_\delta^{k_N})\}$ of $\{\mu_0^N(I_\delta^N)\}$, possibly depending on $\delta > 0$, such that $\mu_0^{k_N}(I_\delta^{k_N}) \in (0,1)$ for all $N \in \mathbb{N}$. Then the restricted probability measures $P^{k_N}(\cdot|\eta_0 \in I_\delta^{k_N})$, $P^{k_N}(\cdot|\eta_0 \notin I_\delta^{k_N})$ are well-defined for all $N \in \mathbb{N}$. Restricting a Markov process $(X_t)_{t\geq 0}$, which starts from a law μ, to start on a particular set of positive probability, i.e. on a set of the form $\{X_0 \in A\}$ for some measurable subset A of its state space with $\mu(A) > 0$, is the same as starting the process from the restricted law $\mu(\cdot|A)$, i.e. $P^\mu(\cdot|X_0 \in A) = P^{\mu(\cdot|A)}(\cdot)$. Therefore

$$P^{k_N}(B_{\varepsilon,T}^{k_N}) \leq P^{\mu_0^{k_N}(\cdot|I_\delta^{k_N})}(B_{\varepsilon,T}^{k_N}) + P^{\mu_0^{k_N}(\cdot|(I_\delta^{k_N})^c)}(B_{\varepsilon,T}^{k_N}).$$

The second term on the right hand side above is the probability of the event $B_{\varepsilon,T}^{k_N}$ with respect to the law of the ZRP starting from a sequence with at least $\delta > 0$ condensing mass, namely the sequence $\{\mu_0^{k_N}(\cdot|(I_\delta^{k_N})^c)\}$, and thus it converges to zero as $N \to +\infty$ by Proposition 4.3. The first term is bounded above by

$$P^{\mu_0^{k_N}(\cdot|I_\delta^{k_N})}(B_{\varepsilon,T}^{k_N}) \leq 2P^{\mu_0^{k_N}(\cdot|I_\delta^{k_N})}\Big\{\sup_{0\leq t\leq T}\Big(\frac{M_t^{k_N}}{k_N^d} - \delta\Big) > \frac{\varepsilon}{2} - \delta\Big\}.$$

Since the sequence $\{\mu_0^{k_N}(\cdot|I_\delta^{k_N})\}$ is a sequence of initial states with at most δ condensing mass and we have chosen $\delta < \varepsilon/2$, it follows by Proposition 4.4 that the right hand side above converges to zero. Thus the sequence $\{P^{k_N}(B_{\varepsilon,T}^{k_N})\}$ is the required subsequence converging to zero and the proof of (2) is complete.

Relative compactness of the rescaled maximum process. We prove in this section that the sequence $\{Q^N\} := \{((N^{-d}M_t^N)_{t\geq 0})_\sharp P^N\}_{N\in\mathbb{N}}$ of the laws of the rescaled maximum processes is relatively compact in the Skorokhod space $D(0,T;\mathbb{R}_+)$. According to standard criteria for compactness in Skorokhod spaces ([11, Theorem 4.1.3 and Remark 4.1.4]) it suffices to prove the tightness of $\{(N^{-d}M_t^N)_\sharp P^N\}_{N\in\mathbb{N}}$ at each fixed time $t \geq 0$ and then control the ordinary modulus of continuity,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} P^N\Big\{\sup_{0\leq s,t\leq T:|t-s|\leq\delta}\Big|\frac{M_t^N - M_s^N}{N^d}\Big| > \varepsilon\Big\} = 0, \quad \forall \varepsilon > 0. \tag{31}$$

Controlling the ordinary modulus of continuity implies in addition that all limit points are concentrated on continuous curves ([11, Remark 4.1.4]). This does not yield any additional information in dimensions $d \geq 3$, since we have proved that the maximum is asymptotically constant in the hydrodynamic scale, but it does so in dimension $d = 2$.

First, since we assume that the ZRP starts from initial states with at most $\bar\rho$ mass and the total number of particles is conserved, we have that

$$\limsup_{N\to+\infty} P^N\Big\{\sup_{t\geq 0}\frac{M_t^N}{N^d} > r\Big\} = 0, \quad \forall r \geq \bar\rho, \tag{32}$$

which is stronger than the pointwise in time tightness. In the case that $d \geq 3$, since we have proved that the maximum is asymptotically constant in the hydrodynamic scale, the limit (31) follows immediately by (2), which was proved in the previous section.

We prove now Remark 2.1 regarding the relative compactness of the rescaled maximum of the Beltrán-Landim model with $b > 3$ in dimension $d = 2$. In this case we can still control the ordinary modulus of continuity. By (11), and in view of Remark 4.1 which states that the martingale A^N is asymptotically negligible in dimension $d = 2$, it suffices to show that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} P^N\Big\{\sup_{0\le s,t\le T:|t-s|\le\delta}\Big|\int_s^t L_N M_N(\eta_r)\,\mathrm{d}r\Big| > \varepsilon\Big\} = 0, \quad \forall\varepsilon > 0. \tag{33}$$

By (15), for each $0 \le s, t \le T$ with $|t-s| \le \delta$ we have

$$\begin{aligned}\Big|\int_s^t L_N M_N(\eta_r)\,\mathrm{d}r\Big| &\le 2\|\mathfrak{g}\|_\infty \int_s^t \mathbb{1}_{\{\sharp K_r^N = 1\}}\,\mathrm{d}r + \|\mathfrak{g}\|_\infty \int_s^t \sharp K_N \mathbb{1}_{\{\sharp K_r^N \ge 2\}}\,\mathrm{d}r \\ &\le 2\|\mathfrak{g}\|_\infty\delta + \|\mathfrak{g}\|_\infty N^2 \int_0^T \mathbb{1}_{\{\sharp K_r^N \ge 2\}}\,\mathrm{d}r.\end{aligned}$$

Therefore, by Lemma 4.2 there exists a constant $C < +\infty$ such that for any pair of conjugate exponents $p \in [1, +\infty)$, $q \in (1, +\infty]$,

$$\mathbb{E}^N\Big(\sup_{0\le s,t\le T:|t-s|\le\delta}\Big|\int_s^t L_N M_N(\eta_r)\,\mathrm{d}r\Big|\Big) \le 2\|\mathfrak{g}\|_\infty\delta + \frac{C^{\frac{1}{p}}T}{N^{\frac{b-2}{p}2-2}}\Big\|\frac{\mathrm{d}\mu_0^N}{\mathrm{d}\nu_N}\Big\|_{L^q(\nu_N)}.$$

By assumption (4) the right hand side above converges to 0 as $N \to \infty$ and then $\delta \to 0$ and thus the limit in (33) follows by Chebyshev's inequality.

Weak convergence of the rescaled maximum. Having proved the relative compactness of the sequence $\{Q^N\}_{N\in\mathbb{N}}$ of the laws of the rescaled maximum of a diffusively rescaled symmetric ZRP, it remains to identify its limit points. So let Q^* be a limit point of $\{Q^N\}$ and let $M_t\colon D(\mathbb{R}_+;\mathbb{R}) \to \mathbb{R}$, $t \ge 0$, be the evaluation maps. The map M_t is continuous at each càdlàg path x that is continuous at t. Since $\{Q^N\}$ converges (modulo a subsequence) to Q^*, we know by [5, Theorem 7.8] that for each $0 \le s < t$ there exist sequences $\{s_N\} \subseteq [s, +\infty)$ and $\{t_N\} \subseteq [t, +\infty)$ converging to s and t respectively, such that $\{(M_{s_N}, M_{t_N})_\sharp Q^N\}$ converges weakly to $(M_s, M_t)_\sharp Q^*$. For any $\varepsilon > 0$ the set $U_\varepsilon := \big\{(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \;\big|\; |t-s| > \varepsilon\big\}$ is open. Therefore, if $T > \max\{s, t\}$ is such that $[0, T]$ contains the sequences $\{s_N\}$ and $\{t_N\}$, then by the Portmanteau theorem and (2),

$$\begin{aligned}Q^*\{|M_t - M_s| > \varepsilon\} &\le \liminf_{N\to+\infty}(M_{s_N}, M_{t_N})_\sharp Q^N(U_\varepsilon) = \liminf_{N\to+\infty} P^N\Big\{\Big|\frac{M_{t_N}^N - M_{s_N}^N}{N^d}\Big| > \varepsilon\Big\} \\ &\le \lim_{N\to+\infty} P^N\Big\{\sup_{0\le s,t\le T}\Big|\frac{M_t^N - M_s^N}{N^d}\Big| > \varepsilon\Big\} = 0.\end{aligned}$$

We assume finally that the sequence $\{(N^{-d}M_N)_\sharp\mu_0^N\}$ converges to a measure m_0 on $\mathbb{R}_+$ and we will to show that the arbitrary limit point Q^* of $\{Q^N\}$ is equal to the measure Q_{m_0}. By what we have shown so far, $Q^*\{|M_t - M_0| = 0\} = 1$ for all $t \ge 0$. Consequently for any $0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$,

$$Q^*(M_0 = M_{t_1} = \ldots = M_{t_n}) = Q^*\Big(\bigcap_{i=1}^n\{|M_{t_i} - M_0| = 0\}\Big) = 1.$$

By the right continuity of paths in $D(0,T;\mathbb{R})$ the evaluation map M_0 is continuous with respect to the Skorokhod topology. Therefore, for any limit point Q^* of $\{Q^N\}$ we have that $M_{0\sharp}Q^* = m_0$ and so for any Borel sets $A_1,\dots,A_n \subseteq \mathbb{R}_+$

$$Q^*\{M_{t_1} \in A_1,\dots,M_{t_n} \in A_n\} = Q^*\{M_0 \in A_1 \cap \dots \cap A_n\} = m_0(A_1 \cap \dots \cap A_n).$$

But this is the measure $\mathfrak{c}_\sharp m_0$ since $\mathfrak{c}^{-1}(\{M_{t_1} \in A_1,\dots M_{t_n} \in A_n\}) = A_1 \cap \dots \cap A_n$. □

4.2. **On the size of the strict second maximum.** In the first subsection we introduce an appropriate martingale associated to the rescaled strict second maximum and prove its asymptotic negligibility. In the second subsection we prove Theorem 2.2 and in the last one we prove Corollary 2.1.

A stopped martingale associated to the second maximum. The martingale associated to the rescaled strict second maximum $(N^{-d}M_t^{2,N})_{t\geq 0}$ via the martingale problem is the process

$$A^{2,N} := \frac{M_t^{2,N}}{N^d} - \frac{M_0^{2,N}}{N^d} - \frac{1}{N^{d-2}}\int_0^t L_N M_{2,N}(\eta_s)\,\mathrm{d}s, \quad t \geq 0.$$

In Theorem 2.2 we assume that the ZRP starts from a sequence of initial states for which the rescaled maximum is attained on a singleton in probability as $N \to +\infty$, in the sense of (7), and differs from the rescaled strict second maximum by a positive amount δ_0, as in (6). In our approach we will stop the ZRP prior to the first time this ceases to be true, at an appropriate time τ_N. We will first obtain the required results up to this time τ_N and we will then extend them to all times by showing that the time τ_N converges in probability to $+\infty$, in the sense that $\lim_{N\to+\infty} P^N(\tau_N \leq T) = 0$ for every $T > 0$. The time τ_N is defined as the exit time from the event

$$E_N := \{M_{2,N} \leq M_N - n\} \cap \{\sharp K_N = 1\} \tag{34}$$

for some fixed $n \geq 3$.

In this approach, we need to obtain an explicit formula for the martingale associated to the rescaled strict second maximum only up to the stopping time τ_N and thus it suffices to compute $L_N M_{2,N}$ on the event E_N. The advantage is that on the event E_N the formula of $M_{2,N}^{x,y}(\eta) := M_{2,N}(\eta^{x,y})$ is simplified considerably compared to the general case: for jumps from $x \in \mathbb{T}_N^d$ to $y \in K_{2,N}$ we have $M_{2,N}^{x,y} = M_{2,N} + 1$, for jumps from $x \notin K_{2,N}$ to $y \notin K_{2,N}$ we have $M_{2,N}^{x,y} = M_{2,N}$, while, finally, for jumps from $x \in K_{2,N}$ to $y \notin K_{2,N}$,

$$M_{2,N}^{x,y} = \begin{cases} M_{2,N} - 1 & \text{if } \sharp K_{2,N} = 1 \text{ and } \eta(y) \neq M_{2,N} - 1 \\ M_{2,N} & \text{otherwise} \end{cases}.$$

Therefore, on the event E_N we have that

$$L_N M_{2,N} = \sum_{\substack{x\in\mathbb{T}_N^d \\ y \in K_{2,N}}} \mathfrak{g}(\eta(x))p(y-x) - \mathbb{1}_{\{\sharp K_{2,N}=1\}}\mathfrak{g}(M_{2,N}) \sum_{\eta(y)\neq M_{2,N}-1} p(y - x_{2,N}).$$

Note that this formula of $L_N M_{2,N}$ on the event E_N is similar to the one for $L_N M_N$. Thus, if we set $\overline{L_N}M_{2,N} := \mathbb{1}_{E_N} L_N M_{2,N}$, we have the bounds

$$-\|\mathfrak{g}\|_\infty \mathbb{1}_{E_N} \leq \overline{L_N}M_{2,N} \leq \|\mathfrak{g}\|_\infty N^d \mathbb{1}_{E_N}. \tag{35}$$

Consequently, similar reasoning to that used in the study of M_N will apply to $M_{2,N}$ as long as the ZRP stays in the event E_N defined in (34).

Lemma 4.6. *Let τ_N be the exit time of the ZRP from the set E_N defined in* (34). *If the initial distributions $\{\mu_0^N\}$ of the ZRP have finite first order moments then the stopped process $(A^{2,N})^{\tau_N}$ defined by*

$$(A^{2,N})_t^{\tau_N} := A_{t\wedge\tau_N}^{2,N} = \frac{M_{t\wedge\tau_N}^{2,N}}{N^d} - \frac{M_0^{2,N}}{N^d} - \frac{1}{N^{d-2}} \int_0^{t\wedge\tau_N} \overline{L_N} M_{2,N}(\eta_s)\,\mathrm{d}s$$

is a square integrable martingale with quadratic variation

$$\langle A^{2,N}\rangle_{t\wedge\tau_N} = \frac{1}{N^{2d-2}} \int_0^{t\wedge\tau_N} \sum_{x,y\in\mathbb{T}_N^d} (M_{2,N}(\eta_s^{x,y}) - M_{2,N}(\eta_s))^2 \mathfrak{g}(\eta_s(x))p(y-x)\,\mathrm{d}s$$

and it is asymptotically negligible.

Proof. The square integrability of the stopped martingale $(A^{2,N})^{\tau_N}$ and the expression of its quadratic variation are obtained by the dominated convergence theorem similarly to the case of the martingale A^N associated to the rescaled maximum process. Since $(M_{2,N}(\eta^{x,y}) - M_{2,N}(\eta))^2 \in \{0,1\}$ for η in the event E_N, the quadratic variation satisfies $\langle A^{2,N}\rangle_{T\wedge\tau_N} \leq \frac{\|\mathfrak{g}\|_\infty T}{N^{d-2}}$. Thus, since $d \geq 3$, the asymptotic negligibility of the stopped martingale $(A^{2,N})^{\tau_N}$ follows by the Chebyshev-Doob inequality as in Proposition 4.2 □

Evolution of the second maximum. We prove in this section that under the assumptions (5) and (6) the rescaled strict second maximum $(N^{-d}M_t^{2,N})_{t\geq 0}$ is asymptotically constant in the hydrodynamic scale in dimensions $d \geq 3$. As a preliminary step we show first that the rescaled second maximum remains asymptotically constant in the hydrodynamic scale up to the exit time τ_N.

Lemma 4.7. *Assume that the initial distributions $\{\mu_0^N\}$ of the ZRP have finite first order moments and total mass bounded above by $\bar\rho < +\infty$ in the sense of* (1) *and let τ_N be the exit time from the set E_N defined in* (34). *Then, the rescaled strict second maximum is asymptotically non-decreasing in the hydrodynamic scale up to the time τ_N, i.e.*

$$\lim_{N\to+\infty} \Big\{ \inf_{0\leq s\leq t\leq T} \frac{M_{t\wedge\tau_N}^{2,N} - M_{s\wedge\tau_N}^{2,N}}{N^d} < -\varepsilon \Big\} = 0, \quad \forall \varepsilon, T > 0. \tag{36}$$

If the rescaled strict second maximum has initially at least some condensing mass $\gamma_2 > 0$, in the sense that

$$\lim_{N\to+\infty} \mu_0^N \Big\{ \frac{M_{2,N}}{N^d} < \gamma_2 - \varepsilon \Big\} = 0, \quad \forall \varepsilon > 0, \tag{37}$$

then the rescaled strict second maximum is asymptotically constant in the hydrodynamic scale up to the time τ_N, i.e.

$$\lim_{N\to+\infty} \Big\{ \sup_{0\leq s\leq t\leq T} \Big| \frac{M_{t\wedge\tau_N}^{2,N} - M_{s\wedge\tau_N}^{2,N}}{N^d} \Big| > \varepsilon \Big\} = 0, \quad \forall \varepsilon, T > 0. \tag{38}$$

Proof. The stopped martingale $(A^{2,N})^{\tau_N}$ is asymptotically negligible by Lemma 4.6. So, as in the case of the maximum, it follows by (35) that $(N^{-d}M_t^{2,N})_{t\geq 0}^{\tau_N}$ is non-decreasing in the hydrodynamic scale as $N \to +\infty$. Indeed, by the definition of the martingale $(A^{2,N})^{\tau_N}$,

$$P^N\Big\{\inf_{0\le s\le t\le T}\frac{M^{2,N}_{t\wedge\tau_N}-M^{2,N}_{s\wedge\tau_N}}{N^d}<-\varepsilon\Big\}\le P^N\Big\{\inf_{0\le s\le t\le T}(A^{2,N}_{t\wedge\tau_N}-A^{2,N}_{s\wedge\tau_N})<-\frac{\varepsilon}{2}\Big\}$$
$$+P^N\Big\{\inf_{0\le s\le t\le T}\frac{1}{N^{d-2}}\int_{s\wedge\tau_N}^{t\wedge\tau_N}\overline{L_N}M_{2,N}(\eta_r)\,\mathrm{d}r<-\frac{\varepsilon}{2}\Big\}.$$

The first term in the right hand side vanishes as $N \to +\infty$ in dimensions $d \ge 3$ by the asymptotic negligibility of the stopped martingale $(A^{2,N})^{\tau_N}$. By (35) and Chebyshev's inequality the second term is bounded above by

$$P^N\Big\{\sup_{0\le s\le t\le T}\frac{\|\mathfrak{g}\|_\infty(t\wedge\tau_N-s\wedge\tau_N)}{N^{d-2}}>\frac{\varepsilon}{2}\Big\}\le\frac{2\|\mathfrak{g}\|_\infty T}{\varepsilon N^{d-2}},$$

which converges to 0 as $N \to +\infty$ for $d \ge 3$. This proves (36). Thus, as in the case of the maximum, if $N^{-d}M^{2,N}$ starts with some condensing mass $\gamma_2 > 0$ at time $t = 0$, then the stopped process $(N^{-d}M^{2,N})^{\tau_N}$ has at least γ_2 condensing mass in the hydrodynamic scale in any finite time horizon:

$$\lim_{N\to+\infty}P^N\Big\{\inf_{0\le t\le T}\Big(\frac{M^{2,N}_{t\wedge\tau_N}}{N^d}-\gamma_2\Big)<-\varepsilon\Big\}=0. \tag{39}$$

In view of the elementary inequality $M^{2,N}_t\cdot\sharp K^{2,N}_t\le|\eta_0|_1$ this yields the upper bound $\frac{\bar\rho}{\gamma_2}$ on the cardinality of $K_{2,N}$ up to the time τ_N in the hydrodynamic scale:

$$\lim_{N\to+\infty}P^N\Big\{\sup_{0\le t\le T}\sharp K^{2,N}_{t\wedge\tau_N}>\frac{\bar\rho}{\gamma_2}+\delta\Big\}=0,\quad\forall\delta,T>0.$$

The proof is entirely similar to that of Lemma 4.4. Then, again as in the case of the maximum, this in turn shows that the paths of $(N^{-d}M^{2,N}_t)^{\tau_N}_{t\ge0}$ must also be non-increasing in the hydrodynamic scale up to the time τ_N, i.e.

$$\lim_{N\to+\infty}P^N\Big\{\sup_{0\le s\le t\le T}\frac{M^{2,N}_{t\wedge\tau_N}-M^{2,N}_{s\wedge\tau_N}}{N^d}>\varepsilon\Big\}=0,\quad\forall\varepsilon,T>0.$$

Combining this with (36) we obtain (38). $\square$

Lemma 4.8. *Assume that the initial distributions $\{\mu^N_0\}$ of the ZRP have first order moments, total mass bounded above by $\bar\rho<+\infty$ and satisfy assumptions* (5) *and* (6).
(a) If the strict second maximum has initially at least some condensing mass $\gamma_2>0$ in the sense of (37)*, then the rescaled strict second maximum is asymptotically constant in the hydrodynamic scale i.e.* (8) *holds.*
(b) Let δ_0 be the number in (6)*. If the second maximal coordinate has initially at most $\gamma_2\in[0,\delta_0/4)$ condensing mass, in the sense that*

$$\lim_{N\to+\infty}\mu^N_0\Big\{\frac{M_{2,N}}{N^d}-\gamma_2>\varepsilon\Big\}=0,\quad\forall\varepsilon>0,$$

then the same is true in the hydrodynamic scale in any finite time horizon:

$$\lim_{N\to+\infty}P^N\Big\{\sup_{0\le t\le T}\Big(\frac{M^{2,N}_t}{N^d}-\gamma_2\Big)>\varepsilon\Big\}=0,\quad\forall\varepsilon,T>0. \tag{40}$$

Proof. For the number $n \geq 3$ appearing in the set defined in (34) we consider the stopping time

$$\sigma_N = \inf\big\{t \geq 0 \;\big|\; M_t^{2,N} \geq M_t^N - (n-1)\big\}.$$

Whenever the process starts from a configuration in which the maximum is achieved on a singleton and the second maximum has at least n fewer particles than the maximum then in order for the maximizing set to stop being a singleton the difference $M_N - M_{2,N}$ must first become less or equal to $n-1$. Thus $\tau_N = \sigma_N$ on the event $\{\tau_N > 0\}$ and it is immediate from the assumptions (5) and (6) on the initial data that in both cases (a) and (b) it holds that $\lim_{N\to+\infty} P^N\{\tau_N > 0\} = 1$.

(a) We will show first that Lemma 4.7 implies that $\lim_{N\to+\infty} P^N\{\tau_N \leq T\} = 0$ for every $T > 0$. Indeed, let $T > 0$. We fix $\varepsilon \in (0, \frac{\delta_0}{2})$, where δ_0 is the lower bound of the initial distance between M_N and $M_{2,N}$ in (6), and define the event

$$F_{2,N,\varepsilon} := \Big\{ \sup_{0\leq t\leq T} \frac{|M_{t\wedge\tau_N}^{2,N} - M_0^{2,N}|}{N^d} \leq \varepsilon,\; \sup_{0\leq t\leq T} \frac{|M_t^N - M_0^N|}{N^d} \leq \varepsilon \Big\}.$$

By (38) and (2) this is an asymptotically sure event. The event $G_N := \{M_0^N - M_0^{2,N} \geq \delta_0 N^d\}$ is also asymptotically sure by assumption (6) and on the event $F_{2,N,\varepsilon} \cap G_N$

$$M_t^{2,N} \leq M_0^{2,N} + \varepsilon N^d \leq M_0^N + (\varepsilon - \delta_0)N^d \leq M_t^N + (2\varepsilon - \delta_0)N^d, \quad \forall\, 0 \leq t \leq \tau_N \wedge T.$$

Since on the event $\{\tau_N > 0\}$ we have that $\tau_N = \sigma_N$, it follows by the definition of the stopping time σ_N that on the event

$$F_{2,N,\varepsilon} \cap G_N \cap \{0 < \tau_N \leq T\} = F_{2,N,\varepsilon} \cap G_N \cap \{\tau_N > 0\} \cap \{\sigma_N \leq T\} \tag{41}$$

we have

$$M_{\sigma_N}^{2,N} \leq M_{\sigma_N}^N + (2\varepsilon - \delta_0)N^d = M_{\sigma_N}^{2,N} + n - 1 + (2\varepsilon - \delta_0)N^d. \tag{42}$$

But since we have chosen $\varepsilon < \delta_0/2$ inequality (42) can not hold if $N \in \mathbb{N}$ is large enough. Thus the event in (41) must be empty for large enough N and so

$$\{\tau_N \leq T\} \subseteq \big(F_{2,N,\varepsilon} \cap G_N \cap \{\tau_N > 0\}\big)^c.$$

Therefore

$$\lim_{N\to+\infty} P^N\{\tau_N \leq T\} \leq \lim_{N\to+\infty} \big(P^N(F_{2,N,\varepsilon}^c) + P^N(G_N^c) + P^N(\{\tau_N = 0\})\big) = 0.$$

Thus, for given $\varepsilon, T > 0$, we can take the limit as $N \to +\infty$ in the inequality

$$P^N\Big\{ \sup_{0\leq t\leq T} \frac{|M_t^{2,N} - M_0^{2,N}|}{N^d} > \varepsilon \Big\} \leq P^N\Big\{ \sup_{0\leq t\leq T} \frac{|M_{t\wedge\tau_N}^{2,N} - M_0^{2,N}|}{N^d} > \varepsilon \Big\} + P^N\{\tau_N \leq T\}$$

to obtain

$$\lim_{N\to+\infty} P^N\Big\{ \sup_{0\leq t\leq T} \frac{|M_t^{2,N} - M_0^{2,N}|}{N^d} > \varepsilon \Big\},$$

which is equivalent to (8).

(b) We show first that if $(M_t^{2,N})_{t\geq 0}$ starts with at most $\gamma_2 \in [0, \delta_0/4)$ condensing mass then it will have at most γ_2 condensing mass up to the time τ_N in the hydrodynamic scale. The proof is similar to the one in Proposition 4.4, but, in order to apply part (a), an additional argument is needed to show that (for some sufficiently small $\varepsilon > 0$) the laws of the ZRP at the stopping times $\tau_{2,N}^\varepsilon := \inf\{t \geq 0 | M_t^{2,N} \geq [(\gamma_2 + \varepsilon)N^d]\}$ satisfy the assumptions (5) and (6).

Note that $\lim_{N\to+\infty} P^N\{\tau^\varepsilon_{2,N}=0\}=0$ for all $\varepsilon>0$ by the assumption that $M_{2,N}$ has at most γ_2 condensing mass at time $t=0$. Our first goal is to prove that

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\le t\le T}\frac{M^{2,N}_{t\wedge\tau_N}}{N^d}>\gamma_2+\varepsilon\Big\}=0. \tag{43}$$

Since $(M^{2,N}_t)^{\tau_N}_{t\ge0}$ is asymptotically non-decreasing by Lemma 4.7, in order to show (43) it suffices to show that

$$\lim_{N\to+\infty} a_{2,N}(\varepsilon+\delta,T)=0,\quad \forall\varepsilon<\delta_0/2 \text{ and } \delta,T>0. \tag{44}$$

Here for each $N\in\mathbb{N}$ and $r,T>0$ we have set $a_{2,N}(r,T):=P^N\big\{M^{2,N}_{\tau_N\wedge T}/N^d>\gamma_2+r\big\}$. To prove (44) let $\varepsilon<\delta_0/2$ and $\delta,T>0$. We will show that any subsequence of $\{a_{2,N}\}\equiv\{a_{2,N}(\varepsilon+\delta,T)\}$ has a further subsequence which converges to zero. So let $\{a_{2,N}\}_{N\in\mathbb{N}}$ denote an arbitrary subsequence (which we continue to denote by the same index). We set $\widetilde{\tau}^\varepsilon_{2,N}:=\tau^\varepsilon_{2,N}\wedge\tau_N$. First, if there exists an increasing sequence $\{k_N\}_{N\in\mathbb{N}}$ such that $\lim_{N\to+\infty}P^{k_N}\{\widetilde{\tau}^\varepsilon_{2,k_N}<+\infty\}=0$, then, since

$$\Big\{\frac{M^{2,N}_{T\wedge\tau_N}}{N^d}>\gamma_2+\varepsilon+\delta\Big\}\cap\{\widetilde{\tau}^\varepsilon_{2,N}=+\infty\}=\emptyset,$$

we have for the corresponding subsequence $\{a_{2,k_N}\}$ of $\{a_{2,N}\}$ that

$$a_{2,k_N}=P^{k_N}\Big(\Big\{\frac{M^{2,k_N}_{T\wedge\tau_{k_N}}}{k_N^d}>\gamma_2+\varepsilon+\delta\Big\}\cap\{\widetilde{\tau}^\varepsilon_{2,k_N}<+\infty\}\Big)\le P^{k_N}\big\{\widetilde{\tau}^\varepsilon_{2,k_N}<+\infty\big\}.$$

Thus in this case $\{a_{2,k_N}\}$ is the further subsequence converging to zero. So in what follows we assume that $\liminf_{N\to+\infty}P^N\{\widetilde{\tau}^\varepsilon_{2,N}<+\infty\}>0$. Then the filtered space $\Omega^{2,N,\varepsilon}=(\Omega^{2,N,\varepsilon},(\mathcal{F}^{2,N,\varepsilon}_t)_{t\ge0},P^{2,N,\varepsilon})$ resulting by conditioning the canonical space Ω^N of the ZRP on the event $\{\widetilde{\tau}^\varepsilon_{2,N}<+\infty\}$ is well-defined for large N. Since $\sup_{0\le t\le T}M^{2,N}_t\le[(\gamma_2+\varepsilon)N^d]$ on the event $\{\widetilde{\tau}^\varepsilon_{2,N}>T\}$

$$\begin{aligned}a_{2,N}&=P^N\Big(\Big\{\frac{M^{2,N}_{\widetilde{\tau}^\varepsilon_{2,N}+(T\wedge\tau_N-\widetilde{\tau}^\varepsilon_{2,N})}}{N^d}>\gamma_2+\varepsilon+\delta\Big\}\cap\{\widetilde{\tau}^\varepsilon_{2,N}\le T\}\Big)\\&\le P^N\big(H^{2,N}_T(\varepsilon+\delta)\cap\{\widetilde{\tau}^\varepsilon_{2,N}\le T\}\big)\end{aligned}$$

where we have set

$$H^{2,N}_T(r):=\Big\{\sup_{0\le t\le T}\frac{M^{2,N}_{\widetilde{\tau}^\varepsilon_{2,N}+t}}{N^d}>\gamma_2+r\Big\}.$$

We note next that

$$\{\widetilde{\tau}^\varepsilon_{2,N}\le T\}\subseteq\{\tau^\varepsilon_{2,N}<\tau_N,\ \tau^\varepsilon_{2,N}\le T\}\cup\{\tau_N\le\tau^\varepsilon_{2,N},\ \tau_N\le T\}$$

and therefore

$$a_{2,N}\le P^N\big(H^{2,N}_T(\varepsilon+\delta)\cap\{\tau^\varepsilon_{2,N}<\tau_N,\ \tau^\varepsilon_{2,N}\le T\}\big)+P^N\big\{\tau_N\le\tau^\varepsilon_{2,N},\ \tau_N\le T\big\}. \tag{45}$$

We will prove first that the second term in the right hand side above converges to 0. Since $\tau_N=\sigma_N$ on the asymptotically sure event $\{\tau_N>0\}$ it suffices to show this for the time σ_N in place of τ_N. By assumption (6) the maximal coordinate has at least $\gamma_1:=3\delta_0/4$ condensing mass. We choose $\theta > 0$ such that $\varepsilon + 2\theta < \delta_0/2$, so that $\gamma_2 + \varepsilon + 2\theta < \gamma_1$, and for each $t > 0$ we consider the asymptotically sure event

$$F^T_{N,\theta} := \Big\{ \inf_{0\le t\le T} \Big(\frac{M^N_t}{N^d} - \gamma_1 \Big) \ge -\theta \Big\}.$$

On the event $\{\sigma_N \le \tau^\varepsilon_{2,N},\ \sigma_N \le T\} \cap F^T_{N,\theta}$ we have that

$$(\gamma_2 + \varepsilon + \theta) < (\gamma_1 - \theta) \le \frac{M^N_{\sigma_N}}{N^d} = \frac{M^{2,N}_{\sigma_N} + (n-1)}{N^d} \le \frac{[(\gamma_2+\varepsilon)N^d]}{N^d} + \frac{(n-1)}{N^d}.$$

So the event $\{\sigma_N \le \tau^\varepsilon_{2,N},\ \sigma_N \le T\} \cap F^T_{N,\theta}$ is empty for large enough N. Since $F^T_{N,\theta}$ has asymptotically vanishing complement, this proves that the second term in the right hand side of (45) converges to 0 as claimed.

Since $F^T_{N,\theta}$ has asymptotically vanishing complement, it follows thus by (45) that

$$\limsup_{N\to+\infty} a_{2,N} \le \limsup_{N\to+\infty} P^N\big(H^{2,N}_T(\varepsilon+\delta) \cap \{\tau^\varepsilon_{2,N} < \tau_N,\ \tau^\varepsilon_{2,N} \le T\} \cap F^T_{N,\theta} \big).$$

But $\{\tau^\varepsilon_{2,N} < \tau_N,\ \tau^\varepsilon_{2,N} \le T\} \cap F^T_{N,\theta} \subseteq \{\widetilde{\tau}^\varepsilon_{2,N} < +\infty\} \cap \{\eta_{\widetilde{\tau}^\varepsilon_{2,N}} \in E^{\varepsilon,\theta}_N\}$ where

$$E^{\varepsilon,\theta}_N := \{M_N \ge [(\gamma_1 - \theta)N^d],\ M_{2,N} = [(\gamma_2+\varepsilon)N^d],\ \sharp K_N = 1\}$$

and therefore (recall that $P^{2,N,\varepsilon}$ is the law P^N conditioned on the event $\{\widetilde{\tau}^\varepsilon_{2,N} < +\infty\}$)

$$\begin{aligned}\limsup_{N\to+\infty} a_{2,N} &\le \limsup_{N\to+\infty} P^N\Big(H^{2,N}_T(\varepsilon+\delta) \cap \big\{\widetilde{\tau}^\varepsilon_{2,N} < +\infty\big\} \cap \{\eta_{\widetilde{\tau}^\varepsilon_{2,N}} \in E^{\varepsilon,\theta}_N\}\Big)\\ &\le \limsup_{N\to+\infty} P^{2,N,\varepsilon}\Big(\Big\{ \sup_{0\le t\le T} \frac{M^{2,N}_{\widetilde{\tau}^\varepsilon_{2,N}+t}}{N^d} > \gamma_2 + \varepsilon + \delta \Big\} \bigcap \big\{\eta_{\widetilde{\tau}^\varepsilon_{2,N}} \in E^{\varepsilon,\theta}_N\big\}\Big).\end{aligned}$$

Let now $\mu^{2,N,\varepsilon}_0 := (\eta_{\widetilde{\tau}^\varepsilon_{2,N}})_\sharp P^{2,N,\varepsilon}$ denote the law of the ZRP at the stopping time $\widetilde{\tau}^\varepsilon_{2,N}$. By the strong Markov property, the law of the process $(\eta_{\widetilde{\tau}^\varepsilon_{2,N}+t})_{t\ge0}$ is equal to the law of the ZRP starting from $\mu^{2,N,\varepsilon}_0$, that is

$$\big((\eta_{\widetilde{\tau}^\varepsilon_{2,N}+t})_{t\ge0}\big)_\sharp P^{2,N,\varepsilon} = P^{\mu^{2,N,\varepsilon}_0}.$$

Therefore, if $\eta_0\colon D(\mathbb{R}_+;\mathbb{M}^d_N) \to \mathbb{M}^d_N$ denotes the evaluation map at $t=0$, we can write the last inequality as (with the convention $\frac{0}{0}\cdot 0 = 0$)

$$\begin{aligned}\limsup_{N\to+\infty} a_{2,N} &\le \limsup_{N\to+\infty} P^{\mu^{2,N,\varepsilon}_0}\Big(\Big\{ \sup_{0\le t\le T} \frac{M^{2,N}_t}{N^d} > \gamma_2+\varepsilon+\delta \Big\} \Big| \eta_0 \in E^{\varepsilon,\theta}_N \Big)\\ &= \limsup_{N\to+\infty} P^{\mu^{2,N,\varepsilon}_0(\cdot|E^{\varepsilon,\theta}_N)} \Big\{ \sup_{0\le t\le T} \Big| \frac{M^{2,N}_t}{N^d} - (\gamma_2+\varepsilon) \Big| > \delta \Big\}.\end{aligned}$$

It follows that $\limsup_{N\to+\infty} a_{2,N}(\varepsilon+\delta) = 0$ since the initial states $\mu^{2,N,\varepsilon}_0(\cdot|E^{\varepsilon,\theta}_N)$ satisfy the assumptions of (a). This proves (43).

As in part (a) now, the limit (43) implies that τ_N converges in probability to $+\infty$ as $N \to +\infty$. Indeed similarly to part (a) one can check that for sufficiently small $\varepsilon > 0$ and sufficiently large $N \in \mathbb{N}$

$$\{\tau_N \le T\} \subseteq \Big(\Big\{ \sup_{0\le t\le T} \frac{M^{2,N}_{t\wedge\tau_N}}{N^d} \le \gamma_2 + \varepsilon,\ \inf_{0\le t\le T} \frac{M^N_t}{N^d} \ge \gamma_1 - \varepsilon \Big\} \cap \{\tau_N > 0\} \Big)^c.$$

But the event in the right hand side is asymptotically of zero probability and therefore, as in part (a) again, we have that for all $\varepsilon, T > 0$

$$P^N\Big\{\sup_{0\le t\le T}\frac{M_t^{2,N}}{N^d} > \varepsilon\Big\} \le P^N\Big\{\sup_{0\le t\le T}\frac{M_{t\wedge\tau_N}^{2,N}}{N^d} > \varepsilon\Big\} + P^N\{\tau_N \le T\} \stackrel{N\to+\infty}{\longrightarrow} 0.$$

Since $\sigma_N = \tau_N$ on the asymptotically sure event $\{\tau_N > 0\}$ we also obtain (7). $\square$

We combine finally parts (a) and (b) of Lemma 4.8 to conclude the proof of Theorem 2.2. We fix arbitrary $\varepsilon, T > 0$ and set

$$B_{\varepsilon,T}^{2,N} := \Big\{\sup_{0\le s,t\le T}\Big|\frac{M_t^{2,N} - M_s^{2,N}}{N^d}\Big| > \varepsilon\Big\}, \quad N \in \mathbb{N}.$$

It suffices to show that any subsequence of $\{P^N(B_{\varepsilon,T}^{2,N})\}$, which we continue to denote by the same subscript, has a further subsequence that converges to zero. For each $\theta \in (0, \delta_0/4)$, where δ_0 is the number in (6), we define the set $J_\theta^N := \{M_{2,N} \le [\theta N^d]\}$. As in the case of the maximum we can assume that there exists $\theta_0 \in (0, \delta_0/4)$ such that $p_{2,\theta} := \liminf_{N\to+\infty}\mu_0^N(J_\theta^N) \in (0,1)$ for all $\theta \in (0, \theta_0)$.

We choose then $\theta \in (0, \theta_0 \wedge \frac{\varepsilon}{2})$, a subsequence $\{P^{k_N}(J_\theta^{k_N})\}$ of $\{P^N(J_\theta^N)\}_N$, possibly depending on $\theta > 0$, such that $p_{2,\theta} = \lim_{N\to+\infty}\mu_0^{k_N}(J_\theta^{k_N}) \in (0,1)$ and $N_0 \in \mathbb{N}$ such that $\mu_0^{k_N}(J_\theta^{k_N})$ is bounded away from 0 and 1 for all $N \ge N_0$. Then, for any sequence of events $A_N \subseteq \mathbb{M}_N^d$ such that $\lim_{N\to+\infty}\mu_0^N(A_N) = 1$, we have that

$$1 \ge \mu_0^{k_N}(A_{k_N} \cup J_\theta^{k_N}) \ge \mu_0^{k_N}(A_{k_N}) \stackrel{N\to+\infty}{\longrightarrow} 1$$

and so, by taking the the limit as $N \to +\infty$ in the equality

$$\mu_0^{k_N}(A_{k_N} \cap J_\theta^{k_N}) = \mu_0^{k_N}(A_{k_N}) + \mu_0^{k_N}(J_\theta^{k_N}) - \mu_0^{k_N}(A_{k_N} \cup J_\theta^{k_N}),$$

we obtain that $\lim_{N\to+\infty}\mu_0^{k_N}(A_{k_N} \cap J_\theta^{k_N}) = \lim_{N\to+\infty}\mu_0^{k_N}(J_\theta^{k_N})$. Since $\mu_0^{k_N}(J_\theta^{k_N}) > 0$ for large N, it follows that $\lim_{N\to+\infty}\mu_0^{k_N}(A_{k_N}|J_\theta^{k_N}) = 1$. Likewise we obtain that $\lim_{N\to+\infty}$ $\mu_0^{k_N}(A_{k_N}|(J_\theta^{k_N})^c) = 1$. Applying this for the sets $A_N = \{\sharp K_N = 1\}$ and $A_N = \{M_N - M_{2,N} \ge \delta_0 N^d\}$ we obtain that each one of the sequences $\{\mu_0^{k_N}(\cdot|J_\theta^{k_N})\}$ and $\{\mu_0^{k_N}(\cdot|(J_\theta^{k_N})^c)\}$ satisfies assumptions (5) and (6).

The restricted probability measures $P^{k_N}(\cdot|\eta_0 \in J_\theta^{k_N})$, $P^{k_N}(\cdot|\eta_0 \in (J_\theta^{k_N})^c)$ are well-defined for large N and therefore

$$P^{k_N}(B_{\varepsilon,T}^{2,k_N}) \le P^{\mu_0^{k_N}(\cdot|J_\theta^{k_N})}(B_{\varepsilon,T}^{2,k_N}) + P^{\mu_0^{k_N}(\cdot|(J_\theta^{k_N})^c)}(B_{\varepsilon,T}^{2,k_N}).$$

The second term on the right hand side above is the probability of the event $B_{\varepsilon,T}^{2,k_N}$ with respect to the law of the ZRP starting with at least $\theta > 0$ condensing mass in the strict second maximum and as such it converges to zero as $N \to +\infty$ by Lemma 4.8(a). The first term is the probability of the event $B_{\varepsilon,T}^{2,k_N}$ with respect to a sequence of initial states with at most θ condensing mass and thus, since $\theta < \min\{\delta_0/4, \varepsilon/2\}$, the first term also converges to zero by by Lemma 4.8(b). Thus the sequence $\{P^{k_N}(B_{\varepsilon,T}^{2,k_N})\}$ is the required subsequence.

Evolution of the maximizing set. Recall the definition of the processes (x_t^N) and (u_t^N) in the paragraph prior to the statement of Corollary 2.1. In order for the maximum to change position when starting as a singleton it has to be surpassed first by the second maximum. As shown in Theorem 2.2, if $\{\mu_0^N\}$ satisfies assumptions (5) and (6), then this never happens in the hydrodynamic scale. So, if τ_N is the exit time of the ZRP from the set E_N defined in (34), we have on the event $\{\tau_N > T\}$ that $x_t^N \equiv x_0^N$ for all $0 \le t \le T$. Consequently, for any $\varepsilon > 0$

$$\lim_{N\to+\infty} P^N\Big\{\sup_{0\le s,t\le T} \bar{d}(u_s^N, u_t^N) > \varepsilon\Big\} \le \lim_{N\to+\infty} P^N\{\tau_N \le T\} = 0 \tag{46}$$

and the relative compactness conditions [11, Theorem 4.1.3, Remark 4.1.4] are trivially satisfied by the processes $(u_t^N)_{0\le t\le T}$. Thus, the sequence of their laws is relatively compact and all of its limit points are supported on $C(0,T;\bar{\mathbb{T}}^d)$.

Let now U^N be the law of the process $(u_t^N)_{0\le t\le T}$ on $D(0,T;\bar{\mathbb{T}}^d)$, let U^* be any limit point of $\{U^N\}$ and let $u_t\colon D(0,T;\bar{\mathbb{T}}^d) \to \bar{\mathbb{T}}^d$, $t \ge 0$, denote the natural projections. The map $\bar{d}(u_0,u_t)\colon D(0,T;\bar{\mathbb{T}}^d) \to \mathbb{R}_+$ is continuous at each path which is continuous at t. Since U^* is concentrated on continuous trajectories, the map $\bar{d}(u_0,u_t)$ is U^*-a.s. continuous and thus by the Portmanteau theorem and the limit (46), for any $\varepsilon > 0$

$$U^*\big\{\bar{d}(u_0,u_t) > \varepsilon\big\} \le \liminf_{N\to\infty} U^N\big\{\bar{d}(u_0,u_t) > \varepsilon\big\} = \liminf_{N\to\infty} P^N\big\{\bar{d}(u_0^N,u_t^N) > \varepsilon\big\} = 0.$$

Since this holds for all $\varepsilon > 0$ it follows that for any $0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$, $U^*\{u_0 = u_{t_1} = \ldots = u_{t_n}\} = 1$ and thus U^* is concentrated on constant trajectories.

The point $\mathfrak{d}$ is an isolated point of $\bar{\mathbb{T}}^d$, e.g. $D(\mathfrak{d}, \frac{1}{2}) = \{\mathfrak{d}\}$. This implies that the singleton $\{c_\mathfrak{d}\}$ of the trajectory that is constant at $\mathfrak{d}$ is also isolated in the Skorokhod metric. In fact $D(c_\mathfrak{d}, \frac{1}{2}) = \{c_\mathfrak{d}\}$ and thus $\{c_\mathfrak{d}\}$ is open. Consequently, by the Portmanteau theorem and assumption (5),

$$\begin{aligned} U^*\big\{(u_t)_{0\le t\le T} = c_\mathfrak{d}\big\} &\le \liminf_{N\to+\infty} P^N\big\{(u_t^N)_{0\le t\le T} = c_\mathfrak{d}\big\} \\ &\le \liminf_{N\to+\infty} P^N\big\{x_0^N = \mathfrak{d}\big\} \le \lim_{N\to+\infty} \mu_0^N\{\sharp K_N \ge 2\} = 0. \end{aligned}$$

This implies that U^* is supported by the set of constant paths in $\mathbb{T}^d$.

We assume finally that $\lambda_N := u_{N\sharp}\mu_0^N$ converges weakly to a measure λ_0 on $\bar{\mathbb{T}}^d$. The projection u_0 at time 0 is continuous on the Skorokhod space and therefore $u_{0\sharp}U^* = \lambda_0$. Thus, since U^* is concentrated in constant trajectories, we have that for any Borel sets $B_1,\ldots,B_n \subseteq \bar{\mathbb{T}}^d$ and any $0 \le t_1 \le \ldots \le t_n$

$$U^*\{u_{t_1} \in B_1, \ldots, u_{t_n} \in B_n\} = U^*\{u_0 \in B_1 \cap \ldots \cap B_n\} = \lambda_0(B_1 \cap \ldots \cap B_n).$$

This proves that $U^* = (\mathfrak{c}_{\bar{\mathbb{T}}^d})_\sharp \lambda_0$ and completes the proof of Corollary 2.1.

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