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FOURIER AND TAYLOR-LAURENT SERIES WITH UNIVERSAL APPROXIMATION PROPERTIES

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Dedicated to Professor Vassili Nestoridis.

Abstract. It has long been known that there exist trigonometric series, the partial sums of which possess universal approximation properties on the unit circle. This paper shows that, for most smooth functions on the unit circle, the partial sums of the associated Fourier series, when extended to the plane, have universal approximation properties off the circle. Related results are also established for pairs of Taylor and Laurent series arising from functions that are holomorphic off a Jordan curve.

1. Introduction

Classical work of Menshov [9] established the existence of trigonometric series $\sum_{-\infty}^{\infty} a_n e^{int}$ which are "universal" in the following sense: for any Lebesgue measurable function g on the unit circle $\mathbb{T}$, there is an increasing sequence (λ_n) of natural numbers such that $\sum_{-\lambda_n}^{\lambda_n} a_k e^{ikt} \to g(e^{it})$ as $n \to \infty$ for almost every point e^{it} of $\mathbb{T}$. (See also Chapter 15 of Bary [1].) More recently, Kahane [4] showed that such universality is a generic property of trigonometric series with respect to the topology of convergence of each of the coefficients a_n.

Our first result below also concerns universal approximation by trigonometric series. However, it differs from Menshov's work in two important respects: it concerns Fourier series of smooth functions, and the approximation takes place off $\mathbb{T}$ (by the natural extension to $\mathbb{C}\backslash\{0\}$ of the partial sums).

If $K \subset \mathbb{C}$ is compact, then we define $K^c = \mathbb{C}\backslash K$, and $A(K)$ to be the collection of functions in $C(K)$ that are holomorphic on K°. We denote by $C^\infty(\mathbb{T})$ the usual space of infinitely differentiable functions h on $\mathbb{T}$, endowed with the topology generated by the seminorms $\max_{\mathbb{T}} \left|h^{(k)}\right|$ $(k \geq 0)$.

Theorem 1. *(i) There exists $h \in C^\infty(\mathbb{T})$, the Fourier series $\sum_{-\infty}^{\infty} a_n e^{int}$ of which has the following property: for every compact set $K \subset (\mathbb{T}\cup\{0\})^c$ with K^c connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that $\sum_{-\lambda_n}^{\lambda_n} a_k z^k \to g$ uniformly on K as $n \to \infty$.*
(ii) The set of all $h \in C^\infty(\mathbb{T})$ with the above property is G_δ and dense in $C^\infty(\mathbb{T})$, and contains a dense vector subspace of $C^\infty(\mathbb{T})$ apart from the zero function.

Let $\omega \subset \mathbb{C}$ be an open set with compact boundary. We denote by $H(\omega)$ the collection of all holomorphic functions on ω, and by $A^\infty(\omega)$ the subset comprising those functions f for which each derivative $f^{(k)}$ $(k = 0, 1, ...)$ has a continuous extension to $\partial\omega$. If ω is unbounded, then we add the requirement that for each f in $A^\infty(\omega)$ all its derivatives are bounded. We endow $H(\omega)$

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with the topology of local uniform convergence, and $A^\infty(\omega)$ with the topology induced by the seminorms $\sup_\omega |f^{(k)}|$ $(k \geq 0)$.

Let us describe the idea that led us to the statement of Theorem 1. The results in [7] imply the following: If $\mathbb{D}$ is the open unit disc then there exist two functions $f_1 \in A^\infty(\mathbb{D})$ and $f_2 \in A^\infty(\overline{\mathbb{D}}^c)$, $f_1(z) = \sum_{n=0}^\infty a_n z^n$, $|z| < 1$ and $f_2(z) = \sum_{n=1}^\infty b_n z^{-n}$, $|z| > 1$, such that their partial sums $S_N(z) = \sum_{n=0}^N a_n z^n$, $T_N(z) = \sum_{n=1}^N b_n z^{-n}$, $N = 1, 2, \dots$ have the following properties: for any compact sets $K_1 \subset \overline{\mathbb{D}}^c$, $K_2 \subset \mathbb{D}\backslash\{0\}$, with connected complements and any functions $\phi_1 \in A(K_1)$, $\phi_2 \in A(K_2)$ there exist sequences N_m, M_m in $\{1, 2, ..\}$ such that $S_{N_m}(z) \to \phi_1(z)$ uniformly on K_1, $T_{M_m}(z) \to \phi_2(z)$ uniformly on K_2 as $m \to +\infty$. Furthermore, a general result in [14] implies that generically we can have $N_m = M_m$. For two such universal functions f_1, f_2 which carry out approximations with the same indices, we consider $h(z) = f_1(z) + f_2(z)$, $z \in \mathbb{T}$, which is a C^∞ function on the unit circle $\mathbb{T}$. Obviously the function $t_{N_m}(e^{i\theta}) = S_{N_m}(e^{i\theta}) + T_{N_m}(e^{i\theta})$ is a symmetric partial sum of the Fourier series of h. Let now $K \subset (\mathbb{T} \cup \{0\})^c$ be a compact set with K^c connected, and $g \in A(K)$ as in Theorem 1. Set $K_1 = K \cap \overline{\mathbb{D}}^c$, $K_2 = K \cap \mathbb{D}$, $\phi_1 = g - f_2$ on K_1 and $\phi_2 = g - f_1$ on K_2. Then, applying the above arguments, one can easily prove that there exists a sequence N_m such that $t_{N_m}(z)$ converges uniformly on K to g as $m \to +\infty$. This gives the generic result of Theorem 1. We aim to use the same approach to prove similar results to Theorem 1. However in section 4 we present a different proof of Theorem 1.

In our next result universal approximation occurs both on and off $\mathbb{T}$. In fact, we will generalize $\mathbb{T}$ to a rectifiable Jordan curve J, denote the interior and exterior domains of J by ω_i and ω_e respectively, and assume without loss of generality that $0 \in \omega_i$. Let $H_0(J^c)$ denote the subspace of $H(J^c)$ comprising those functions f for which $\lim_{z\to\infty} f(z) = 0$. Given any $f \in H_0(J^c)$, we denote its Taylor series about 0 by $\sum_{n=0}^\infty a_n(f) z^n$ and its Laurent expansion near ∞ by $\sum_{n=1}^\infty a_{-n}(f) z^{-n}$. Further, let B_J denote the space of those functions $f \in H_0(J^c)$ such that $f|_{\omega_i} \in A^\infty(\omega_i)$, endowed with the topology induced by the seminorms

$$\sup_{\omega_i} |f^{(k)}|, \quad \max_{L_k} |f| \quad (k \geq 0), \quad \text{where} \quad L_k = \{2^{-k} \leq \operatorname{dist}(z, \overline{\omega}_i) \leq 2^k\}.$$

Theorem 2. *(i) There exists $f \in B_J$ with the following property: for every compact set $K \subset \{0\}^c$ such that $K \cap \omega_e$ is compact and K^c is connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that $\sum_{-\lambda_n}^{\lambda_n} a_k(f) z^k \to g$ uniformly on K as $n \to \infty$.*
(ii) The set of all functions $f \in B_J$ with the above property is G_δ and dense in B_J, and contains a dense vector subspace of B_J apart from the zero function.

Remark 3. *A slight modification of the proof of Theorem 2 yields the obvious companion result, in which B_J is replaced by the space of functions $f \in H_0(J^c)$ such that $f|_{\omega_e} \in A^\infty(\omega_e)$ (endowed with the analogous topology), and the sets $K \subset \{0\}^c$ are now chosen so that $K \cap \omega_i$ is compact (and K^c is again connected).*

Using Theorem 2 and Remark 3 we shall prove the following:

Theorem 4. *(i) There exists $f \in H_0(J^c)$ with the following property: for every compact set $K \subset \{0\}^c$ such that K^c is connected and at least one of the sets $K \cap \omega_i, K \cap \omega_e$ is compact, and for every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that $\sum_{-\lambda_n}^{\lambda_n} a_k(f) z^k \to g$ uniformly on K as $n \to \infty$.*

(ii) The set of all functions $f \in H_0(J^c)$ with the above property is G_δ and dense in $H_0(J^c)$, and contains a dense vector subspace of $H_0(J^c)$ apart from the zero function.

If the Jordan curve J is not rectifiable, then part (i) of Theorem 2 still holds. We do not know if part (ii) also holds in this case, but can prove an analogue where B_J is replaced by a smaller space $B_J^{'}$ in which $f|_{\omega_i}$ is required to belong to $X^\infty(\omega_i)$, the closure in $A^\infty(\omega_i)$ of the set of all polynomials (see [5]). Furthermore, Theorem 4 still holds (exactly as it is formulated) even if J is not rectifiable. To prove this, one should work on $X^\infty(\omega_i), X^\infty(\omega_e)$ instead of $A^\infty(\omega_i), A^\infty(\omega_e)$ and get similar results to Theorem 2 and Remark 3 as we mentioned above. It was shown in [12] that $X^\infty(\omega_i) = A^\infty(\omega_i)$ if there is a constant M such that any two points z, w in ω_i can be joined by a curve in ω_i with length at most M. This last condition is equivalent to the requirement that every bounded holomorphic function on the simply connected domain ω_i has a bounded primitive (see [13]).

2. Preliminary results

Let $(S_n(f,\zeta))_{n\geq 0}$ denote the partial sums of the Taylor series about ζ of a holomorphic function f. The following result concerning "universal Taylor series" is due to Melas and Nestoridis [8].

Theorem A. *Let ω be a Jordan domain with rectifiable boundary and $\zeta \in \omega$. There exists $f \in H(\omega)$ with the following property: for every compact set $K \subset \mathbb{C}$, $K \cap \omega = \emptyset$, K^c connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that $S_{\lambda_n}(f,\zeta) \to g$ uniformly on K, $S_{\lambda_n}(f,\zeta) \to f$ locally uniformly on ω as $n \to \infty$.*

Let $(T_n(f))_{n>0}$ denote the partial sums of the Laurent expansion near ∞ of a function f that is holomorphic in the complement of some closed disc centred at 0 and is bounded near ∞.

Proposition 5. *Let J be a rectifiable Jordan curve and $0 \in \omega_i$. There exists $f \in H_0(J^c)$ with the following property: for every compact set $K \subset \{0\}^c$ with K^c connected and for every $g \in A(K)$ there is an increasing sequence (λ_n) in $\mathbb{N}$ such that*
$S_{\lambda_n}(f,0) \to g$ uniformly on $K \backslash \omega_i$, $T_{\lambda_n}(f) \to g$ uniformly on $K \backslash \omega_e$,
$S_{\lambda_n}(f,0) \to f$ locally uniformly on ω_i , $T_{\lambda_n}(f) \to f$ locally uniformly on ω_e as $n \to \infty$.

Proof. Using the simultaneous approximation by universal series (see [14]), we may fix two functions $\alpha(z) \in H(\omega_i)$ and $\beta(z) \in H(\frac{1}{\omega_e} \cup \{0\})$, with $\beta(0) = 0$, which are universal in the sense of Theorem A and carry out approximations with the same indices. Let

$$f(z) = \begin{cases} \alpha(z), & z \in \omega_i \\ \beta(\frac{1}{z}), & z \in \omega_e \ . \end{cases}$$

Since $\beta(\frac{1}{z})$ gives rise to f's Laurent expansion $(T_n(f))_{n>0}$, we use twice Theorem A (i.e. for $\alpha(z), \beta(z)$) and we are done. □

Next we recall a result of Kariofillis, Konstadilaki and Nestoridis [5] (see also Melas and Nestoridis [7]).

Theorem B. *Let ω be a Jordan domain with rectifiable boundary and $\zeta \in \omega$. There exists $f \in A^\infty(\omega)$ with the following property: for every compact set $K \subset (\overline{\omega})^c$ such that K^c is connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that, as $n \to \infty$,*

$$S_{\lambda_n}(f,\zeta) \to g \text{ uniformly on } K, \quad S_{\lambda_n}(f,\zeta) \to f \text{ uniformly on } \omega.$$

We can argue analogously to Proposition 5 but now the universal functions $\alpha(z)$, $\beta(z)$ are chosen in the sense of Theorem B. Then the same argument gives the following:

Proposition 6. *Let J be a rectifiable Jordan curve and $0 \in \omega_i$. There exists $f \in H_0(J^c)$ such that $f|_{\omega_i} \in A^\infty(\omega_i)$ and $f|_{\omega_e} \in A^\infty(\omega_e)$, with the following property: for every compact set $K \subset (J \cup \{0\})^c$ with K^c connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that, as $n \to \infty$,*

$$\begin{aligned} S_{\lambda_n}(f,0) &\to g \text{ uniformly on } K \cap \omega_e, \quad S_{\lambda_n}(f,0) \to f \text{ uniformly on } \omega_i, \\ T_{\lambda_n}(f) &\to g \text{ uniformly on } K \cap \omega_i, \quad T_{\lambda_n}(f) \to f \text{ uniformly on } \omega_e. \end{aligned}$$

If now we fix a universal function $\alpha(z) \in A^\infty(\omega_i)$ in the sense of Theorem B and a universal function $\beta(z) \in H(\omega_e)$ in the sense of Theorem A which carry out approximations with the same indices, then the same argument as that in the proof of Proposition 5 gives the following:

Proposition 7. *Let J be a rectifiable Jordan curve and $0 \in \omega_i$. There exists $f \in B_J$ with the following property: for every compact set $K \subset \{0\}^c$ such that $K \cap \omega_e$ is compact and K^c is connected, and for every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that, as $n \to \infty$,*

$$\begin{aligned} S_{\lambda_n}(f,0) &\to g \text{ uniformly on } K \cap \omega_e, \quad S_{\lambda_n}(f,0) \to f \text{ uniformly on } \omega_i, \\ T_{\lambda_n}(f) &\to g \text{ uniformly on } K \cap \overline{\omega_i}\ , \quad T_{\lambda_n}(f) \to f \text{ locally uniformly on } \omega_e. \end{aligned}$$

3. Abstract theory of universal series

Let $(X_m)_{m\geq 1}$ be a sequence of metrizable topological vector spaces over $\mathbb{C}$. For each m we suppose that X_m is equipped with a topology induced by a translation-invariant metric, and fix a (doubly infinite) sequence $(x_{m,k})_{k\in\mathbb{Z}}$ in X_m. Let E be a complete metrizable topological vector space with topology induced by a translation-invariant metric, and with $(e_0, e_1, e_{-1}, e_2, e_{-2}, ...)$ as a basis. Suppose further that $(\phi_k : E \to \mathbb{C})_{k\in\mathbb{Z}}$ is a sequence of continuous linear mappings satisfying $\phi_k(e_n) = \delta_{k,n}$ $(k, n \in \mathbb{Z})$. We say that an element f in E belongs to the collection $\mathcal{U}_E$ if, for each $m \geq 1$ and each $x \in X_m$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that, as $n \to \infty$,

$$\sum_{k=-\lambda_n}^{\lambda_n} \phi_k(f) x_{m,k} \to x \text{ in } X_m, \text{ and } \sum_{k=-\lambda_n}^{\lambda_n} \phi_k(f) e_k \to f \text{ in } E.$$

The following result is implied by Theorem 3 in [2] (see also [11] for a simplified version), which is formulated for sequences indexed by $\mathbb{N}$. This $\mathbb{Z}$-indexed version can be deduced by, for example, replacing pairs of the form e_n, e_{-n} by e_{2n}, e_{2n+1} $(n \geq 1)$ and then putting $\mu = (2j+1)_{j\geq 1}$ in that result.

Theorem C. *The following statements are equivalent:*
(i) $\mathcal{U}_E \neq \emptyset$;
(ii) $\mathcal{U}_E$ is dense and G_δ in E;
(iii) $\mathcal{U}_E \cup \{0\}$ contains a dense vector subspace of E.

If $f \in \mathcal{U}_E$ we say that f is *universal of type $\mathcal{U}_E$*.

If, in the definition of $\mathcal{U}_E$, we drop the condition of approximation of f in E, then it is defined a larger class of universality which is denoted by $\mathcal{U}$. Thus, an element f belongs to the collection

$\mathcal{U}\cap E$ if, $f \in E$ and for each $m \geq 1$ and each $x \in X_m$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that

$$\sum_{k=-\lambda_n}^{\lambda_n} \phi_k(f) x_{m,k} \to x \text{ in } X_m, \text{ as } n \to \infty.$$

This type of universality is called *universality of type* $\mathcal{U}\cap E$.

Since the set $G = \{\sum_{j=-n}^{n} a_j e_j : a_j \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in E we get the following:

Theorem D. *The following statements are equivalent:*
(i) $\mathcal{U}\cap E \neq \emptyset$;
(ii) $\mathcal{U}\cap E$ *is dense and* G_δ *in* E;
(iii) $\mathcal{U}\cap E$ *contains a dense vector subspace of* E *except for* 0.

Remark 8. *The universality in Theorems 1, A and B is universality of type* $\mathcal{U}_E$. *The universality in Theorems 2 and 4 and Remark 3 is universality of type* $\mathcal{U}\cap E$.

4. Proofs of the main results

Proof of Theorem 1. It is known ([3]) that there exists a sequence K_m, $m = 1, 2, \ldots,$ of compact subsets of $\mathbb{C}\backslash(\mathbb{T}\cup\{0\})$ with connected complements such that the following holds: Every compact set $K \subset \mathbb{C}\backslash(\mathbb{T}\cup\{0\})$ with connected complement is contained in some K_m. We consider the classes of functions

$$\mathcal{U}_m = \{f \in C^\infty(\mathbb{T}) : \forall g \in A(K_m) \ \exists \text{ a sequence } (\lambda_N) \text{ in } \mathbb{N} \text{ such that}$$

$$\sum_{k=-\lambda_N}^{\lambda_N} \hat{f}(k) z^k \to g(z) \ , \text{ as N } \to \infty, \text{ uniformly on } K_m\}.$$

Obviously $\cap_m \mathcal{U}_m$ is the class of Theorem 1.
We shall apply the abstract theory of the previous Section 3 having $X_m = A(K_m)$, $E = C^\infty(\mathbb{T})$, $x_{m,k} = z^k$, $e_k = e^{ikt}$, $\phi_k(f) = \hat{f}(k)$.
Since $\sum_{-N}^{N} \hat{f}(k) e^{ikt} \to f(e^{it})$, as $N \to +\infty$, in the topology of $C^\infty(\mathbb{T})$ for all $f \in C^\infty(\mathbb{T})$, we have that the two classes of universality $\mathcal{U}_E$ and $\mathcal{U}\cap E$ coincide. Furthermore $\cap_m \mathcal{U}_m = \mathcal{U}_{C^\infty(\mathbb{T})} = \mathcal{U}\cap C^\infty(\mathbb{T})$.
Fix $m = 1, 2, \ldots$ Then, there is an annulus $D_m = \{z \in \mathbb{C} : a_m < |z| < b_m\}$, $0 < a_m < 1 < b_m$, such that $D_m \cap K_m = \emptyset$.
According to [3] there is $f \in H(D_m)$ such that $f|_{\mathbb{T}} \in \mathcal{U}_m$. Thus, $\mathcal{U}_m \neq \emptyset$.
Applying either Theorem C or D for just one $X_m = A(K_m)$ we get that $\mathcal{U}_m$ is G_δ and dense in $C^\infty(\mathbb{T})$. By Baire's Theorem the same holds for their intersection $\cap_m \mathcal{U}_m$. Hence, we get from either Theorem C or D that $\cap_m \mathcal{U}_m$ contains a dense vector subspace of $C^\infty(\mathbb{T})$ except for 0.

Proof of Theorem 2. We choose f as in Proposition 7 and write $a_k = a_k(f)$, $k \in \mathbb{Z}$ (i.e. for $k \geq 0$, a_k are the coefficients of f's Taylor expansion about 0 and for $k < 0$, a_k are the coefficients of f's Laurent expansion near ∞.) Let $\varepsilon > 0$ and K be a compact set with the properties stated in the theorem. Then K can be written as $K_i \cup K_e$ with $K_i \subset \overline{\omega_i}\backslash\{0\}$ and $K_e \cap \overline{\omega_i} = \emptyset$. If $g \in A(K)$, then there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{K_e} \left| \sum_{k=0}^{n_0} a_k z^k - (g-f)(z) \right| < \frac{\varepsilon}{2}, \quad \sup_{K_i} \left| \sum_{k=1}^{n_0} \frac{a_{-k}}{z^k} - (g-f)(z) \right| < \frac{\varepsilon}{2},$$

$$\sup_{K_i}\left|f(z)-\sum_{k=0}^{n_0}a_kz^k\right|<\frac{\varepsilon}{2}\quad\text{and}\quad\sup_{K_e}\left|f(z)-\sum_{k=1}^{n_0}\frac{a_{-k}}{z^k}\right|<\frac{\varepsilon}{2},$$

where f is defined on $K_i \cap J$ by continuous extension from ω_i. Hence

$$\sup_{K}\left|\sum_{k=-n_0}^{n_0}a_kz^k-g(z)\right|<\varepsilon.$$

Thus part (i) of the theorem holds.

Part (ii) follows using Theorem D. We note that $E = B_J$,

$$e_k=\begin{cases} z\mapsto z^k\chi_{\omega_i}(z) & (k\geq 0)\\ z\mapsto z^k\chi_{\omega_e}(z) & (k<0)\end{cases}\ ,\text{ or briefly } e_k=z^k,$$

$x_{m,k} = z^k$, $\phi_k(f) = a_k(f)$ and $X_m = A(K_m)$ where K_m is defined as follows:
Let $(K_{i,n})_n$ be a sequence of compact sets, $K_{i,n} \cap \omega_e = \emptyset$, $\mathbb{C}\setminus K_{i,n}$ connected such that for every compact set K, $K\cap\omega_e = \emptyset$ and $\mathbb{C}\setminus K$ connected, there exists $n = 1, 2, \ldots$ with $K \subset K_{i,n}$. This was established in [8]. Moreover, from [6] we fix a sequence $(K_{e,n})_n$ of compact sets, $K_{e,n} \cap \overline{\omega_i} = \emptyset$, $\mathbb{C}\setminus K_{e,n}$ connected such that for every compact set K, $K \cap \overline{\omega_i} = \emptyset$ and $\mathbb{C}\setminus K$ connected, there exists $n = 1, 2, \ldots$ with $K \subset K_{e,n}$. We define (K_m) to be the sequence consisting of all possible unions $K_{i,n} \cup K_{e,k}$, $n, k \in \mathbb{N}$.
With all the above considerations, it is clear that $\mathcal{U} \cap B_J$, as defined in Section 3, is the class of Theorem 2. Since we proved above that $\mathcal{U} \cap B_J \neq \emptyset$, it follows from Theorem D that this class is also G_δ and dense in B_J and contains a dense vector subspace of B_J apart from 0.

Remark 9. *We note that if we immitate the above proof of Theorem 2 but now setting $J = \mathbb{T}$ and choosing f as in Proposition 6 instead of Proposition 7 then we get a second proof of Theorem 1.*

Remark 10. *For the proof of Remark 3 we slightly modify the proof of Theorem 2. For example in the sequence $K_m = K_{i,n} \cup K_{e,k}$ the compact sets $K_{i,n} \subset \omega_i$ are not allowed now to meet J while $K_{e,k}$ may meet J from outside J ($K_{e,k} \subset \overline{\omega_e}$.) The existence of a such sequence $(K_{i,n})_n$ was proved in* [7] *while the existence of a sequence $(K_{e,k})_k$ as above was established in* [10].

Proof of Theorem 4. Let $\mathcal{U}_e$ (respectively, $\mathcal{U}_i$) denote the collection of functions $f \in H_0(J^c)$ with the property: for every compact set $K \subset \{0\}^c$ such that $K \cap \omega_e$ (respectively, $K \cap \omega_i$) is compact and K^c is connected, and every $g \in A(K)$, there is an increasing sequence (λ_n) in $\mathbb{N}$ such that $\sum_{-\lambda_n}^{\lambda_n} a_k(f)z^k \to g(z)$ uniformly on K as $n \to \infty$. It follows from Theorem 2 and Remark 3 that each of these collections (i.e. $\mathcal{U}_i$, $\mathcal{U}_e$) is non-empty. Using Theorem D we deduce that each of these collections is G_δ and dense in $H_0(J^c)$. By Baire's theorem, $\mathcal{U}_e \cap \mathcal{U}_i$ is also G_δ and dense in $H_0(J^c)$. Finally, a further application of Theorem D shows that $\mathcal{U}_e \cap \mathcal{U}_i$ also contains a dense vector subspace of $H_0(J^c)$ apart from the zero function. We mention that in this proof we used the abstract theory of section 3 with $E = H_0(J^c)$ and $X_m = A(K_m)$ where $(K_m)_m$ is defined as follows:
Let $(K_{i,n})_n$ be the sequence we used in the proof of Theorem 2 (i.e. $K_{i,n} \subset \overline{\omega_i}$ may meet J) and $(K_{e,k})_k$ be the sequence we used in Remark 10 (i.e. $K_{e,k} \subset \overline{\omega_e}$ may meet J). We define (K_m) to be the sequence of all possible unions $K_{i,n} \cup K_{e,k}$, $n, k \in \mathbb{N}$, where $K_{i,n}, K_{e,k}$ are not allowed to meet J simultaneously, that is, at most one of them may meet J in their union.

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