

AN ALTERNATIVE APPROACH TO SHARP L^1 ESTIMATES FOR THE DYADIC MAXIMAL OPERATOR

ELEFThERIOS N. NIKOLIDAKIS AND ANDREAS G. TOLIAS

ABSTRACT. We provide alternative proofs of sharp L^1 inequalities for the dyadic maximal function $\mathcal{M}_\tau\phi$ when ϕ satisfies certain L^1 and L^∞ conditions (see [4]).

1. INTRODUCTION

The dyadic maximal operator on \mathbb{R}^n is a useful tool in analysis and is defined by the formula

$$\mathcal{M}_d\phi(x) = \sup \left\{ \frac{1}{|S|} \int_S |\phi(u)| du : x \in S, S \subset \mathbb{R}^n \text{ is a dyadic cube} \right\}, \quad (1)$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where $|\cdot|$ denotes the Lebesgue measure on \mathbb{R}^n , and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, for $N = 0, 1, 2, \dots$

It is well known that the operator defined above satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(u)| du \quad (2)$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$, from which it is easy to get the following L^p -inequality

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1} \|\phi\|_p \quad (3)$$

for every $p > 1$, and every $\phi \in L^p(\mathbb{R}^n)$. It is easy to see that the weak type inequality (2) is the best possible. For refinements of this inequality see [7].

It has also been proved that (3) is best possible (see [1] and [2] for general martingales and [10] for dyadic ones). An approach for the study of the behaviour of this maximal operator in more depth is the introduction of the so called Bellman functions which play the role of generalized norms of \mathcal{M}_d . Such functions related to the L^p -inequality (3) have been precisely evaluated in [3], [4] and [6]. For the study of the Bellman functions of \mathcal{M}_d , we use the notation $\text{Av}_E(\psi) = \frac{1}{|E|} \int_E \psi$, whenever E is a Lebesgue measurable subset of \mathbb{R}^n of positive measure and ψ is a real valued integrable function defined on E . For a fixed dyadic cube Q the localized maximal operator $\mathcal{M}'_d\phi$ is defined as in (1) but with the dyadic cubes S being assumed to be

Date: Received July 7, 2021. Accepted May 17, 2022.

2020 Mathematics Subject Classification. 42B25.

Key words and phrases. Dyadic maximal operator, Bellman type functions.

contained in Q . Then for every $p > 1$ let

$$B_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}'_d \phi)^p : \phi \geq 0, \text{Av}_Q(\phi) = f, \text{Av}_Q(\phi^p) = F \right\}, \quad (4)$$

where the variables f, F satisfy $0 < f^p \leq F$. This is the well known Bellman function of two integral variables of the dyadic maximal operator. By a scaling argument it is easy to see that (4) is independent of the choice of Q , so we may choose Q to be the unit cube $[0, 1]^n$. In [3], the function (4) has been precisely evaluated for the first time. The proof has been given in a much more general setting of tree-like structures on probability spaces.

More precisely for a non-atomic probability space (X, \mathcal{A}, μ) and \mathcal{T} a family of measurable subsets of X that has a tree-like structure similar to the one of the dyadic case (the exact definition is given in Section 2) the dyadic maximal operator associated to \mathcal{T} is defined by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\} \quad (5)$$

for every $\phi \in L^1(\mu)$ and $x \in X$.

This operator is related to the theory of martingales and satisfies essentially the same inequalities as \mathcal{M}_d does. Now we define the corresponding Bellman function of three variables of $\mathcal{M}_{\mathcal{T}}$, by

$$B_p^{\mathcal{T}}(f, F, k) = \sup \left\{ \int_K (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F, K \subset X \text{ measurable with } \mu(K) = k \right\}, \quad (6)$$

the variables f, F, k satisfying $0 < f^p \leq F$ and $k \in (0, 1]$. The exact evaluation of (6) is given in [3].

It is well known that in general $\mathcal{M}_{\mathcal{T}}\phi$ does not belong to $L^1(\mu)$ when $\phi \in L^1(\mu)$. In [11] it is proved that if ϕ satisfies the condition $\int_X |\phi| \log^+ |\phi| d\mu < +\infty$ then $\mathcal{M}_{\mathcal{T}}\phi \in L^1(\mu)$. In [9] it is shown that this condition is also necessary for the integrability of $\mathcal{M}_{\mathcal{T}}\phi$. In [5] the corresponding to (6) function with respect to certain $L \log L$ conditions has been precisely evaluated. As a matter of fact in [5] more general conditions on ϕ have been considered. An application of this result is the evaluation of the following Bellman type function

$$B_1^{\mathcal{T}}(f, M) = \sup \left\{ \int_X \mathcal{M}_{\mathcal{T}}\phi d\mu : \phi \geq 0, \int_X \phi d\mu = f, \|\phi\|_{\infty} = M \right\} \quad (7)$$

when $0 < f \leq M$.

In the subsequent sections we provide proofs of Theorem 1.1 and Theorem 1.2 that are stated right below. The results that we present are special cases of deep results concerning the study of more general Bellman type functions that are considered in [4] by A. Melas. However the approach that we give in the present paper is more simple and elementary and thus easily accessible to the reader.

Theorem 1.1. For all real variables f, M_1, M_2 with $M_1 \geq f > M_2 \geq 0$ the following holds:

$$\sup \left\{ \int_X \mathcal{M}_{\mathcal{T}} \phi d\mu : \phi : X \rightarrow \mathbb{R}^+ \text{ is measurable,} \right. \\ \left. \int_X \phi d\mu = f, \|\phi\|_{\infty} = M_1, \operatorname{ess\,inf}_X(\phi) = M_2 \right\} \\ = f + (f - M_2) \log \left(\frac{M_1 - M_2}{f - M_2} \right). \quad (8)$$

For the proof of the above theorem we study the respective Hardy operator problem which is connected to the dyadic maximal operator problem and we use a symmetrization principle which appears in [6].

Theorem 1.2. For all f, M, k that satisfy $0 < f \leq M$ and $k \in (0, 1]$ it holds that

$$\sup \left\{ \int_K \mathcal{M}_{\mathcal{T}} \phi d\mu : \phi : X \rightarrow \mathbb{R}^+ \text{ is measurable,} \int_X \phi d\mu = f, \|\phi\|_{\infty} = M, \right. \\ \left. K \text{ measurable, } \mu(K) = k \right\} \\ = \begin{cases} kM & \text{if } 0 < k \leq \frac{f}{M} \\ f + f \log\left(\frac{Mk}{f}\right) & \text{if } \frac{f}{M} < k \leq 1. \end{cases} \quad (9)$$

The values of the suprema that appear in Theorem 1.1 and Theorem 1.2 are independent of the probability measure space (X, \mathcal{A}, μ) and the tree \mathcal{T} (see also [4]).

At this points we should comment on the methods that we use in the proofs of Theorems 1.1 and 1.2 compared to the methods used in the proofs of more general results in [4] by A. Melas.

In [4] A. Melas studies a more general problem by considering integrals of ϕ and $\mathcal{M}_{\mathcal{T}}\phi$ related to two increasing convex functions G and H that satisfy certain growth conditions. His approach is given in several steps. In the first one he provides a combinatorial rearrangement inequality on subtrees of the initial tree \mathcal{T} , and several technical lemmas that uses in the sequel. In the second step he applies a linearization for $\mathcal{M}_{\mathcal{T}}\phi$ which permits him to study this maximal function on a certain subtree of \mathcal{T} related to ϕ . By using the rearrangement inequality proved in the first step he reduces the evaluation of the Bellman function of interest to the evaluation of a respective Bellman type function involving decreasing functions. The proofs of his results involve techniques from ODEs and from the theory of calculus of variations. At the third step he finds extremals for the Bellman functions that he studies, again using ODEs and several techniques on extremization of integral expressions. Finally he provides examples, considering specific functions G and H .

In our approach we use independent results (appearing in [6] and [8]) that allow us to reduce the evaluation of the Bellman type functions that we study, to the corresponding problem for the Hardy operator acting on decreasing functions on $(0, 1]$. We determine the upper bound that is described in (8) (see Theorem 1.1) by using Riemann-Stieltjes integrals in a direct way. The sharpness of this upper bound in (8) is proved by using the results in [8]. Then we prove Theorem 1.2 by considering integrals of $\mathcal{M}_{\mathcal{T}}\phi$ on certain subsets of X (related to the distribution function of $\mathcal{M}_{\mathcal{T}}\phi$) which can be decomposed as pairwise almost disjoint unions of elements of

\mathcal{T} . Then applying Theorem 1.1 we reach the upper bound that is stated in (9). Finally we prove the sharpness of this upper bound by providing functions ϕ which satisfy the conditions that are settled on our problem, for which the value of $\int_K \mathcal{M}_{\mathcal{T}}\phi d\mu$ on certain suitable subsets K of X is arbitrarily close to the right side of (9).

2. PRELIMINARIES

Definition 2.1. Let (X, \mathcal{A}, μ) be a non-atomic probability measure space. We recall that a collection of measurable sets \mathcal{T} is called a tree in \mathcal{A} provided that the following conditions are satisfied:

- (i) $X \in \mathcal{T}$ and every $I \in \mathcal{T}$ has positive measure.
- (ii) To every $I \in \mathcal{T}$ corresponds a countable (finite or infinite) family $C(I) \subset \mathcal{T}$, containing at least two elements, such that:
 - (a) The elements of $C(I)$ are almost pairwise disjoint, i.e. for $J, J' \in C(I)$ with $J \neq J'$ we have that $\mu(J \cap J') = 0$.
 - (b) $I = \cup C(I)$.
- (iii) If we define $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(n+1)} = \cup\{C(I) : I \in \mathcal{T}_{(n)}\}$ for all n then $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_{(n)}$.
- (iv) For $(\mathcal{T}_{(n)})_{n \in \mathbb{N}}$ as defined above, $\lim_n [\sup\{\mu(I) : I \in \mathcal{T}_{(n)}\}] = 0$.

The maximal operator $\mathcal{M}_{\mathcal{T}}$ associated to the tree \mathcal{T} corresponds to every measurable function $\phi : X \rightarrow \mathbb{R}$ the function $\mathcal{M}_{\mathcal{T}}\phi$ defined by the formula

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}.$$

We also recall that for every measurable function $\phi : X \rightarrow \mathbb{R}$, defining $\phi^* : (0, 1] \rightarrow [0, +\infty)$ by the formula

$$\phi^*(t) = \inf\{y > 0 : \mu(|\phi| > y) < t\}$$

we have that ϕ^* is the unique decreasing and left continuous function on $(0, 1]$ that is equimeasurable to ϕ . The following is proved in [3]:

Lemma 2.2. For every $I \in \mathcal{T}$ and every α such that $0 < \alpha < 1$, there exists a subfamily \mathcal{F} of \mathcal{T} consisting of almost pairwise disjoint subsets of I such that

$$\mu\left(\bigcup_{J \in \mathcal{F}} J\right) = \sum_{J \in \mathcal{F}} \mu(J) = (1 - \alpha)\mu(I).$$

In [8] the following is proved:

Lemma 2.3. For any integrable $\phi : X \rightarrow \mathbb{R}^+$ it holds that

$$(\mathcal{M}_{\mathcal{T}}f)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) du \text{ for every } t \in (0, 1].$$

We will also need the following symmetrization principle which appears in [6].

Theorem 2.4. Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be decreasing and let $G_1, G_2 : [0, +\infty) \rightarrow [0, +\infty)$ be two increasing functions. Then for every $k \in (0, 1]$ the following holds:

$$\begin{aligned} \sup_K \left\{ \int (G_1 \circ \mathcal{M}_\mathcal{T}\phi)(G_2 \circ \phi) d\mu : \phi^* = g, \mu(K) = k \right\} \\ = \int_0^k G_1\left(\frac{1}{t} \int_0^t g(u) du\right) G_2(g(t)) dt. \end{aligned}$$

Let us fix some notation. For $M_1 \geq f > M_2 \geq 0$, we set

$$\begin{aligned} \mathcal{C}_{X,\mathcal{T}}(M_1, f, M_2) = \{ \phi : X \rightarrow \mathbb{R}^+ \text{ measurable, } \|\phi\|_\infty = M_1, \\ \int_X \phi d\mu = f, \text{essinf}_X(\phi) = M_2 \} \end{aligned}$$

and

$$\mathcal{C}_{X,\mathcal{T}}(M_1, f) = \{ \phi : X \rightarrow \mathbb{R}^+ \text{ measurable, } \|\phi\|_\infty = M_1, \int_X \phi d\mu = f \}.$$

In Theorem 1.1 we will compute the quantity

$$\sup_X \left\{ \int \mathcal{M}_\mathcal{T}\phi d\mu : \phi \in \mathcal{C}_{X,\mathcal{T}}(M_1, f, M_2) \right\}$$

for all for all $M_1 \geq f > M_2 \geq 0$ while in Theorem 1.2 we will compute the quantity

$$\sup_K \left\{ \int \mathcal{M}_\mathcal{T}\phi d\mu, \phi \in \mathcal{C}_{X,\mathcal{T}}(M_1, f), K \in \mathcal{A}, \mu(K) = k \right\}$$

for all $M_1 \geq f > 0$ and all $k \in (0, 1]$.

3. PROOF OF THEOREM 1.1

For the proof of Theorem 1.1 we work as follows. Fix $M_1 \geq f > M_2 \geq 0$ and let $\mathcal{A}(M_1, f, M_2)$ be the class of functions

$$\begin{aligned} \mathcal{A}(M_1, f, M_2) = \left\{ g : [0, 1] \rightarrow [0, +\infty) : g \text{ is decreasing,} \right. \\ \left. \text{left continuous, continuous at 0,} \right. \\ \left. g(0) = M_1, \lim_{t \rightarrow 1} g(t) = M_2 \text{ and } \int_0^1 g(t) dt = f \right\}. \end{aligned}$$

For each $g \in \mathcal{A}(M_1, f, M_2)$ we set

$$I_g = \int_0^1 \left(\frac{1}{t} \int_0^t g(u) du \right) dt.$$

Our goal is to maximize I_g over all $g \in \mathcal{A}(M_1, f, M_2)$.

We observe that since $\lim_{t \rightarrow 0} \left(\log(t) \int_0^t g(u) du \right) = 0$ for every $g \in \mathcal{A}(M_1, f, M_2)$, when we integrate by parts we get that

$$I_g = \int_0^1 \left(\frac{1}{t} \int_0^t g(u) du \right) dt = - \int_0^1 \log(t) g(t) dt.$$

We consider the function $h : [0, 1] \rightarrow \mathbb{R}$ defined by the formula

$$h(t) = \begin{cases} t - t \log(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0. \end{cases}$$

Since $h(0) = 0 = \lim_{t \rightarrow 0} h(t)$, $h'(t) = -\log(t) > 0$ and $h''(t) = -\frac{1}{t} < 0$ for every $t \in (0, 1)$, h is continuous, strictly increasing and strictly concave, thus for every $g \in \mathcal{A}(M_1, f, M_2)$ the value $I_g = \int_0^1 g(t) (-\log(t)) dt$ may be expressed as the Riemann-Stieltjes integral $I_g = \int_0^1 g(t) dh(t)$.

Proposition 3.1. The quantity $\sup\{I_g : g \in \mathcal{A}(M_1, f, M_2)\}$ is equal to

$$f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right).$$

Moreover the supremum is uniquely attained by the function g_0 defined as

$$g_0(t) = \begin{cases} M_1 & \text{if } 0 \leq t \leq c \\ M_2 & \text{if } c < t \leq 1, \end{cases}$$

where $c = \frac{f - M_2}{M_1 - M_2}$.

Proof. We start with the second part. A direct calculation shows that

$$\int_0^1 g_0(t) dt = M_1 \frac{f - M_2}{M_1 - M_2} + M_2 \left(1 - \frac{f - M_2}{M_1 - M_2}\right) = f,$$

while $g_0(0) = M_1$, $g_0(1) = M_2$ and g_0 is left continuous; thus $g_0 \in \mathcal{A}(M_1, f, M_2)$. We also have that

$$\begin{aligned} I_{g_0} &= - \int_0^1 \log(t) g_0(t) dt = - \int_0^c M_1 \log(t) dt - \int_c^1 M_2 \log(t) dt \\ &= M_1 (h(c) - h(0)) + M_2 (h(1) - h(c)) \\ &= M_1 (c - c \log(c)) + M_2 (1 - (c - c \log(c))) \\ &= M_2 + (M_1 - M_2) c (1 - \log(c)) \\ &= M_2 + (M_1 - M_2) \frac{f - M_2}{M_1 - M_2} \left(1 - \log\left(\frac{f - M_2}{M_1 - M_2}\right)\right) \\ &= f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right). \end{aligned}$$

Thus $\sup\{I_g : g \in \mathcal{A}(M_1, f, M_2)\} \geq f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right)$.

We will show now that the number $f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right)$ is an upper bound of the set $\{I_g : g \in \mathcal{A}(M_1, f, M_2)\}$.

Let $g \in \mathcal{A}(M_1, f, M_2)$. We will calculate the Riemman Stieltjes integral $I_g = \int_0^1 g(t)dh(t)$ as the limit of the Riemman Stieltjes sums over the net of all partitions $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \dots < t_n = 1\}$ equipped with the right boundaries of the intervals $[t_{i-1}, t_i]$ as intermediate points.

$$\begin{aligned}
I_g &= \int_0^1 g(t)dh(t) \\
&= \lim_{\mathcal{P}} \sum_{i=1}^n g(t_i)(h(t_i) - h(t_{i-1})) \\
&= \lim_{\mathcal{P}} \left(\sum_{i=1}^n g(t_i)h(t_i) - \sum_{i=1}^n g(t_i)h(t_{i-1}) \right) \\
&= \lim_{\mathcal{P}} \left[-g(t_1)h(t_0) + \sum_{i=1}^{n-1} h(t_i)(g(t_i) - g(t_{i+1})) + g(t_n)h(t_n) \right] \\
&= \lim_{\mathcal{P}} \left[M_2 + \sum_{i=0}^{n-1} h(t_i)(g(t_i) - g(t_{i+1})) \right]
\end{aligned}$$

(we took into account that $h(t_0) = h(0) = 0$, $h(t_n) = h(1) = 1$ and $g(t_n) = g(1) = M_2$).

Since g is decreasing, for every partition $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \dots < t_n = 1\}$ we have that $g(t_i) - g(t_{i+1}) \geq 0$ for $i = 0, 1, \dots, n-1$ while $\sum_{i=0}^{n-1} (g(t_i) - g(t_{i+1})) = g(0) - g(1) = M_1 - M_2$ and thus the numbers $\left(\frac{g(t_i) - g(t_{i+1})}{M_1 - M_2}\right)_{i=0}^{n-1}$ serve as coefficients of a convex combination. Since

$$I_g = M_2 + (M_1 - M_2) \lim_{\mathcal{P}} \sum_{i=0}^{n-1} h(t_i) \frac{g(t_i) - g(t_{i+1})}{M_1 - M_2}$$

the fact that the function h is concave and continuous yields

$$I_g \leq M_2 + (M_1 - M_2)h\left(\lim_{\mathcal{P}} \sum_{i=0}^{n-1} \frac{g(t_i) - g(t_{i+1})}{M_1 - M_2} t_i\right). \quad (10)$$

But also

$$\begin{aligned}
&\lim_{\mathcal{P}} \sum_{i=0}^{n-1} (g(t_i) - g(t_{i+1}))t_i \\
&= \lim_{\mathcal{P}} \left(g(t_0)t_0 + \sum_{i=0}^{n-1} g(t_i)(t_i - t_{i-1}) - g(t_n)t_{n-1} \right) \\
&= M_1 \cdot 0 + \int_0^1 g(t)dt - M_2 \cdot 1 = f - M_2
\end{aligned}$$

and thus

$$\begin{aligned}
I_g &\leq M_2 + (M_1 - M_2)h\left(\frac{f - M_2}{M_1 - M_2}\right) \\
&= M_2 + (M_1 - M_2)\frac{f - M_2}{M_1 - M_2}\left(1 - \log \frac{f - M_2}{M_1 - M_2}\right) \\
&= M_2 + (f - M_2) + (f - M_2)\log \frac{M_1 - M_2}{f - M_2} \\
&= f + (f - M_2)\log \frac{M_1 - M_2}{f - M_2}.
\end{aligned}$$

Up to this point we have shown that

$$\sup\{I_g : g \in \mathcal{A}(M_1, f, M_2)\} = f + (f - M_2)\log \frac{M_1 - M_2}{f - M_2}$$

and that the supremum is attained by the function g_0 . It remains to prove that g_0 is the unique function in $\mathcal{A}(M_1, f, M_2)$ with this property. It is enough to show that for each $g \in \mathcal{A}(M_1, f, M_2)$ with $g \neq g_0$ we have that $I_g < f + (f - M_2)\log \frac{M_1 - M_2}{f - M_2}$.

Consider such a g . Since $g \neq g_0$ there exists y_0 a point of continuity of g such that $M_1 > g(y_0) > M_2$. We set $M = g(y_0)$. In the proof that is presented above we may consider only partitions of $[0, 1]$ containing the point y_0 i.e. partitions of the form

$$\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = y_0 < t_{k+1} < \cdots < t_n = 1\}.$$

Then

$$\begin{aligned}
I_g &= M_2 + (M_1 - M_2)\lim_{\mathcal{P}} \left(\sum_{i=0}^n h(t_i) \frac{g(t_i) - g(t_{i+1})}{M_1 - M_2} \right) \\
&= M_2 + (M_1 - M_2)\lim_{\mathcal{P}} \left(\frac{M_1 - M}{M_1 - M_2} \sum_{i=0}^{k-1} h(t_i) \frac{g(t_i) - g(t_{i+1})}{M_1 - M} \right. \\
&\quad \left. + \frac{M - M_2}{M_1 - M_2} \sum_{i=k}^{n-1} h(t_i) \frac{g(t_i) - g(t_{i+1})}{M - M_2} \right).
\end{aligned}$$

Since $\sum_{i=0}^{k-1} \frac{g(t_i) - g(t_{i+1})}{M_1 - M} = 1$ and $\sum_{i=k}^{n-1} \frac{g(t_i) - g(t_{i+1})}{M - M_2} = 1$, using the fact that h is concave we get that

$$\begin{aligned}
I_g &\leq M_2 + (M_1 - M_2)\lim_{\mathcal{P}} \left(\frac{M_1 - M}{M_1 - M_2} h\left(\sum_{i=0}^{k-1} \frac{g(t_i) - g(t_{i+1})}{M_1 - M} t_i\right) \right. \\
&\quad \left. + \frac{M - M_2}{M_1 - M_2} h\left(\sum_{i=k}^{n-1} \frac{g(t_i) - g(t_{i+1})}{M - M_2} t_i\right) \right).
\end{aligned}$$

But

$$\begin{aligned}
\lim_{\mathcal{P}} \sum_{i=0}^{k-1} (g(t_i) - g(t_{i+1}))t_i &= \lim_{\mathcal{P}} \left[g(t_0)t_0 + \sum_{i=1}^{k-1} g(t_i)(t_i - t_{i-1}) - g(t_k)t_{k-1} \right] \\
&= M_1 \cdot 0 + \int_0^{y_0} g(t)dt - My_0 = \int_0^{y_0} g(t)dt - My_0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\mathcal{P}} \sum_{i=k}^{n-1} (g(t_i) - g(t_{i+1}))t_i &= \lim_{\mathcal{P}} g(t_k)t_k + \sum_{i=k+1}^{n-1} g(t_i)(t_i - t_{i-1}) - g(t_n)t_{n-1} \\
&= M \cdot y_0 + \int_{y_0}^1 g(t)dt - M_2,
\end{aligned}$$

as g is continuous at y_0 and $t_{k-1} \xrightarrow{\mathcal{P}} y_0$.

Thus we get that

$$\begin{aligned}
I_g \leq M_2 + (M_1 - M_2) &\left[\frac{M_1 - M}{M_1 - M_2} h \left(\frac{\int_0^{y_0} g(t)dt - My_0}{M_1 - M} \right) \right. \\
&\left. + \frac{M - M_2}{M_1 - M_2} h \left(\frac{My_0 + \int_{y_0}^1 g(t)dt - M_2}{M - M_2} \right) \right].
\end{aligned}$$

We observe that the coefficients $a = \frac{M_1 - M}{M_1 - M_2}$ and $\beta = \frac{M - M_2}{M_1 - M_2}$ are positive, $a + \beta = 1$ while,

$$\frac{\int_0^{y_0} g(t)dt - My_0}{M_1 - M} < y_0 < \frac{My_0 + \int_{y_0}^1 g(t)dt - M_2}{M - M_2}$$

where we took into account that the function g is continuous at the points y_0 and 1 and $M_1 > g(y_0) = M > M_2$.

Therefore the fact that h is strictly concave yields

$$\begin{aligned}
I_g &< M_2 + (M_1 - M_2)h \left(\frac{M_1 - M}{M_1 - M_2} \cdot \frac{\int_0^{y_0} g(t)dt - My_0}{M_1 - M} \right. \\
&\quad \left. + \frac{M - M_2}{M_1 - M_2} \cdot \frac{My_0 + \int_{y_0}^1 g(t)dt - M_2}{M - M_2} \right) \\
&= M_2 + (M_1 - M_2)h \left(\frac{\int_0^1 g(t)dt - M_2}{M_1 - M_2} \right) \\
&= M_2 + (M_1 - M_2)h \left(\frac{f - M_2}{M_1 - M_2} \right) \\
&= f + (f - M_2) \log \left(\frac{M_1 - M_2}{f - M_2} \right).
\end{aligned}$$

The proof is complete. □

Similar arguments to those used in the previous proof lead to the following.

Proposition 3.2. If $(g_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{A}(M_1, f, M_2)$ such that $\lim_n I_{g_n} = f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right)$ then $g_n \xrightarrow{a.e.} g_0$ (and therefore also $g_n \xrightarrow{L^1} g_0$), where g_0 is the function defined in the statement of Proposition 3.1.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. First we observe that for $\phi \in \mathcal{C}_{X, \mathcal{T}}(M_1, f, M_2)$ we have that $\phi^* \in \mathcal{A}(M_1, f, M_2)$. Also from [8] we have that $(\mathcal{M}_{\mathcal{T}}\phi)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(s) ds$ for every $t \in (0, 1]$. Since the functions $\mathcal{M}_{\mathcal{T}}\phi$ and $(\mathcal{M}_{\mathcal{T}}\phi)^*$ are equimeasurable we get that

$$\begin{aligned} \int \mathcal{M}_{\mathcal{T}}\phi d\mu &= \int_0^1 (\mathcal{M}_{\mathcal{T}}\phi)^*(t) dt \\ &\leq \int_0^1 \left(\frac{1}{t} \int_0^t \phi^*(s) ds \right) dt \\ &= I_{\phi^*} \leq f + (f - M_2) \log\left(\frac{M_1 - M_2}{f - M_2}\right). \end{aligned}$$

where the last inequality follows from Proposition 3.1.

The sharpness of the above inequality is a consequence of Theorem 2.4 for $G_1(t) = t$, $G_2(t) = 1$, $t \in [0, +\infty)$ and $k = 1$, where $g = g_0$ is the function in the statement of Proposition 3.1 and the fact that since the probability measure space (X, \mathcal{A}, μ) is non-atomic we may easily find a measurable function $\phi : X \rightarrow \mathbb{R}$ such that $\phi^* = g_0$. The proof of Theorem 1.1 is complete. \square

4. PROOF OF THEOREM 1.2

In this section we will prove Theorem 1.2. We start with the following lemma.

Lemma 4.1. Let $g_1, g_2 : [0, +\infty) \rightarrow [0, +\infty)$ and $0 < k \leq 1$ such that:

- (a) g_1, g_2 are decreasing.
- (b) $g_1(t) \leq g_2(t)$ for every $t \in [0, +\infty)$.
- (c) g_1 is right continuous and g_2 is left continuous.
- (d) $\lim_{t \rightarrow +\infty} g_2(t) = 0$ and $g_2(0) = 1$. Then there exists $u \in [0, +\infty)$ such that $g_1(u) \leq k \leq g_2(u)$.

Proof. We set $u = \sup g_2^{-1}([k, +\infty))$. Since $g_2(0) = 1$ the set $g_2^{-1}([k, +\infty))$ is nonempty, while it is bounded from above since g_2 is decreasing and $\lim_{t \rightarrow +\infty} g_2(t) = 0$. Thus u is a well defined real number.

The left continuity of g_2 yields $g_2(u) \geq k$, thus it suffices to show that $g_1(u) \leq k$. If $g_1(u) > k$ from the right continuity of g_1 we get that $g_1(t) > k$ for all $t \in [u, u + \varepsilon)$ for some $\varepsilon > 0$. Thus $k < g_1(t) \leq g_2(t)$ for all $t \in [u, u + \varepsilon)$, which contradicts the definition of u . \square

Lemma 4.2. Let (X, \mathcal{A}, μ) be a measure space, let $g : X \rightarrow [0, +\infty)$ be a measurable function and $u \geq 0$. Let also D be a measurable set such that $[g > u] \subset D \subset [g \geq u]$. Then for every measurable set K such that $\mu(D) = \mu(K)$ we have that $\int_K g d\mu \leq \int_D g d\mu$.

Proof. Let ν denote the indefinite integral of g with respect to μ , i.e. the measure defined by the formula $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{A}$.

We set $V_1 = [g > u]$ and $V_2 = [g \geq u]$. We have that $\nu(K) = \nu(K \cap V_1) + \nu(K \setminus V_1) = \nu(V_1) - \nu(V_1 \setminus K) + \nu(K \setminus V_1)$ and taking into account that $V_1 \subset D \subset V_2$ we get that

$$\nu(K) - \nu(D) = -\nu(D \setminus V_1) - \nu(V_1 \setminus K) + \nu(K \setminus V_1) \quad (11)$$

Since for every $x \in D \setminus V_1$ we have that $x \in V_2$ and hence $g(x) \geq u$ and since $V_1 \subset D$ we get that

$$-\nu(D \setminus V_1) \leq -u\mu(D \setminus V_1) = -u\mu(D) + u\mu(V_1) \quad (12)$$

For every $x \in V_1 \setminus K$ we have that $g(x) > u$, thus

$$-\nu(V_1 \setminus K) \leq -u\mu(V_1 \setminus K) = -u\mu(V_1) + u\mu(K \cap V_1) \quad (13)$$

Finally for every $x \in K \setminus V_1$ we have that $g(x) \leq u$ thus

$$\nu(K \setminus V_1) \leq u\mu(K \setminus V_1) = u\mu(K) - u\mu(K \cap V_1) \quad (14)$$

Therefore from (11), (12), (13), (14) we get that

$$\nu(K) - \nu(D) \leq u\mu(K) - u\mu(D) = 0$$

i.e. $\int_K g d\mu \leq \int_D g d\mu$. □

Proof of Theorem 1.2. For the case $k \leq \frac{f}{M}$ it is obvious that for every $\phi \in \mathcal{C}_{X,\mathcal{T}}(M, f)$ and $K \in \mathcal{A}$ with $\mu(K) = k$ it holds that $\int_K \mathcal{M}_{\mathcal{T}}\phi d\mu \leq M\mu(K) = kM$.

We examine next the case where $\frac{f}{M} < k \leq 1$. Let $\phi \in \mathcal{C}_{X,\mathcal{T}}(M, f)$. The functions $g_1, g_2 : X \rightarrow [0, +\infty]$ defined as $g_1(t) = \mu([\mathcal{M}_{\mathcal{T}}\phi > t])$ and $g_2(t) = \mu([\mathcal{M}_{\mathcal{T}}\phi \geq t])$ are decreasing, g_1 is right continuous, g_2 is left continuous, $g_1(t) \leq g_2(t)$ for all t , $\lim_{t \rightarrow +\infty} g_2(t) = 0$ and $g_2(0) = 1$. By Lemma 4.1 there exists $u \geq 0$ such that $g_1(u) \leq k \leq g_2(u)$, i.e. $\mu([\mathcal{M}_{\mathcal{T}}\phi > u]) \leq k \leq \mu([\mathcal{M}_{\mathcal{T}}\phi \geq u])$. The fact that the probability measure space (X, \mathcal{A}, μ) is non-atomic yields the existence of a $D \in \mathcal{A}$ with $[\mathcal{M}_{\mathcal{T}}\phi > u] \subset D \subset [\mathcal{M}_{\mathcal{T}}\phi \geq u]$ such that $\mu(D) = k$. Lemma 4.2 implies that for every $K \in \mathcal{A}$ with $\mu(D) = \mu(K)$ the inequality $\int_K \mathcal{M}_{\mathcal{T}}\phi d\mu \leq \int_D \mathcal{M}_{\mathcal{T}}\phi d\mu$ holds.

We set $V_1 = [\mathcal{M}_{\mathcal{T}}\phi > u]$ and $V_2 = [\mathcal{M}_{\mathcal{T}}\phi \geq u]$. Since $V_1 \subset D \subset V_2$ and thus $\mu(V_1) \leq \mu(D) \leq \mu(V_2)$, there exists unique $s \in [0, 1]$ such that $k = \mu(D) = s\mu(V_1) + (1-s)\mu(V_2)$ hence $\mu(D \setminus V_1) = (1-s)\mu(V_2 \setminus V_1)$.

Claim.

$$\int_D \mathcal{M}_{\mathcal{T}}\phi d\mu = s \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + (1-s) \int_{V_2} \mathcal{M}_{\mathcal{T}}\phi d\mu$$

Proof.

$$\begin{aligned}
\int_D \mathcal{M}_{\mathcal{T}}\phi d\mu &= \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + \int_{D \setminus V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu \\
&= \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + u\mu(D \setminus V_1) \\
&= \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + u(1-s)\mu(V_2 \setminus V_1) \\
&= s \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + (1-s) \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + (1-s) \int_{V_2 \setminus V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu \\
&= s \int_{V_1} \mathcal{M}_{\mathcal{T}}\phi d\mu + (1-s) \int_{V_2} \mathcal{M}_{\mathcal{T}}\phi d\mu.
\end{aligned}$$

□

The set $V_1 = [\mathcal{M}_{\mathcal{T}}\phi > u]$ can be written as $V_1 = \bigcup_{i \in A} I_i$ where $(I_i)_{i \in A}$ is a pairwise almost disjoint family of sets in \mathcal{T} , such that for each $i \in I$ the set I_i is a maximal set of the family $\{I \in \mathcal{T} : \frac{1}{\mu(I)} \int_I \phi d\mu > u\}$.

We may also assume that the set $V_2 = [\mathcal{M}_{\mathcal{T}}\phi \geq u]$ can be written $V_2 = \bigcup_{j \in B} J_j$ where $(J_j)_{j \in B}$ is a pairwise almost disjoint family of sets in \mathcal{T} , such that for each $j \in J$ the set J_j is a maximal set of the family $\{J \in \mathcal{T} : \frac{1}{\mu(J)} \int_J \phi d\mu \geq u\}$.

This is possible because we may assume that ϕ satisfies the following additional property:

$$\forall x \in X \quad \exists I_x \in \mathcal{T} : \mathcal{M}_{\mathcal{T}}\phi(x) = \frac{1}{\mu(I_x)} \int_{I_x} \phi d\mu. \quad (15)$$

This is due to the following:

Remark. Let $\phi : X \rightarrow \mathbb{R}^+$ be a measurable function such that $\int_X \phi d\mu = f$ and $\|\phi\|_{\infty} = M$. If we define the sequence $(\phi_n)_{n \in \mathbb{N}}$ of functions as follows

$$\phi_n = \sum_{I \in \mathcal{T}_{(n)}} \text{Av}_I(\phi) \chi_I$$

(where χ_I denotes the characteristic function of I), then $\int_X \phi_n d\mu = f$, $\|\phi_n\|_{\infty} \leq M$ and each ϕ_n satisfies (15). Moreover, $(\mathcal{M}_{\mathcal{T}}\phi_n)_{n \in \mathbb{N}}$ is an increasing sequence of functions that converges pointwise to $\mathcal{M}_{\mathcal{T}}\phi$. This implies that $\int_B \mathcal{M}_{\mathcal{T}}\phi_n \rightarrow \int_B \mathcal{M}_{\mathcal{T}}\phi$ for every measurable subset B of X .

For each $i \in A$ we set $\mathcal{A}_i = \{I \in \mathcal{A} : I \subset I_i\}$, $\mathcal{T}_i = \{I \in \mathcal{T} : I \subset I_i\}$ and we define $\mu_i(I) = \frac{1}{\mu(I_i)} \mu(I)$ for every $I \in \mathcal{A}_i$. Then $(I_i, \mathcal{A}_i, \mu_i)$ is a non-atomic probability measure space and \mathcal{T}_i is a tree in \mathcal{A}_i . Denoting by ϕ_i the restriction of ϕ on I_i , by M_i the essential supremum of ϕ_i and setting $f_i(\phi) = \int \phi_i d\mu_i = \frac{1}{\mu(I_i)} \int_{I_i} \phi d\mu$ we observe that $0 \leq f_i(\phi) \leq M_i \leq M$ and

$\phi_i \in \mathcal{C}_{I_i, \mathcal{T}_i}(M_i, f_i(\phi), M_{2,i})$ for some $M_{2,i}$ with $0 \leq M_{2,i} \leq f_i(\phi)$. In the case where $M_{2,i} < f_i(\phi)$, Theorem 1.1 yields

$$\int_{I_i} \mathcal{M}_{\mathcal{T}_i} \phi_i d\mu_i \leq f_i(\phi) + (f_i(\phi) - M_{2,i}) \log \left(\frac{M_i - M_{2,i}}{f_i(\phi) - M_{2,i}} \right)$$

which implies that

$$\int_{I_i} \mathcal{M}_{\mathcal{T}_i} \phi_i d\mu_i \leq f_i(\phi) + f_i(\phi) \log \left(\frac{M_i}{f_i(\phi)} \right),$$

while in the case $M_{2,i} = f_i(\phi)$ the last inequality is obvious. The fact that each I_i is a maximal set of the family $\{I \in \mathcal{T} : \frac{1}{\mu(I)} \int_I \phi d\mu > u\}$ implies that $\mathcal{M}_{\mathcal{T}_i} \phi_i(x) = \mathcal{M}_{\mathcal{T}} \phi(x)$ for every $x \in I_i$.

Thus we get that

$$\frac{1}{\mu(I_i)} \int_{I_i} \mathcal{M}_{\mathcal{T}} \phi d\mu \leq f_i(\phi) + f_i(\phi) \log \left(\frac{M_i}{f_i(\phi)} \right) \quad \text{for every } i \in A.$$

By defining in a similar way $f_j(\phi)$ and M_j for every $j \in B$, we get that

$$\frac{1}{\mu(J_j)} \int_{J_j} \mathcal{M}_{\mathcal{T}} \phi d\mu \leq f_j(\phi) + f_j(\phi) \log \left(\frac{M_j}{f_j(\phi)} \right) \quad \text{for every } j \in B.$$

From the inequalities above we get that

$$\begin{aligned} \int_{V_1} \mathcal{M}_{\mathcal{T}} \phi d\mu &= \sum_{i \in A} \int_{I_i} \mathcal{M}_{\mathcal{T}} \phi d\mu \\ &\leq \sum_{i \in A} \mu(I_i) \left(f_i(\phi) + f_i(\phi) \log \left(\frac{M_i}{f_i(\phi)} \right) \right), \end{aligned}$$

and

$$\begin{aligned} \int_{V_2} \mathcal{M}_{\mathcal{T}} \phi d\mu &= \sum_{j \in B} \int_{J_j} \mathcal{M}_{\mathcal{T}} \phi d\mu \\ &\leq \sum_{j \in B} \mu(J_j) \left(f_j(\phi) + f_j(\phi) \log \left(\frac{M_j}{f_j(\phi)} \right) \right). \end{aligned}$$

Let now an arbitrary $K \in \mathcal{A}$ with $\mu(K) = k$. We have that

$$\begin{aligned}
\int_K \mathcal{M}_T \phi d\mu &\leq \int_D \mathcal{M}_T \phi d\mu \\
&= s \int_{V_1} \mathcal{M}_T \phi d\mu + (1-s) \int_{V_2} \mathcal{M}_T \phi d\mu \\
&\leq s \sum_{i \in A} \mu(I_i) \left(f_i(\phi) + f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) \right) \\
&\quad + (1-s) \sum_{j \in B} \mu(J_j) \left(f_j(\phi) + f_j(\phi) \log\left(\frac{M_j}{f_j(\phi)}\right) \right) \\
&= s \sum_{i \in A} \mu(I_i) f_i(\phi) + (1-s) \sum_{j \in B} \mu(J_j) f_j(\phi) \\
&\quad + s \sum_{i \in A} \mu(I_i) f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) + (1-s) \sum_{j \in B} \mu(J_j) f_j(\phi) \log\left(\frac{M_j}{f_j(\phi)}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_K \mathcal{M}_T \phi d\mu &\leq s \int_{V_1} \phi d\mu + (1-s) \int_{V_2} \phi d\mu \\
&\quad + s \sum_{i \in A} \mu(I_i) f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) + (1-s) \sum_{j \in B} \mu(J_j) f_j(\phi) \log\left(\frac{M_j}{f_j(\phi)}\right).
\end{aligned}$$

We observe that since $V_1 \subset V_2$ we have that

$$s \int_{V_1} \phi d\mu + (1-s) \int_{V_2} \phi d\mu \leq \int_{V_2} \phi d\mu \leq \int_X \phi d\mu = f. \quad (16)$$

Also, since $M_i \leq M$ for each $i \in A$ we get

$$\sum_{i \in A} \mu(I_i) f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) \leq \mu(V_1) \sum_{i \in A} \frac{\mu(I_i)}{\mu(V_1)} f_i(\phi) \log\left(\frac{M}{f_i(\phi)}\right).$$

Since $\sum_{i \in A} \frac{\mu(I_i)}{\mu(V_1)} = 1$ and the function h_0 defined as $h_0(t) = t \log\left(\frac{M}{t}\right)$ is concave we obtain

$$\begin{aligned}
&\sum_{i \in A} \frac{\mu(I_i)}{\mu(V_1)} f_i(\phi) \log\left(\frac{M}{f_i(\phi)}\right) \\
&= \sum_{i \in A} \frac{\mu(I_i)}{\mu(V_1)} h_0(f_i(\phi)) \\
&\leq h_0\left(\sum_{i \in A} \frac{\mu(I_i)}{\mu(V_1)} f_i(\phi)\right) \\
&= h_0\left(\frac{1}{\mu(V_1)} \int_{V_1} \phi d\mu\right).
\end{aligned}$$

Thus

$$\sum_{i \in A} \mu(I_i) f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) \leq \mu(V_1) h_0\left(\frac{1}{\mu(V_1)} \int_{V_1} \phi d\mu\right).$$

Similarly it is shown that

$$\sum_{j \in B} \mu(J_j) f_j(\phi) \log\left(\frac{M_j}{f_j(\phi)}\right) \leq \mu(V_2) h_0\left(\frac{1}{\mu(V_2)} \int_{V_2} \phi d\mu\right).$$

Therefore

$$\begin{aligned} & s \sum_{i \in A} \mu(I_i) f_i(\phi) \log\left(\frac{M_i}{f_i(\phi)}\right) + (1-s) \sum_{j \in B} \mu(J_j) f_j(\phi) \log\left(\frac{M_j}{f_j(\phi)}\right) \\ & \leq s \mu(V_1) h_0\left(\frac{1}{\mu(V_1)} \int_{V_1} \phi d\mu\right) + (1-s) \mu(V_2) h_0\left(\frac{1}{\mu(V_2)} \int_{V_2} \phi d\mu\right). \end{aligned}$$

Taking into account that the number s has been selected such that $s\mu(V_1) + (1-s)\mu(V_2) = \mu(D) = k$ and using again the fact that the function h_0 is concave we get that the quantity above is

$$\begin{aligned} & = k \left(\frac{s\mu(V_1)}{k} h_0\left(\frac{\int_{V_1} \phi d\mu}{\mu(V_1)}\right) + \frac{(1-s)\mu(V_2)}{k} h_0\left(\frac{\int_{V_2} \phi d\mu}{\mu(V_2)}\right) \right) \\ & \leq k h_0\left(\frac{s}{k} \int_{V_1} \phi d\mu + \frac{(1-s)}{k} \int_{V_2} \phi d\mu\right). \end{aligned}$$

We set $\Gamma = \frac{s}{k} \int_{V_1} \phi d\mu + \frac{(1-s)}{k} \int_{V_2} \phi d\mu$. Until this point we have shown that

$$\int_K \mathcal{M}_{\mathcal{T}} \phi d\mu \leq k\Gamma + k h_0(\Gamma) = k\Gamma + k\Gamma \log\left(\frac{Mk}{k\Gamma}\right).$$

We consider the function $h_1(t) = t + t \log\left(\frac{Mk}{t}\right)$, $t \in (0, Mk]$. The function h_1 is increasing, since $h_1'(t) = \log\left(\frac{Mk}{t}\right) > 0$ for every $t \in (0, Mk)$. Since from (16) we have that $k\Gamma \leq f$ we get that $h_1(k\Gamma) \leq h_1(f)$ (remember that we are studying the case where $\frac{f}{M} < k \leq 1$), thus $k\Gamma + k\Gamma \log\left(\frac{Mk}{k\Gamma}\right) \leq f + f \log\left(\frac{Mk}{f}\right)$.

Therefore

$$\int_K \mathcal{M}_{\mathcal{T}} \phi d\mu \leq f + f \log\left(\frac{Mk}{f}\right).$$

Up to this point we have shown that the left hand side of (9) in the statement of Theorem 1.2 is less than or equal to the right side. It remains to show that equality holds. We will again discern the cases $0 < k \leq \frac{f}{M}$ and $\frac{f}{M} < k \leq 1$.

Let's first consider the case $0 < k \leq \frac{f}{M}$. Using Lemma 2.2 we select a family $(I_i)_{i \in A}$ of almost pairwise disjoint sets in \mathcal{T} such that $\sum_{i \in A} \mu(I_i) = k$. We consider the function $\phi : X \rightarrow \mathbb{R}$ defined as

$$\phi(x) = \begin{cases} M & \text{if } x \in \bigcup_{i \in A} I_i \\ \frac{f-Mk}{1-k} & \text{if } x \in X \setminus \bigcup_{i \in A} I_i. \end{cases}$$

The function ϕ is measurable, while since $k \leq \frac{f}{M}$ we have $\phi \geq 0$. Also, since $f \leq M$ we have $\frac{f-Mk}{1-k} \leq M$ thus $\|\phi\|_\infty = M$. Finally

$$\begin{aligned} \int_X \phi d\mu &= M\mu\left(\bigcup_{i \in A} I_i\right) + \frac{f-Mk}{1-k}\mu\left(X \setminus \bigcup_{i \in A} I_i\right) \\ &= Mk + \frac{f-Mk}{1-k}(1-k) = f, \end{aligned}$$

thus $\phi \in \mathcal{C}_{X,\mathcal{T}}(M, f)$.

Since

$$\int_{\bigcup_{i \in A} I_i} \mathcal{M}_{\mathcal{T}}\phi d\mu = Mk$$

the proof in the case where $0 < k \leq \frac{f}{M}$ is complete.

We treat now the case $\frac{f}{M} \leq k \leq 1$. In this case it is enough to show that for every $\delta > 0$ there exists a $K \in \mathcal{A}$ with $\mu(K) = k$ and a $\phi \in \mathcal{C}_{X,\mathcal{T}}(M, f)$ such that

$$\int_K \mathcal{M}_{\mathcal{T}}\phi d\mu \geq (1-\delta)\left(f + f \log\left(\frac{Mk}{f}\right)\right).$$

As in the first case, using again Lemma 2.2, we select a family $(I_i)_{i \in A}$ of almost pairwise disjoint sets in \mathcal{T} such that $\sum_{i \in A} \mu(I_i) = k$. We set $K = \bigcup_{i \in A} I_i$, $f' = \frac{f}{k}$ (notice that $f' \leq M$) and for every $i \in A$ we set $\mathcal{A}_i = \{I \in \mathcal{A} : I \subset I_i\}$, $\mathcal{T}_i = \{I \in \mathcal{T} : I \subset I_i\}$ and we define $\mu_i(I) = \frac{1}{\mu(I_i)}\mu(I)$ for every $I \in \mathcal{A}_i$. For each $i \in A$, Theorem 1.1 yields the existence of a $\phi_i \in \mathcal{C}_{I_i, \mathcal{T}_i}(M, f', 0)$ such that $\int \mathcal{M}_{\mathcal{T}_i}\phi_i d\mu_i > (1-\delta)(f' + f' \log(\frac{M}{f'}))$.

We consider $\phi : X \rightarrow \mathbb{R}$ with $\phi|_{I_i} = \phi_i$ for each $i \in I$ and $\phi|(X \setminus \bigcup_{i \in A} I_i) = 0$. Then $\phi \geq 0$, $\|\phi\|_\infty = M$ and

$$\int_X \phi d\mu = \sum_{i \in A} \int_{I_i} \phi d\mu = \sum_{i \in A} \mu(I_i) f' = k \frac{f}{k} = f$$

hence $\phi \in \mathcal{C}_{X,\mathcal{T}}(M, f)$.

Finally we have that

$$\begin{aligned}
\int_K \mathcal{M}_{\mathcal{T}} \phi d\mu &= \sum_{i \in A} \int_{I_i} \mathcal{M}_{\mathcal{T}} \phi d\mu \\
&\geq \sum_{i \in A} \int_{I_i} \mathcal{M}_{\mathcal{T}_i} \phi_i d\mu \\
&= \sum_{i \in A} \mu(I_i) \int_{I_i} \mathcal{M}_{\mathcal{T}_i} \phi_i d\mu_i \\
&\geq \sum_{i \in A} \mu(I_i) (1 - \delta) (f' + f' \log(\frac{M}{f'})) \\
&= k(1 - \delta) (\frac{f}{k} + \frac{f}{k} \log(\frac{Mk}{f})) \\
&= (1 - \delta) (f + f \log(\frac{Mk}{f})).
\end{aligned}$$

Letting $\delta \rightarrow 0^+$ we obtain our result. \square

Remark 4.3. In the case that $f < M$ we may add the condition $\text{essinf}_X(\phi) = 0$ in the statement of Theorem 1.2 without affecting the value of the supremum.

Proof. We distinguish the following three cases

Case 1. $k = 1$.

This case is obviously a special case of Theorem 1.1 for $M_2 = 0$.

Case 2. $\frac{f}{M} \leq k < 1$.

In this case the extremal function ϕ that we constructed in the proof of Theorem 1.2 satisfies the desired condition.

Case 3. $0 < k < \frac{f}{M}$.

In this case the extremal function ϕ that we constructed in the proof of Theorem 1.2 does not satisfy the condition $\text{essinf}_X(\phi) = 0$. In order to overcome this difficulty, for every sufficiently small $\varepsilon > 0$ (namely $\varepsilon < 1 - \frac{f}{M}$) we construct a function ϕ_ε as follows. Keeping the notation that was used in the proof of Theorem 1.2, for fixed $i_0 \in A$, we choose a set $J_\varepsilon \in \mathcal{T}$ such that $0 < \mu(J_\varepsilon) < \varepsilon \mu(I_0) < 1 - \frac{f}{M}$ and $\mu(J_\varepsilon) < \frac{\varepsilon}{M}$. We define ϕ_ε by the formula

$$\phi(x) = \begin{cases} M & \text{if } x \in \bigcup_{i \in A} I_i \setminus J_\varepsilon \\ 0 & \text{if } x \in J_\varepsilon \\ \frac{f - M(k - \mu(J_\varepsilon))}{1 - k} & \text{if } x \in X \setminus \bigcup_{i \in A} I_i. \end{cases}$$

We notice that for this choice of ε and J_ε we have that $\frac{f - M(k - \mu(J_\varepsilon))}{1 - k} < M$ and thus $\|\phi_\varepsilon\|_\infty = M$, while $\int \phi_\varepsilon d\mu = f$, $\text{essinf}_X(\phi_\varepsilon) = 0$ and

$$\int_{\bigcup_{i \in A} I_i} \mathcal{M}_{\mathcal{T}} \phi_\varepsilon d\mu \geq \int_{\bigcup_{i \in A} I_i \setminus J_\varepsilon} \mathcal{M}_{\mathcal{T}} \phi_\varepsilon d\mu = M(k - \mu(J_\varepsilon)) \geq Mk - \varepsilon,$$

and this completes the proof of the remark. \square

An application of Theorem 1.2 is the following.

Corollary 4.4. For $0 \leq M_2 < f < M_1$ and $0 \leq k \leq 1$

$$\begin{aligned} & \sup \left\{ \int_K \mathcal{M}_{\mathcal{T}} \phi d\mu : \phi : X \rightarrow \mathbb{R}^+ \text{ is measurable, } \int_X \phi d\mu = f, \|\phi\|_{\infty} = M_1, \right. \\ & \quad \left. \text{essinf}_X(\phi) = M_2, K \text{ measurable, } \mu(K) = k \right\} \\ & = \begin{cases} kM_1 & \text{if } 0 < k \leq \frac{f-M_2}{M_1-M_2} \\ f - M_2(1-k) + (f - M_2) \log\left(\frac{(M_1-M_2)k}{f-M_2}\right) & \text{if } \frac{f-M_2}{M_1-M_2} < k \leq 1. \end{cases} \end{aligned} \quad (17)$$

Proof. Fix M_2, f, M_1 and k as in the statement, and let A be the value of the supremum. Then

$$\begin{aligned} A &= kM_2 + \sup \left\{ \int_K \mathcal{M}_{\mathcal{T}}(\phi - M_2) d\mu : \phi : X \rightarrow \mathbb{R}^+ \text{ is measurable,} \right. \\ & \quad \left. \int_X (\phi - M_2) d\mu = f - M_2, \|\phi - M_2\|_{\infty} = M_1 - M_2, \right. \\ & \quad \left. \text{essinf}_X(\phi - M_2) = 0, K \text{ measurable, } \mu(K) = k \right\} \\ &= kM_2 + \sup \left\{ \int_K \mathcal{M}_{\mathcal{T}} \psi d\mu : \psi : X \rightarrow \mathbb{R}^+ \text{ is measurable, } \int_X \psi d\mu = f - M_2, \right. \\ & \quad \left. \|\psi\|_{\infty} = M_1 - M_2, \text{essinf}_X(\psi) = 0, K \text{ measurable, } \mu(K) = k \right\}. \end{aligned}$$

The last supremum is evaluated using Remark 4.3. In the case that $0 < k \leq \frac{f-M_2}{M_1-M_2}$ we get that

$$A = kM_2 + k(M_1 - M_2) = kM_1$$

while in the case that $\frac{f-M_2}{M_1-M_2} < k \leq 1$ we get that

$$\begin{aligned} A &= kM_2 + (f - M_2) + (f - M_2) \log\left(\frac{(M_1 - M_2)k}{f - M_2}\right) \\ &= f - M_2(1 - k) + (f - M_2) \log\left(\frac{(M_1 - M_2)k}{f - M_2}\right) \end{aligned}$$

and this completes the proof. \square

REFERENCES

- [1] D. L. Burkholder, *Martingales and Fourier analysis in Banach spaces*, Probability and analysis (Varenna 1985), 61–108, Lecture Notes in Math., 61-108, **1206**, Springer, Berlin, 1986.
- [2] D. L. Burkholder, *Explorations in martingale theory and its applications*, École d'Été de Probabilités de Saint-Flour XIX—1989, 1-66, Lecture Notes in Math., **1464**, Springer, Berlin, 1991.
- [3] A. D. Melas, *The Bellman functions of dyadic-like maximal operators and related inequalities*, Adv. Math. **192** (2005), no. 2, 310-340.
- [4] A. D. Melas. *Sharp general local estimates for dyadic-like maximal operators and related Bellman functions*, Adv. in Math. **220** (2009), no. 2, 367-426.

- [5] A. D. Melas, *Dyadic-like maximal operators on $L \log L$ functions*, J. Funct. Anal. **257** (2009), no. 6, 1631-1654.
- [6] E. N. Nikolidakis, A. D. Melas, *A sharp integral rearrangement inequality for the dyadic maximal operator and applications*, Appl. and Comput. Harmon. Anal., **38** (2015), no. 2, 242-261.
- [7] E. N. Nikolidakis, *Extremal problems related to maximal dyadic-like operators*, J. Math. Anal. Appl, **369** (2010), no.1, 377-385.
- [8] E. N. Nikolidakis, *The Bellman function of the dyadic maximal operator in connection with the Dyadic Carleson Imbedding Theorem and related inequalities*, arXiv: 1905.08091.
- [9] E. M. Stein, *Note on the class $L \log L$* , Studia Math. **32** (1969), 305-310.
- [10] G. Wang, *Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion*, Proceedings of the American Mathematical Society **112** (1991), no. 2, 579-586.
- [11] N. Wiener, *The ergodic theorem*, Duke Math. J. **5** (1939), no. 1, 1-18.

(E. Nikolidakis) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, GREECE
Email address: enikolid@uoi.gr

(A. Tolias) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, GREECE
Email address: atolias@uoi.gr