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NUMERICAL SOLUTION OF INTERNAL-WAVE SYSTEMS IN THE INTERMEDIATE LONG WAVE AND THE BENJAMIN-ONO REGIMES

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To the memory of Professor Dougalis

Abstract. The paper is concerned with the numerical approximation of the Intermediate Long Wave and Benjamin-Ono systems, that serve as models for the propagation of interfacial internal waves in a two-layer fluid system in particular physical regimes. The paper focuses on two issues of approximation. First, the spectral Fourier-Galerkin method is used to discretize in space the corresponding periodic initial-value problems, and the error of the semidiscretizations is analyzed. The second issue concerns the numerical generation of solitary-wave solutions of the systems. We use acceleration techniques to improve the computation of the approximate solitary waves and check their performance with numerical examples.

1. Introduction

In this paper we consider the numerical approximation of two one-dimensional, nonlocal systems of nonlinear partial differential equations (PDEs) of dispersive wave type. The systems have been derived in [8] as models describing the propagation of internal waves in a two-layer interface problem with rigid upper and lower boundaries and under two different regimes in the case of a shallow upper layer and small-amplitude deformations in the lower layer. The idealized

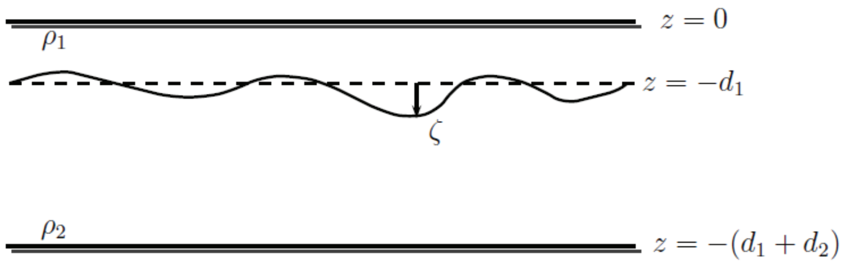

Figure 1. Idealized model of internal wave propagation in a two-layer interface problem: $\rho_2 > \rho_1; d_2 > d_1$; $\zeta(x,t)$ denotes the downward vertical displacement of the interface from its level of rest at (x,t).

model is sketched in Figure 1. It consists of two layers of inviscid, homogeneous, incompressible

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fluids with depths d_1, d_2 and densities $\rho_1 < \rho_2$. The upper and lower layers are bounded above and below, respectively, by rigid lids, with the origin of the vertical coordinate at the top.

In [8] the Euler equations with interface are reformulated in terms of two nonlocal operators linking the velocity potentials associated with the two layers and evaluated at the interface. This approach is then used to derive different asymptotic models, with which the Euler system is consistent. The models are valid in specific physical regimes described in terms of the parameters

$$\epsilon = \frac{a}{d_1},\ \mu = \frac{d_1}{\lambda^2},\ \epsilon_2 = \frac{a}{d_2},\ \mu_2 = \frac{d_2}{\lambda^2},$$

where a and λ denote, respectively, a typical amplitude and wavelength of the interfacial deviation.

In the present paper we focus on two of these regimes. As previously mentioned, the upper layer is assumed to be shallow ($\mu \ll 1$) while for the lower layer the deformations are assumed to be of small amplitude ($\epsilon_2 \ll 1$). Under these conditions, two situations are considered:

(i) The Intermediate Long Wave (ILW) regime: In this case, small amplitude deformations are additionally assumed for the upper layer; specifically, it is supposed that

$$\mu \sim \epsilon^2 \sim \epsilon_2 \ll 1, \mu_2 \sim 1.$$

In 1D, the corresponding system in nondimensional, unscaled form are given by the equations

$$\begin{aligned} &\left[1 + \tfrac{\alpha}{\gamma}|D|\coth|D|\right]\zeta_t + \tfrac{1}{\gamma}\left((1-\zeta)u\right)_x + \tfrac{(\alpha-1)}{\gamma^2}(|D|\coth|D|)u_x = 0\ , \\ &u_t + (1-\gamma)\zeta_x - \tfrac{1}{2\gamma}(u^2)_x = 0\ , \end{aligned} \tag{1.1}$$

where $\zeta = \zeta(x,t)$ denotes the interfacial deviation, $\gamma = \frac{\rho_1}{\rho_2} < 1$, $\alpha \geq 1$ is a modelling parameter, and the nonlocal operator $|D|$ has the Fourier symbol

$$\widehat{|D|f}(k) = |k|\widehat{f}(k),\ k \in \mathbb{R},$$

with $\widehat{f}(k)$ standing for the Fourier transfom of f at k. In (1.1) x and t are proportional to distance along the fluid channel and time respectively, and $u = u(x,t)$ is a velocity variable.

(ii) The Benjamin-Ono (B-O) regime: This corresponds to the range of parameters

$$\mu \sim \epsilon^2 \sim \epsilon_2 \ll 1, \mu_2 = \infty,$$

and the resulting 1D version of the corresponding systems in nondimensional, unscaled form is

$$\begin{aligned} &\left[1 + \tfrac{\alpha}{\gamma}|D|\right]\zeta_t + \tfrac{1}{\gamma}\left((1-\zeta)u\right)_x + \tfrac{(\alpha-1)}{\gamma^2}|D|u_x = 0\ , \\ &u_t + (1-\gamma)\zeta_x - \tfrac{1}{\gamma}uu_x = 0\ , \end{aligned} \tag{1.2}$$

The parameter α is introduced in the derivation of (1.1) and (1.2), [8] (see also [6]) as a way to remodeling the dispersive effects with a BBM-type term, [5]. The range of values of α is also related to modelling requirements. In this sense, the linear well-posedness, studied along with the consistency of the Euler system with (1.1) and (1.2) in [8], requires $\alpha \geq 1$. On the other hand, the Cauchy problem for (1.1) and (1.2) has been studied by Xu, [18]. In the case of (1.1), Xu showed, among other points, local and long-time existence of solutions for $\alpha > 1$. In addition, and for $\alpha > 1$ as well, it is noted in [18], Remark 4.2, that these properties also hold for (1.2). Note that similar systems to (1.1), (1.2) have been considered in [10].

The connection of (1.1) and (1.2) with the corresponding classical, unidirectional models is also treated in [8], and a more detailed description is made in [6]. The derivation starts from considering the linear wave equation obtained from neglecting higher-order terms in the scaled versions of (1.1) and (1.2); this allows to decouple the propagation of the waves to the left and to the right, and to reduce the dynamics to that of two unidirectional models. In each case (1.1) and (1.2), the introduction of suitable higher-order corrections of u, involving ζ, leads to a family of one-way models (the so-called regularized ILW and B-O equations) for the interfacial deviation, depending on the parameter α and which includes the classical ILW and B-O equations, corresponding to $\alpha = 0$.

Existence of smooth solitary-wave solutions of (1.1) and (1.2) was recently proved in [4], with arguments based on the implicit function theorem. Furthermore, the solitary waves of the ILW systems were proved to decay exponentially, while those of the B-O systems to decay like $1/x^2$. The numerical generation of solitary waves of (1.1) and (1.2) was studied in [6], where three iterative techniques, two of them based on the Petviashvili method, [14, 12], and the third given by the Conjugate-Gradient-Newton (CGN) method, [19], were introduced and their performance was compared. Approximations of some solitary-wave solutions for (1.1) and (1.2), whose existence was not covered by the results in [4], were computed, and the resulting profiles were compared with solitary waves of the corresponding unidirectional ILW and BO equations.

The present paper aims at contributing to the study of the numerical approximation of (1.1) and (1.2) in a two-fold way:

(i) We discretize in space the periodic initial-value problem (IVP) for the systems (1.1) and (1.2), using the spectral Fourier-Galerkin method, and prove, in section 2, error estimates for the ensuing semidiscretizations. While there exist error analyses of spectral discretizations of the one-way ILW and B-O equations, cf. e. g. [13], we are not aware of any similar results in the case of the systems. The estimates are much along the lines of those concerning other models for internal waves. like e. g. [11], but the presence of nonlocal terms in (1.1) and (1.2) makes, in our view, the main difference in the proofs.

(ii) Section 3 is devoted to the numerical generation of solitary waves of the systems. Here we modify the Petviashvili iteration, implemented in [6], by introducing a vector extrapolation method, [15], to accelerate the convergence. The use of vector extrapolation techniques for the numerical generation of traveling waves was proposed in [3] and the aim here is improving the performance obtained in [6] with their inclusion in the numerical procedure of the generation of the profiles. This is illustrated with some numerical examples.

The following notation will be used. On the interval $(0,1)$, the inner product and norm on $L^2 = L^2(0,1)$ are denoted by $(\cdot,\cdot)$ and $||\cdot||$, respectively. For real $\mu \geq 0$, H^μ will stand for the L^2-based periodic Sobolev spaces on $[0,1]$ with the norm given by

$$||g||_\mu = \left(\sum_{k\in\mathbb{Z}}(1+k^2)^\mu|\widehat{g}(k)|^2\right)^{1/2}, \; g \in H^\mu,$$

where $\widehat{g}(k)$ denotes the kth Fourier coefficient of g. For $1 \leq p \leq \infty$ $W^{\mu,p} = W^{\mu,p}(0,1)$ stands for the Sobolev space of periodic functions on $(0,1)$ of order μ, whose generalized derivatives are in $L^p = L^p(0,1)$. The norm on L^∞ will be denoted by $|\cdot|_\infty$, and that on $W^{\mu,\infty}$ by $||\cdot||_{\mu,\infty}$. For an integer $N \geq 1$, $(\cdot,\cdot)_N$ will denote the Euclidean inner product in $\mathbb{C}^{2N}$, and the associated norm will be denoted by $||\cdot||_N$.

2. Error estimates of the spectral semidiscretizations

In order to describe and analyze the spectral semidiscretizations of (1.1) and (1.2) some preliminaries are needed. In the sequel we will assume that $\alpha > 1$, so that the theory in [18] is valid. We write the nonlocal operator in (1.1) as

$$g(D) := \frac{\alpha}{\gamma}|D|\coth|D|, \tag{2.1}$$

with symbol $g(k) = \frac{\alpha}{\gamma}|k|\coth|k|$. We observe that $g(k)$ behaves like $\frac{\alpha}{\gamma}|k|$ for $|k| \gg 1$, and like $\frac{\alpha}{\gamma}\left(1 + \frac{k^2}{3} + O(k^4)\right)$ for small $|k|$. Consequently, the symbol of $(1 + g(D))^{-1}$, i. e.

$$\left(1 + \frac{\alpha}{\gamma}|k|\coth|k|\right)^{-1},$$

is bounded for all $k \in \mathbb{R}$, and is of $O\left(\frac{1}{|k|}\right)$ as $|k| \to \infty$.

On the other hand, since in the B-O case the nonlocal term is

$$g(D) = \frac{\alpha}{\gamma}|D|, \tag{2.2}$$

whose symbol is simply $\frac{\alpha}{\gamma}|k|$, we see again that the symbol of $(1 + g(D))^{-1}$, i. e. $\left(1 + \frac{\alpha}{\gamma}|k|\right)^{-1}$, is bounded for all $k \in \mathbb{R}$ and is of $O\left(\frac{1}{|k|}\right)$ as $|k| \to \infty$.

We consider the periodic IVP for (1.1) and (1.2) on the spatial interval $[0, 1]$ and written for $0 \le x \le 1,\ 0 \le t \le T$ in the form

$$\begin{aligned}
&\zeta_t + \frac{1}{\gamma}(1 + g(D))^{-1}\left(1 + \frac{(\alpha - 1)}{\alpha}g(D)\right)u_x = \frac{1}{\gamma}(1 + g(D))^{-1}(\zeta u)_x,\\
&u_t + (1 - \gamma)\zeta_x = \frac{1}{2\gamma}\partial_x(u^2),\\
&\zeta(x, 0) = \zeta_0(x),\ u(x, 0) = u_0(x),
\end{aligned} \tag{2.3}$$

where ζ_0, u_0 are given 1-periodic smooth functions on $[0, 1]$. We note again that for the ILW system (1.1) $g(D)$ is given by (2.1), while, for the B-O system, by (2.2). We assume that the IVP (2.3) has a unique solution which is sufficiently smooth for the purposes of the error estimation.

We introduce the nonlocal operators

$$\mathcal{T} := (1 + g(D))^{-1}, \quad \mathcal{J} := (1 + g(D))^{-1}\left(1 + \frac{(\alpha - 1)}{\alpha}g(D)\right).$$

We have previously examined the symbol of $\mathcal{T}$. The symbol of $\mathcal{J}$ is $\frac{1+\frac{(\alpha-1)}{\alpha}g(k)}{1+g(k)}$, which, in view of our assumptions on α and γ, is well defined and bounded for $k \in \mathbb{R}$.

Let $N \ge 1$ be an integer, and consider the finite dimensional space

$$S_N = \mathrm{span}\{e^{2\pi ikx},\ k \in \mathbb{Z}, -N \le k \le N\}.$$

We recall several properties of the L^2-projection operator onto S_N

$$P_N v = \sum_{|k| \le N} \widehat{v}(k)e^{2\pi ikx},$$

where $\widehat{v}(k)$ is the kth Fourier coefficient of v.

- P_N commutes with ∂_x.

- Given integers $0 \leq j \leq \mu$, and for any $v \in H^\mu, \mu \geq 1$,
$$||v - P_N v||_j \leq CN^{j-\mu}||v||_\mu, \tag{2.4}$$
$$|v - P_N v|_\infty \leq CN^{1/2-\mu}||v||_\mu, \tag{2.5}$$
for some constant C independent of N.

We will also use the following inverse inequalities. Given $0 \leq j \leq \mu$, there exists a constant C independent of N, such that for any $\psi \in S_N$
$$||\psi||_\mu \leq CN^{\mu-j}||\psi||_j,\ ||\psi||_{\mu,\infty} \leq CN^{1/2+\mu-j}||\psi||_j. \tag{2.6}$$

In what follows, C will denote a constant independent of N.

Now we may define the semidiscretizations. The semidiscrete spectral Fourier-Galerkin approximation of (2.3) is defined by the real-valued functions $\zeta_N, u_N : [0,T] \to S_N$ satisfying for $0 \leq t \leq T$
$$\begin{aligned}
&\zeta_{N,t} + \frac{1}{\gamma}(1+g(D))^{-1}\left(1 + \frac{(\alpha-1)}{\alpha}g(D)\right)u_{N,x} = \frac{1}{\gamma}(1+g(D))^{-1}\partial_x P_N(\zeta_N u_N),\\
&u_{N,t} + (1-\gamma)\zeta_{N,x} = \frac{1}{2\gamma}\partial_x P_N(u_N^2),\\
&\zeta_N\big|_{t=0} = P_N\zeta_0,\ u_N\big|_{t=0} = P_N u_0.
\end{aligned} \tag{2.7}$$

The IVP (2.7) is implemented in its Fourier component form
$$\begin{aligned}
&\widehat{\zeta}_{N,t} + \frac{1}{\gamma(1+g(2\pi k))}\left(1 + \frac{(\alpha-1)}{\alpha}g(2\pi k)\right)(2\pi ik)\widehat{u}_N = \frac{2\pi ik}{\gamma(1+g(2\pi k))}\widehat{\zeta_N u_N},\\
&\widehat{u}_{N,t} + (1-\gamma)(2\pi ik)\widehat{\zeta}_N = \frac{1}{2\gamma}(2\pi ik)\widehat{u_N^2},\\
&\widehat{\zeta}_N(k,0) = \widehat{\zeta}_0(k),\ \widehat{u}_N(k,0) = \widehat{u}_0(k),
\end{aligned}$$
where $\widehat{\zeta}_N = \widehat{\zeta}_N(k,t), \widehat{u}_N = \widehat{u}_N(k,t), -N \leq k \leq N, t \geq 0$ denote the Fourier coefficients of ζ_N and u_N respectively.

The IVP (2.7) clearly has a local in time solution; part of the proof of the following proposition is showing that this solution can be extended up to $t = T$.

Proposition 2.1. *(ILW systems.) Assume that the solution ζ, u of (2.3) for $\alpha > 1$, and g given by (2.1), is such that $\zeta, u \in H^\mu, \mu > 3/2$ for $0 \leq t \leq T$. Then, for N sufficiently large,*
$$\max_{0\leq t\leq T}(||\zeta_N - \zeta|| + ||u_N - u||) \leq CN^{1-\mu}. \tag{2.8}$$

Proof. The general plan of the proof resembles that of [11], where another class of asymptotic models for internal waves was analyzed. We let $\theta = \zeta_N - P_N\zeta, \rho = P_N\zeta - \zeta$, so that $\zeta_N - \zeta = \theta + \rho$, and $\xi = u_N - P_N u, \sigma = P_N u - u$, so that $u_N - u = \xi + \sigma$. Applying P_N on both sides of the PDEs in (2.3) and subtracting from the respective semidiscrete equations in (2.7) we obtain
$$\begin{aligned}
&\theta_t + \frac{1}{\gamma}\mathcal{J}\xi_x = \frac{1}{\gamma}\mathcal{T}\partial_x P_N A,\\
&\xi_t + (1-\gamma)\theta_x = \frac{1}{2\gamma}\partial_x P_N B, \qquad t \geq 0\\
&\theta\big|_{t=0} = 0,\ \xi\big|_{t=0} = 0,
\end{aligned} \tag{2.9}$$

where it is straightforward to see that

$$\begin{aligned} A &= u\rho + \zeta\sigma + u\theta + \zeta\xi + \sigma\theta + \rho\xi + \rho\sigma + \theta\xi, \\ B &= u\sigma + u\xi + \sigma\xi + \frac{1}{2}\sigma^2 + \frac{1}{2}\xi^2. \end{aligned}$$

Using a standard trick for rational functions, cf. e. g. [18], we may write

$$\mathcal{J} = (1+g(D))^{-1}\left(1 + \frac{(\alpha-1)}{\alpha}g(D)\right) = \frac{\alpha-1}{\alpha} + \frac{1}{\alpha}(1+g(D))^{-1} = \frac{\alpha-1}{\alpha} + \frac{1}{\alpha}\mathcal{T},$$

and, therefore, may simplify the equations somewhat having just one nonlocal operator in the problem. We rewrite accordingly (2.9) as

$$\begin{aligned} &\theta_t + \frac{1}{\gamma}\frac{\alpha-1}{\alpha}\xi_x + \frac{1}{\alpha\gamma}\mathcal{T}\xi_x = \frac{1}{\gamma}\mathcal{T}\partial_x P_N A, \\ &\xi_t + (1-\gamma)\theta_x = \frac{1}{2\gamma}\partial_x P_N B, \qquad t \geq 0, \\ &\theta\big|_{t=0} = 0,\ \xi\big|_{t=0} = 0. \end{aligned} \tag{2.10}$$

We will use the standard energy method to estimate θ and ξ. Taking the L^2 inner products of the semidiscrete equations in (2.10) with θ and ξ, respectively, we have

$$(\theta_t, \theta) + \frac{1}{\gamma}\frac{\alpha-1}{\alpha}(\xi_x, \theta) + \frac{1}{\alpha\gamma}(\mathcal{T}\xi_x, \theta) = \frac{1}{\gamma}(\mathcal{T}A_x, \theta), \tag{2.11}$$

$$(\xi_t, \xi) - (1-\gamma)(\theta, \xi_x) = \frac{1}{2\gamma}(B_x, \xi). \tag{2.12}$$

Hence, multiplying (2.12) by $\frac{\alpha-1}{\alpha\gamma(1-\gamma)}$ and adding to (2.11) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(||\theta||^2 + \frac{\alpha-1}{\alpha\gamma(1-\gamma)}||\xi||^2\right) &= -\frac{1}{\alpha\gamma}(\mathcal{T}\xi_x, \theta) + \frac{1}{\gamma}(\mathcal{T}A_x, \theta) \\ &\quad + \frac{\alpha-1}{2\alpha\gamma^2(1-\gamma)}(B_x, \xi). \end{aligned} \tag{2.13}$$

Note that $\frac{\alpha-1}{2\alpha\gamma^2(1-\gamma)} > 0$. Also note that the operator $\mathcal{T}\partial_x$, with symbol $\frac{ik}{1+g(k)}$, is bounded in L^2. Therefore we have by (2.13), as long as the solution of (2.10) (or (2.7)) exists, that

$$\frac{d}{dt}\left(||\theta||^2 + ||\xi||^2\right) \leq C\left(||\xi||||\theta|| + ||A||||\theta|| + |(B_x, \xi)|\right). \tag{2.14}$$

We bound now the two last terms in the right-hand side of (2.14). By the definition of A we have

$$\begin{aligned} ||A|| \leq\ & |u|_\infty||\rho|| + |\zeta|_\infty||\sigma|| + |u|_\infty||\theta|| + |\zeta|_\infty||\xi|| + |\sigma|_\infty||\theta|| \\ & + |\rho|_\infty||\xi|| + |\rho|_\infty||\sigma|| + |\theta|_\infty||\xi||. \end{aligned}$$

Let $t_N \in (0, T]$ be the maximal temporal instance such that

$$|\theta|_\infty \leq 1,\ 0 \leq t \leq t_N. \tag{2.15}$$

Then, by the approximation properties of S_N (2.4), (2.5), the inverse inequalities (2.6), and the fact that $\mu \geq 1$, we get by the above estimate of $||A||$ that for $0 \leq t \leq t_N$

$$||A|| \leq C\left(N^{-\mu} + ||\theta|| + ||\xi||\right), \tag{2.16}$$

where C is independent of N. To bound the term (B_x, ξ) note that by periodicity

$$(B_x, \xi) = ((u\sigma)_x, \xi) + ((u\xi)_x, \xi) + ((\sigma\xi)_x, \xi) + (\sigma\sigma_x, \xi). \tag{2.17}$$

Since $\mu \geq 3/2$

$$|((u\sigma)_x, \xi)| \leq |u_x|_\infty ||\sigma|| ||\xi|| + |u|_\infty ||\sigma_x|| ||\xi|| \leq CN^{1-\mu}||\xi||.$$

Also, for the same reason

$$\begin{aligned} |((u\xi)_x, \xi)| &= \frac{1}{2}|(u_x\xi, \xi)| \leq \frac{1}{2}|u_x|_\infty ||\xi||^2 \leq C||\xi||^2, \\ |((\sigma\xi)_x, \xi)| &= \frac{1}{2}|(\sigma_x\xi, \xi)| \leq \frac{1}{2}|\sigma_x|_\infty ||\xi||^2 \leq C||\xi||^2, \end{aligned}$$

and

$$|(\sigma\sigma_x, \xi)| \leq |\sigma_x|_\infty ||\sigma|| ||\xi|| \leq CN^{\frac{3}{2}-2\mu}||\xi|| \leq CN^{-\mu}||\xi||.$$

From these estimates and (2.17) we conclude that

$$|(B_x, \xi)| \leq C\left(N^{2(1-\mu)} + ||\xi||^2\right), \tag{2.18}$$

as long as the solution of (2.10) exists. Therefore, (2.14), (2.16) and (2.18) give for $0 \leq t \leq t_N$

$$\frac{d}{dt}(||\theta||^2 + ||\xi||^2) \leq C\left(N^{2(1-\mu)} + ||\theta||^2 + ||\xi||^2\right),$$

from which, by Gronwall's inequality we get

$$||\theta|| + ||\xi|| \leq CN^{1-\mu}, \tag{2.19}$$

for $0 \leq t \leq t_N$, where C is independent of N and t_N. Since by (2.19) $|\theta|_\infty \leq CN^{3/2-\mu}$ and $\mu > 3/2$, we infer that t_N was not maximal in (2.15) for N sufficiently large, and in the customary way the existence of solutions of (2.9) and the validity of (2.19) may be extended to $t = T$. The estimate (2.8) follows. □

As far as the B-O case is concerned, recall that the symbol of $(1 + g(D))^{-1}$ is also bounded for all $k \in \mathbb{R}$ and is of $O\left(\frac{1}{|k|}\right)$ as $|k| \to \infty$. This was the basic property that we used in the error analysis of the spectral semidiscretization of (1.1). Hence the proof of Proposition 2.1 can be easily adapted to the B-O case for $\alpha > 1$. Without proof we state:

Proposition 2.2. *(B-O systems.) Assume that the solution ζ, u of the periodic IVP for (1.2) for $\alpha > 1$ is such that $\zeta, u \in H^\mu, \mu > 3/2$ for $0 \leq t \leq T$. Let (ζ_N, u_N) be the solution of the Fourier-Galerkin semidiscretization of the periodic IVP, defined by (2.7), where now g is given by (2.2). Then (ζ_N, u_N) exists uniquely up to $t = T$ and satisfy for N sufficiently large*

$$\max_{0\leq t\leq T}(||\zeta_N - \zeta|| + ||u_N - u||) \leq CN^{1-\mu},$$

for some constant C independent of N.

3. Solitary wave solutions

The ILW and B-O systems (1.1) and (1.2) have been shown to possess solitary-wave solutions. These are solutions $\zeta = \zeta(x-ct), u = u(x-ct), c \neq 0$, where $\zeta(x), u(x) \to 0$ as $|x| \to \infty$, and that satisfy the system

$$-c(1+g(D))\zeta + \frac{1}{\gamma}\left(1 + \frac{(\alpha-1)}{\alpha} g(D)\right) u = \frac{1}{\gamma}\zeta u,$$
$$-cu + (1-\gamma)\zeta = \frac{1}{2\gamma}u^2, \tag{3.1}$$

where $g(D)$ is given by (2.1) or (2.2). The existence of smooth solutions of (3.1) was proved by Angulo-Pava and Saut, [4], for some range of speeds c, using the implicit function theorem. Properties of their asymptotic decay were also proved in the same paper, ensuring that in the ILW case the solitary waves decay exponentially, while in the B-O case, the decay is algebraic, like $1/|x|^2$. (Note that, in both cases, the asymptotic behaviour is the same as that of the solitary wave solutions of the corresponding unidirectional models.)

The numerical generation of approximate solutions of (3.1) was studied in [6]. We summarize the numerical technique used. Let $l > 0$ be large enough, $N \geq 1$ be an even integer, and discretize the periodic problem for (3.1) on $[-l, l]$ with a Fourier collocation method based on the N collocation points $x_j = -l + jh, j = 0, \ldots, N-1, h = 2l/N$. The approximation to the solitary wave (ζ, u) is then represented by the nodal values $\zeta_h = (\zeta_{h,0}, \ldots, \zeta_{h,N-1})^T$ and $u_h = (u_{h,0}, \ldots, u_{h,N-1})^T$, where $\zeta_{h,j} \approx \zeta(x_j), u_{h,j} \approx u(x_j), j = 0, \ldots, N-1$, and ζ_h, u_h satisfy the system

$$S\begin{pmatrix}\zeta_h\\ u_h\end{pmatrix} = F(\zeta_h, u_h) := \frac{1}{\gamma}\begin{pmatrix}\zeta_h.u_h\\ (u_h.^2)/2\end{pmatrix}, \tag{3.2}$$

where S is the $2N$-by-$2N$ matrix

$$S := \begin{pmatrix}-c(I_N + g(D_N)) & \frac{1}{\gamma}(I_N + \frac{\alpha-1}{\alpha} g(D_N))\\ (1-\gamma)I_N & -cI_N\end{pmatrix}, \tag{3.3}$$

being I_N the N-by-N identity matrix and D_N the N-by-N Fourier pseudospectral differentiation matrix. The dots on the right-hand side of (3.2) signify Hadamard, component-wise, products. The system (3.2), (3.3) is implemented in its Fourier component form. Thus for $-N/2 \leq k \leq N/2-1$, the kth discrete Fourier components of ζ_h and u_h, denoted by $\widehat{\zeta_h}(k), \widehat{v_h}(k)$, resp., satisfy the fixed point system

$$\underbrace{\begin{pmatrix}-c(1+g(\widetilde{k})) & \frac{1}{\gamma}(1+\frac{\alpha-1}{\alpha}g(\widetilde{k}))\\ (1-\gamma) & -c\end{pmatrix}}_{S(\widetilde{k})}\begin{pmatrix}\widehat{\zeta_h}(\widetilde{k})\\ \widehat{v_h}(\widetilde{k})\end{pmatrix} = \underbrace{\frac{1}{\gamma}\begin{pmatrix}\widehat{\zeta_h.u_h}(\widetilde{k})\\ \widehat{(u_h.^2)/2}(\widetilde{k})\end{pmatrix}}_{\widehat{F(\zeta_h,u_h)}_{\widetilde{k}}}, \tag{3.4}$$

where $\widetilde{k} = \pi k/l, -N/2 \leq k \leq N/2-1$.

In [6] three methods for the iterative resolution of (3.4) were proposed: The Petviashvili iteration, the CGN method, and a variant of Petviashvili's method (called e-Petviashvili's method) obtained from solving ζ in terms of u in the second equation of (3.1) and substituting into the first one. The resulting equation for u (with quadratic and cubic terms) is iteratively solved with the method proposed in [2], an extension of the Petviashvili scheme for nolinearities which are superpositions of homogeneous functions with different degree of homogeneity, cf. [6] for details.

3.1. **Numerical generation of solitary waves with acceleration methods.** In the present paper we propose an alternative technique based on implementing the Petviashvili iteration combined with a vector extrapolation method, [15]. The inclusion of the extrapolation has the general benefit of accelerating the convergence of the basic method used for the iteration. (In some cases, the process changes from divergent to convergent.)

3.1.1. *The minimal polynomial extrapolation (MPE) method.* We start with a brief description of the technique of acceleration of convergence that will be used in the sequel, called the minimal polynomial extrapolation method (MPE), [9]. A typical derivation (see e. g. [15] and references therein for details) comes from the resolution of linear systems by iterative processes of fixed-point form

$$Z^{[\nu+1]} = TZ^{[\nu]} + d, \quad \nu = 0, 1, \ldots, \tag{3.5}$$

for some vector d and matrix T such that $I - T$ is nonsingular, and from some initial iteration $Z^{[0]}$. Let Z be the (unique) solution of the system, fixed point of (3.5). The first observation is that Z can be obtained exactly from (3.5) as follows. Let $e^{[\nu]} = Z^{[\nu]} - Z$ and let

$$p(z) = \sum_{i=0}^{n} c_i z^i, \quad c_n = 1,$$

be the minimal polynomial of T with respect to $e^{[\nu]}$. (This means that $p(z)$ is the polynomial of minimal degree $n = n(\nu)$ satisfying $p(T)e^{[\nu]} = 0$.) Then, [15], it holds that $\sum_{i=0}^{n} c_i \neq 0$ and

$$Z = \sum_{i=0}^{n} \gamma_i Z^{[\nu+i]}, \quad \gamma_i = \frac{c_i}{\sum_{j=0}^{n} c_j}, \quad i = 0, \ldots, n. \tag{3.6}$$

A key point to the approach is the fact that the minimal polynomial of T with respect to $e^{[\nu]}$ is that of T with respect of $U^{[\nu]} = \Delta Z^{[\nu]} := Z^{[\nu+1]} - Z^{[\nu]}$. In theory, this enables us to compute the coefficients c_j and to obtain the solution Z from (3.6). The approach, however, has several drawbacks in practice. One is the availability of the coefficients c_j. Note that they are obtained from the condition $p(T)U^{[\nu]} = 0$, which can be written as a linear system

$$U_{n-1}c' = -U^{[\nu+n]}, \tag{3.7}$$

where U_{n-1} is the matrix of columns $U^{[\nu]}, U^{[\nu+1]}, \ldots, U^{[\nu+n-1]}$, $c' = (c_0, \ldots, c_{n-1})^T$ (recall that $c_n = 1$). Even if the coefficients c_j are computable, it may not be efficient to solve (3.7) since the degree n may be as large as the size of the matrix T. Finally, in (3.6) the iterates $Z^{[\nu+i]}$ come from the linear iteration (3.5) and nothing is said in the case of a sequence generated from a nonlinear process.

A way to overcome these points, that conforms the derivation of the method, is as follows. We choose some value κ, arbitrary but small, and $\nu \geq 0$. Then the system (3.7) is solved in the least squares sense, that is we solve the minimization problem

$$\min_{c_0,\ldots,c_{\kappa-1}} \Big|\Big| \sum_{i=0}^{\kappa-1} c_i U^{[\nu+i]} + U^{[\nu+\kappa]} \Big|\Big|. \tag{3.8}$$

Then we take $c_\kappa = 1$ and, assuming $\sum_{i=0}^{\kappa} c_i \neq 0$, we compute

$$Z_{\nu,\kappa} = \sum_{i=0}^{\kappa} \gamma_i Z^{[\nu+i]}, \quad \gamma_i = \frac{c_i}{\sum_{j=0}^{\kappa} c_j}, \quad i = 0, \ldots, \kappa. \tag{3.9}$$

According to the arguments above, (3.9) will be an approximation to the limit (or antilimit, [15]) Z of the original sequence $\{Z^{[j]}\}_j$. Note that the previous derivation of (3.8), (3.9) is independent of the way (linearly or nonlinearly) in which the $Z^{[j]}$ were generated, and only depends on the differences $U^{[j]}$ to solve (3.8).

The literature contains two types of convergence results for (3.9): results about the behaviour of $\{Z_{\nu,\kappa}\}_{\nu\geq 0}$ for fixed κ (mainly in the case of linearly generated sequences as in (3.5)) and about (3.9) for fixed ν and κ. In this last sense, a relevant feature concerns the quadratic convergence when the method is implemented in the so-called cycling mode, [17, 16, 15]. For fixed κ, a cycle consists of the following steps:

- Step 1: From $Z^{[0]}$, compute $Z^{[1]},\ldots,Z^{[\kappa+1]}$.
- Step 2: Compute the corresponding extrapolation $Z_{0,\kappa}$ from (3.8),(3.9).
- Set $Z^{[0]}=Z_{0,\kappa}$ and go to Step 1.

The cycling process is controlled by imposing some stopping criterion on the corresponding residual error between the steps 2 and 3. The quantity $mw=\kappa+1$ is called the width of extrapolation, [17].

3.1.2. *The Petviashvili method.* We now describe the procedure to solve numerically (3.4) and to generate iterative approximations $Z^{[\nu]}:=(\zeta_h^{[\nu]},u_h^{\nu]}),\nu=0,1,\ldots,$ to (ζ_h,u_h). This is formulated as, cf. [6],

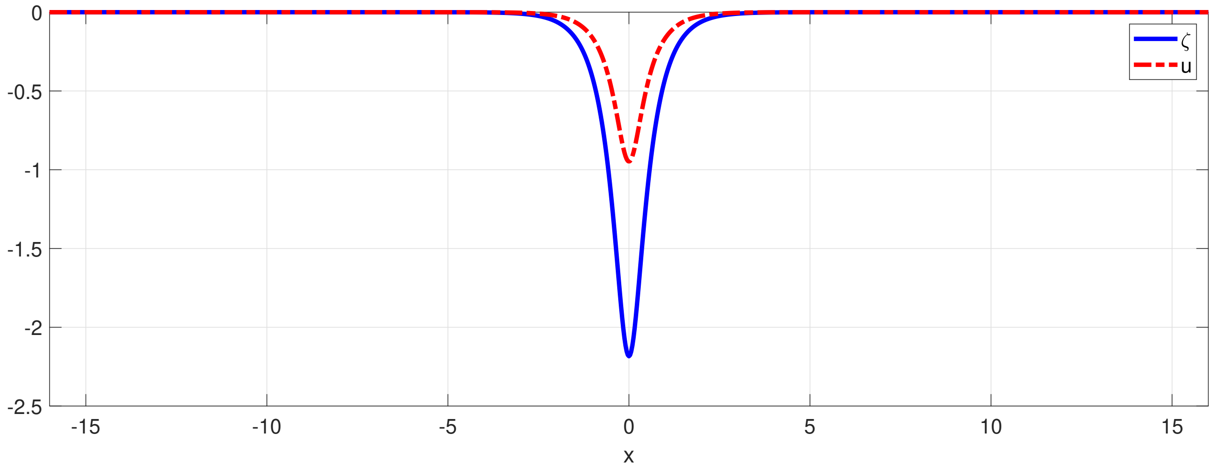
$$\begin{aligned} m_\nu &= \frac{\langle S_N Z^{[\nu]}, Z^{[\nu]}\rangle_N}{\langle F(Z^{[\nu]}), Z^{[\nu]}\rangle_N}, \\ SZ^{[\nu+1]} &= m_\nu^2 F(Z^{[\nu]}),\ \nu=0,1,\ldots \end{aligned} \tag{3.10}$$

The Petviashvili method, originally proposed for computing solitary wave solutions of the KPI equation, [14], is, from a numerical point of view, motivated by the fact that the classical fixed point algorithm, applied to (3.4), is not convergent in general. This is due to the presence of an eigenvalue of the iteration matrix of magnitude above one, corresponding to the degree two of homogeneity of the nonlinear term F in (3.2), as well as the eigenvalue equals one because of the translational symmetry, [12, 1]. The Petviashvili method modifies the classical fixed point algorithm with the inclusion of a stabilizing factor m_ν in (3.10). The spectrum of the resulting iteration matrix coincides with that of the classical algorithm, except the eigenvalue equals two associated to the degree of homogeneity, which is transformed to zero. The details and some examples of this process can be seen in [12, 1] for more general equations and in [6] for the specific cases of the ILW and B-O systems.

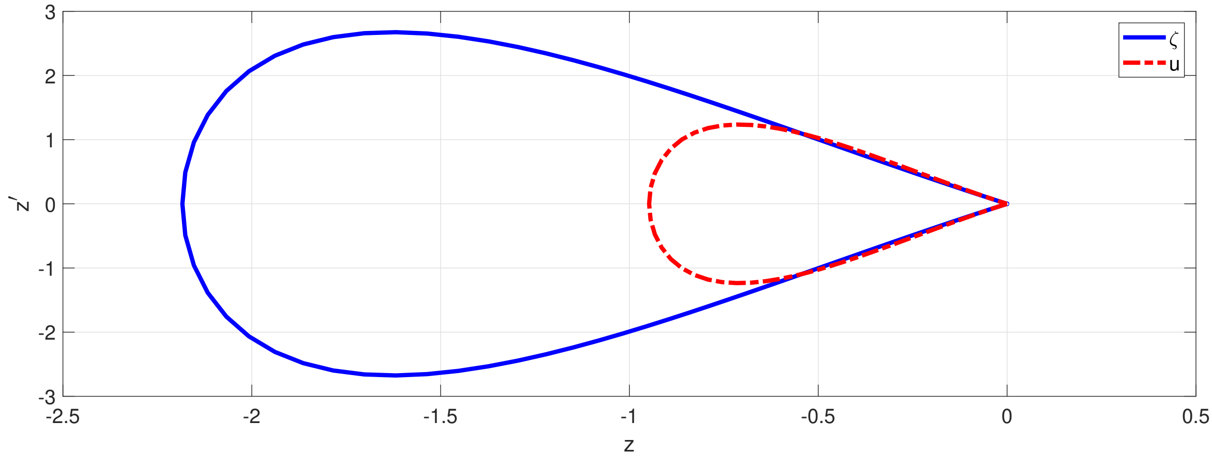
3.2. **Numerical experiments.** We illustrate the iterative method described in the previous section with some numerical experiments. The resulting procedure (Petviashvili's method plus acceleration in cycling mode) is controlled by iterating while the residual error

$$RES(\nu)=||S_N Z^{[\nu]}-F(Z^{[\nu]})||_N, \tag{3.11}$$

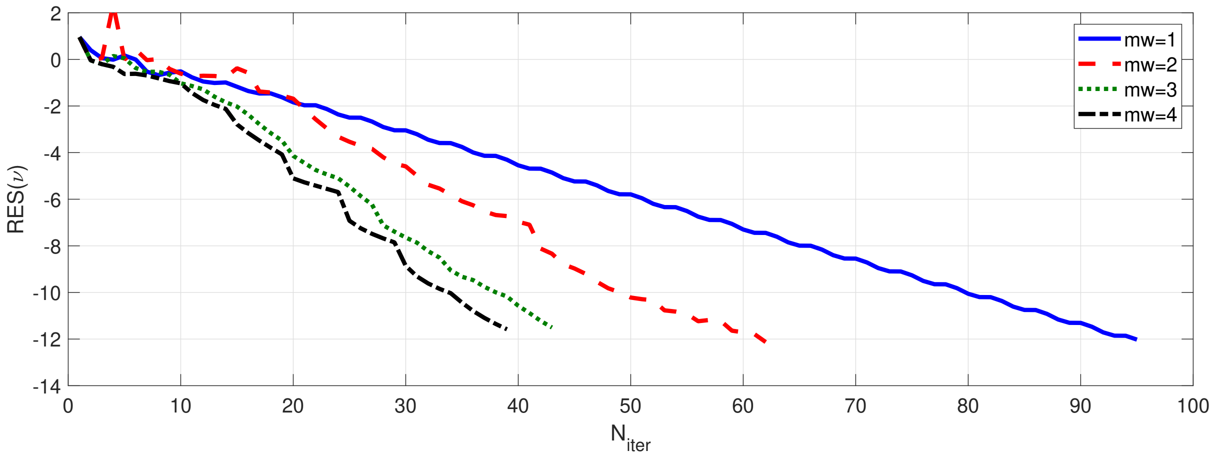
is above some tolerance. In the computations below $N=4096,\gamma=0.8,\alpha=1.2$ were taken. In the case of the ILW system (1.1), Figure 2(a) shows the numerical solitary wave profile of depression obtained with the Petviashvili method (without extrapolation) for $c=0.52$, while the corresponding phase plots are shown in Figure 2(b). The effect of the extrapolation technique is observed in Figure 2(c), which shows the behaviour of the residual error (3.11) as a function of the number of iterations, for several values of the width mw ($mw=1$ would correspond to



(a)



(b)



(c)

FIGURE 2. Numerical generation of solitary waves. ILW case with $\gamma = 0.8, \alpha = 1.2, c_s = 0.52$. (a) ζ and u numerical profiles; (b) Phase plot; (c) Residual error (3.11) vs. number of iterations for several values of the width of extrapolation mw.

the iteration without extrapolation). Note that the inclusion of the vector extrapolation accelerates the convergence of the iteration by diminishing the number of iterations required for the residual to become smaller than a fixed error. The illustration of the B-O case is given in Figure 3, where an approximate solitary wave solution of (3.1), in the B-O case, for $c = 0.57$ is shown in Figure 3(a), and with the corresponding phase plot in Figure 3(b). The acceleration of the convergence for the BO case is shown in Figure 3(c). The most remarkable reduction in the number of iterations for a given residual error is observed when we let $mw = 1$ (no acceleration) and then $mw = 2$ (cycling mode extrapolation with two steps of the Petviashvili method). After that, for larger values of mw, the method continues accelerating the convergence, with a milder reduction in the number of iterations (see e. g. [17, 3] for discussions about an optimal choice for mw). In order to complete the numerical study, two additional experiments are included. The first one concerns the spectral accuracy of the method. Figure 4 shows, in semi-log scale, the Fourier mode amplitudes of the corresponding ζ and u numerical profiles (for both the ILW and B-O systems) with no extrapolation ($mw = 1$) confirming the exponential decay to machine precision. (Indeed, the spectral accuracy is preserved when the extrapolation is incorporated, taking $mw > 1$.) On the other hand, by comparison with Figure 2(b), we can observe the different type of decay to zero at infinity (algebraic versus exponential, cf. [4]) of the solitary waves as trajectories homoclinic to the origin. A second experiment illustrating this behaviour is displayed in Figure 5. This shows, in log-log scale, the deviation of the absolute values of the ζ and u numerical profiles corresponding to panels (a) of Figures 2 and 3. From Figure 5(a), the exponential decay is clear in the ILW case, while in Figure 5(b) we considered several values of l and compared the slopes of the resulting lines with that of the dashed line, confirming a decay in the B-O case like $1/|x|^2$, [4]. In each case, the decay of the solitary waves is similar to that of the solitary waves of the corresponding unidirectional version. The dependence of the cutoff l makes necessarily influence on the accuracy of the method for the B-O case, and must be taken into account when using the profiles in numerical studies of the stability, where long time computations are required, [7].

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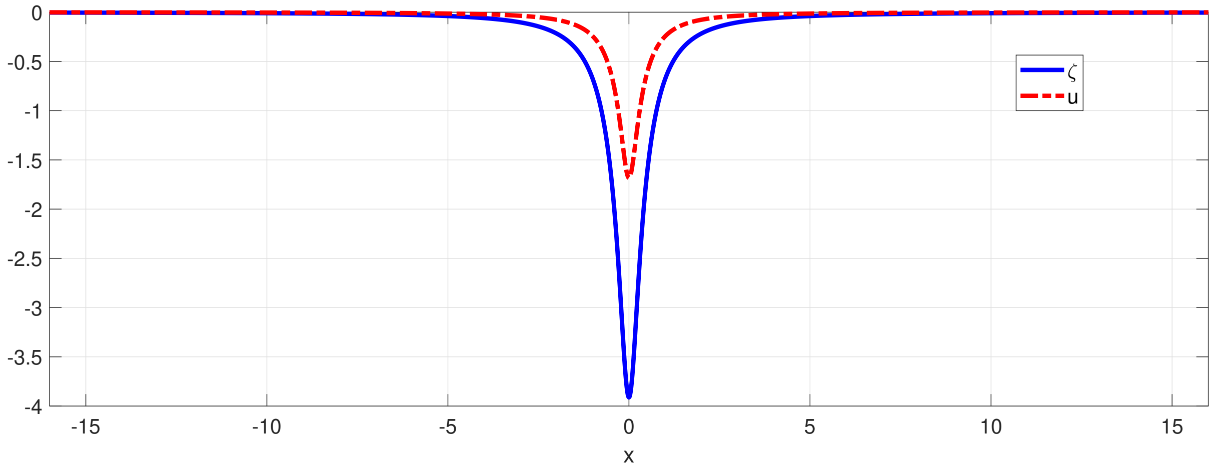
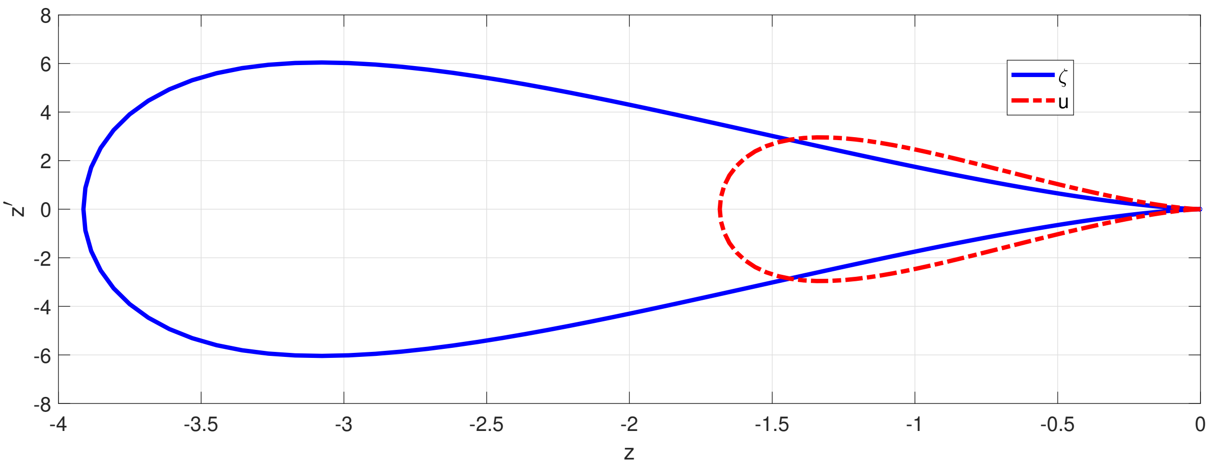
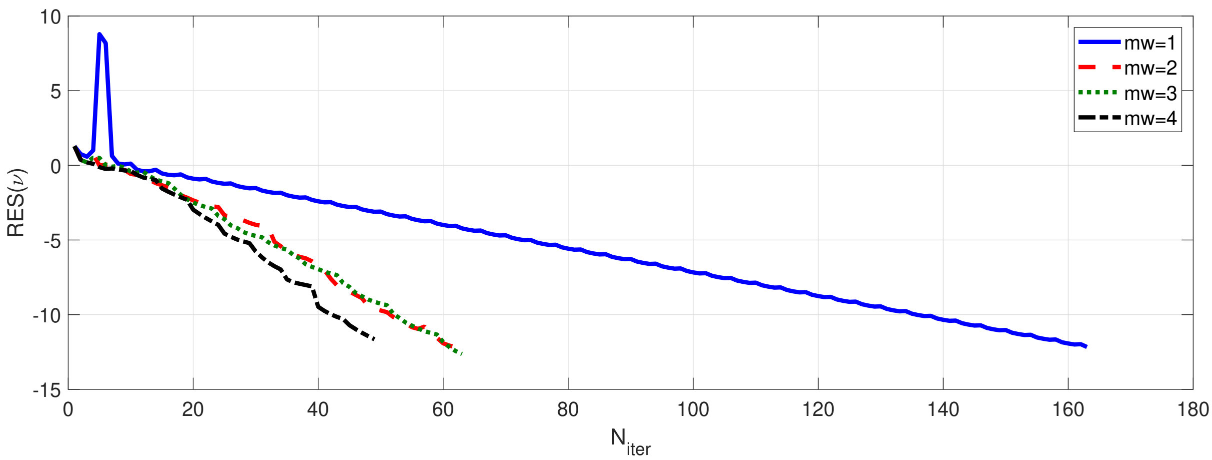


(a)

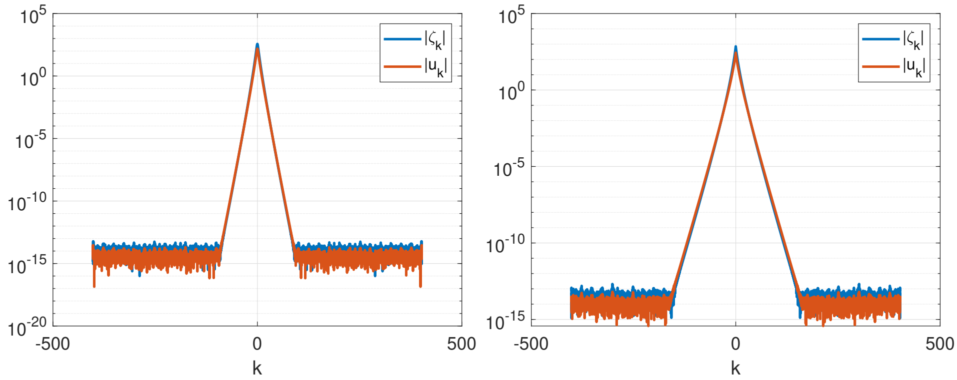


(b)



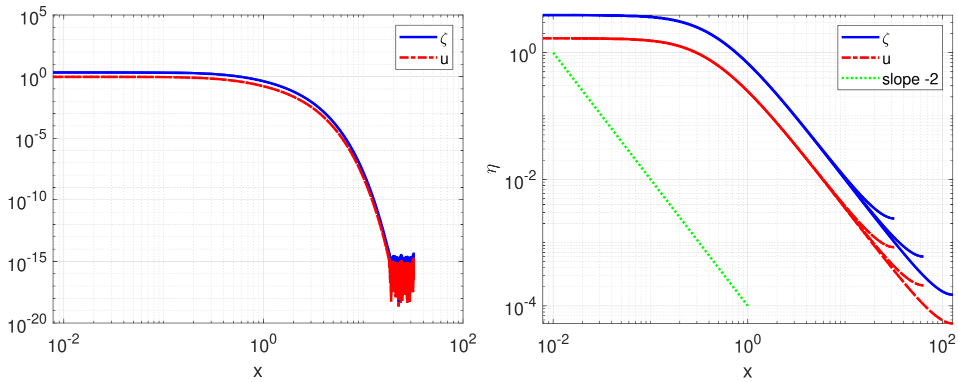
(c)

FIGURE 3. Numerical generation of solitary waves. B-O case with $\gamma = 0.8, \alpha = 1.2, c_s = 0.57$. (a) ζ and u numerical profiles; (b) Phase plot; (c) Residual error (3.11) vs. number of iterations for several values of the width of extrapolation mw.



(a) (b)

FIGURE 4. Logarithm of the Fourier mode amplitudes $|\zeta_k|, |u_k|$ where $\zeta_k = \widehat{\zeta_h}(\widetilde{k}), u_k = \widehat{u_h}(\widetilde{k})$ of ζ and u numerical profiles with $N = 4096, l = 32, mw = 1$. (a) ILW system; (b) B-O system.



(a) (b)

FIGURE 5. Absolute values of ζ and u numerical profiles in log-log scale. (a) ILW case with $N = 4096, mw = 3$, and $l = 32$. The profiles are given in Figure 2(a). (b) B-O case with $N = 4096, mw = 3$, and $l = 32, 64, 128$. The profiles are given in Figure 3(a). The slope of the dashed line is -2.

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