TEMPERED DISTRIBUTIONS WITH DISCRETE SUPPORT AND SPECTRUM

S. YU. FAVOROV

ABSTRACT. We investigate properties of tempered distributions with discrete or countable support such that their Fourier transforms are distributions with discrete or countable support as well. We find sufficient conditions for the support of the distribution to be a finite union of translations of a full-rank lattice. We also find conditions for a distribution to be almost periodic.

1. INTRODUCTION

Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\varphi \in C^\infty(\mathbb{R}^d)$ with finite norms

$$N_{n,m}(\varphi) = \sup_{\mathbb{R}^d}\{\max\{1, |x|^n\} \max_{\|k\| \leq m} |D^k \varphi(x)|\}, \quad n, m = 0, 1, 2, \dots,$$

where

$$k = (k_1, \dots, k_d) \in (\mathbb{N} \cup \{0\})^d, \; \|k\| = k_1 + \dots + k_d, \; D^k = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d}.$$

These norms generate the topology on $S(\mathbb{R}^d)$. Elements of the space $S^*(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called tempered distributions. For each tempered distribution f there are $c < \infty$ and $n, m \in \mathbb{N} \cup \{0\}$ such that for all $\varphi \in S(\mathbb{R}^d)$

$$|f(\varphi)| \leq cN_{n,m}(\varphi). \tag{1}$$

Moreover, this estimate is sufficient for a distribution f to belong to $S^*(\mathbb{R}^d)$ (see [23, Ch. 3]).

The Fourier transform of a tempered distribution f is defined by the equality

$$\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all} \quad \varphi \in S(\mathbb{R}^d),$$

where

$$\hat{\varphi}(y) = \int\limits_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i \langle x, y \rangle\} dx$$

is the Fourier transform of the function φ. Note that the Fourier transform of every tempered distribution is also a tempered distribution.

An element $f \in S^*(\mathbb{R}^d)$ is called *a Fourier quasicrystal* if f and $\hat{f}$ are measures on $\mathbb{R}^d$ with closed discrete supports. In this case the support of $\hat{f}$ is called *spectrum* of f. These notions were inspired by the experimental discovery made in the middle of the 80's of non-periodic atomic structures with diffraction patterns consisting of spots. There are a lot of papers devoted to the investigation of properties of Fourier quasicrystals (see, for example, collections of works [1], [20], papers [3]-[18], and so on).

We will say that a set $A \subset \mathbb{R}^d$ is *strongly discrete* if it has no finite limit points (in other words, A is closed and discrete), and A is *uniformly discrete* if it has a strictly positive separating constant

$$\eta(A) := \inf\{|x - x'| : x, x' \in A, x \neq x'\}.$$

A complex Radon measure or tempered distribution is *strongly discrete (uniformly discrete)* if its support is strongly discrete (uniformly discrete). We will call the support of the Fourier transform of a tempered distribution f the spectrum of f as well. Following [16], we will say that a discrete set $A \subset \mathbb{R}^d$ is *a set of finite type*, if the set

$$A - A = \{x - x' : x, x' \in A\}$$

is strongly discrete. A set $L \subset \mathbb{R}^d$ is called *a full-rank lattice* if $L = T\mathbb{Z}^d$ for some nondegenerate linear operator T on $\mathbb{R}^d$. The lattice $L^* = (T^t)^{-1}\mathbb{Z}^d$ is called the *conjugate lattice* for L. A set A is a *pure crystal* with respect to a full-rank lattice L if it is a finite union of cosets of L.

We begin with the following result of N. Lev and A. Olevskii [8] on quasicrystals.

Theorem 1.1. *Let μ be a uniformly discrete positive Fourier quasicrystal on $\mathbb{R}^d$ with uniformly discrete spectrum. Then the support of μ is a subset of a pure crystal with respect to a full-rank lattice L, and the spectrum of μ is a subset of a pure crystal with respect to the conjugate lattice L^*. In the dimension $d = 1$ the assertion is valid without the positivity assumption.*

Several results stronger than Theorem 1.1 were obtained in [9].

Note that for all d there are Fourier quasicrystals with strongly discrete support and spectrum such that the above assertion is not valid [11]. Also, for dimension $d > 1$ there are non-positive Fourier quasicrystal with uniformly discrete support and spectrum such that their support is not a pure crystal [4].

There is another type of result.

Theorem 1.2 (Favorov, [5]). *Let μ be a complex measure on $\mathbb{R}^d$ with strongly discrete support Λ of finite type such that $\inf_{x \in \Lambda} |\mu(x)| > 0$. Let the Fourier transform be a measure $\hat{\mu} = \sum_{y \in \Gamma} b(y)\delta_y$ with the countable $\Gamma \subset \mathbb{R}^d$ such that*

$$\sum_{|y| < r} |b(y)| = O(r^T), \quad r \to \infty,$$

with some $T < \infty$. Then Λ is a pure crystal.

Let us now go over from measures to distributions. V. Palamodov [19] obtained the following theorem, close in spirit to Theorem 1.1 but under considerably stronger discreteness assumptions on the support and the spectrum.

Theorem 1.3. *Let $f \in S^*(\mathbb{R}^d)$ be such that its support Λ and spectrum Γ are strongly discrete sets of finite type and, moreover, one of the differences $\Lambda - \Lambda$ and $\Gamma - \Gamma$ is uniformly discrete. Then Λ is a pure crystal with respect to a lattice L and Γ is a pure crystal with respect to the conjugate lattice L^*.*

In the present paper we obtain two analogs of Theorem 1.2 for tempered distributions.

2. The main results

By [22], every distribution f with strongly discrete support Λ has the form

$$f = \sum_{\lambda \in \Lambda} P_\lambda(D)\delta_\lambda, \quad P_\lambda(x) = \sum_{\|k\| \le K_\lambda} p_{\lambda,k} x^k, \quad x \in \mathbb{R}^d, \ p_{\lambda,k} \in \mathbb{C}, \ K_\lambda < \infty.$$

Here δ_y means, as usual, the unit mass at the point $y \in \mathbb{R}^d$ and $x^k = x_1^{k_1} \cdots x_d^{k_d}$.

Moreover, $\operatorname{ord} f = \sup_\lambda \deg P_\lambda < \infty$ (see Proposition 3.1 below). Therefore we will consider distributions

$$\text{(2)} \qquad\qquad f = \sum_{\lambda \in \Lambda} \sum_{\|k\| \le m} p_{\lambda,k} D^k \delta_\lambda, \quad k \in (\mathbb{N} \cup \{0\})^d.$$

If the Fourier transform $\hat{f}$ has a strongly discrete support Γ, we also have

$$\text{(3)} \qquad\qquad \hat{f} = \sum_{\gamma \in \Gamma} \sum_{\|j\| \le m'} q_{\gamma,j} D^j \delta_\gamma, \quad j \in (\mathbb{N} \cup \{0\})^d.$$

We will suppose that $m = \operatorname{ord} f$ and $m' = \operatorname{ord} \hat{f}$. Also, we will consider the case of distributions f and $\hat{f}$ of forms (2) and (3) with arbitrary countable Λ and Γ. If this is the case, we will also say that Λ is the support and Γ is the spectrum of f.

Denote by $B(x,r)$ the ball in $\mathbb{R}^d$ of radius r with the center in x, $B(r) = B(0,r)$. By $\#A$ denote a number of elements of the finite set A, and put $n_A(r) = \#(A \cap B(r))$. We say that the discrete set A is of *bounded density* if

$$\sup_{x \in \mathbb{R}^d} \#(A \cap B(x,1)) < \infty.$$

Clearly, every uniformly discrete set A is of bounded density, and every set A of bounded density satisfies the condition

$$n_A(r) = O(r^d), \quad r \to \infty.$$

A set $A \subset \mathbb{R}^d$ is *relatively dense* if there is $R < \infty$ such that every ball of radius R intersects with A.

Also, for any f of the form (2) set

$$\kappa_f(\lambda) = \sup_{\|k\| \le m} |p_{\lambda,k}|, \qquad \rho_f(r) = \sum_{|\lambda| < r} \kappa_f(\lambda).$$

Clearly, if f is a measure μ, then $\rho_f(r)$ equals the variation $|\mu|(B(r))$ of the measure on the ball $B(r)$.

In our article we prove the following theorems

Theorem 2.1. *Let f_1, f_2 be tempered distributions on $\mathbb{R}^d$ with strongly discrete supports Λ_1, Λ_2, respectively, such that $\Lambda_1 - \Lambda_2$ is a strongly discrete set and*

$$\text{(4)} \qquad\qquad \inf_{\lambda \in \Lambda_j} \kappa_{f_j}(\lambda) > 0, \quad j = 1, 2.$$

If $\hat{f}_1, \hat{f}_2$ are both measures with discrete supports Γ_1, Γ_2, respectively, such that

$$\text{(5)} \qquad \exists h < \infty, \ c > 0 \quad \text{such that} \quad |\gamma - \gamma'| > c \min\{1, |\gamma|^{-h}\} \quad \forall \gamma, \gamma' \in \Gamma_j, \ j = 1, 2, \ \gamma \ne \gamma',$$

then Λ_1, Λ_2 are pure crystals with respect to a single full-rank lattice.

The same is true if $\hat{f}_1$, $\hat{f}_2$ are both purely atomic measures such that

$$\exists\, T < \infty \quad |\hat{f}_1|(B(r)) + |\hat{f}_2|(B(r)) = O(r^T), \quad r \to \infty.$$

Corollary 2.2. *Let f be a tempered distribution on $\mathbb{R}^d$ with a discrete support Λ of finite type such that $\inf_{\lambda \in \Lambda} \kappa_f(\lambda) > 0$. If $\hat{f}$ is a measure with discrete support Γ such that*

$$(6) \qquad \exists h < \infty,\ c > 0 \quad \text{such that} \quad |\gamma - \gamma'| > c\min\{1, |\gamma|^{-h}\} \quad \forall \gamma, \gamma' \in \Gamma,\ \gamma \neq \gamma',$$

then Λ is a pure crystal.

  The same is valid if $\hat{f}$ is a purely atomic measure such that

$$\exists\, T < \infty \quad |\hat{f}|(B(r)) = O(r^T), \quad r \to \infty.$$

Theorem 2.3. *Let f_1, f_2 be tempered distributions on $\mathbb{R}^d$ with strongly discrete relatively dense supports Λ_1, Λ_2 and strongly discrete spectrums Γ_1, Γ_2, let $\Lambda_1 - \Lambda_2$ be a strongly discrete set, and let*

$$\exists\, T < \infty \quad n_{\Gamma_1}(r) + n_{\Gamma_2}(r) = O(r^T), \quad r \to \infty.$$

If conditions (4) and

$$(7) \qquad\qquad \sup_{\lambda \in \Lambda_j} \kappa_{f_j}(\lambda) < \infty, \quad j = 1, 2,$$

are satisfied, then Λ_1, Λ_2 are pure crystals with respect to a single full-rank lattice.

Corollary 2.4. *Let f be a tempered distribution on $\mathbb{R}^d$ with a strongly discrete support Λ of finite type and a strongly discrete spectrum Γ such that*

$$\exists\, T < \infty \quad n_\Gamma(r) = O(r^T), \quad r \to \infty.$$

If

$$0 < \inf_{\lambda \in \Lambda} \kappa_f(\lambda) \leq \sup_{\lambda \in \Lambda} \kappa_f(\lambda) < \infty,$$

then Λ is a pure crystal.

  We leave proofs of Theorems 2.1 and 2.3 until Section 6. These proofs are based on properties of almost periodic distributions, i.e., distributions $f \in S^*(\mathbb{R})$ such that the functions $(\psi \star f)(t) = f(\psi(\cdot - t))$ are almost periodic in $t \in \mathbb{R}^d$ for each $\psi \in S(\mathbb{R}^d)$. In Section 5 we investigate the notion of almost periodicity and prove the following theorems.

Theorem 2.5. *Let one of the following conditions for $f \in S^*(\mathbb{R}^d)$ be satisfied:*

  i) $\hat{f}$ is a measure, Γ is strongly discrete and satisfies (6),

  ii) $\hat{f}$ is a measure $\sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma$, Γ is countable, and $|\hat{f}|(B(r)) = O(r^T)$ as $r \to \infty$,

  iii) $\hat{f}$ is a measure $\sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma$, Γ is strongly discrete, $n_\Gamma(r) = O(r^{T_1})$ as $r \to \infty$, and $q_\gamma = O(|\gamma|^{T_2})$ as $|\gamma| \to \infty$,

  iv) Γ is strongly discrete, $n_\Gamma(r) = O(r^T)$ as $r \to \infty$, and f satisfies (1) with $n = d$,

  v) f has the form (2) with countable Λ, Γ is strongly discrete, $n_\Gamma(r) = O(r^T)$ and $\rho_f(r) = O(r^d)$ as $r \to \infty$,

  vi) f has the form (2) with countable Λ, Γ is strongly discrete, $n_\Gamma(r) = O(r^T)$ as $r \to \infty$, $\kappa_f(\lambda)$ is bounded and Λ is of bounded density,

where T, T_1, T_2 are some constants.

  Then f is an almost periodic distribution.

  Also, there are some converse results.

Theorem 2.6. *i) If $f \in S^*(\mathbb{R}^d)$ is an almost periodic distribution with strongly discrete spectrum Γ, then $\hat{f}$ is a measure,*

ii) if $f \in S^*(\mathbb{R}^d)$ is an almost periodic distribution and $\hat{f}$ is a measure such that $|\hat{f}|(B(r)) = O(r^T)$ as $r \to \infty$, then $\hat{f}$ is a purely atomic measure,

iii) if A is a strongly discrete subset of $\mathbb{R}^d$ and every $f \in S^*(\mathbb{R}^d)$ with spectrum contained in A is almost periodic, then A satisfies (6).

Note that for $d = 1$, $\Gamma = \mathbb{Z} + \alpha\mathbb{Z}$ part i) of Theorem 2.5 and part i) of Theorem 2.6 is contained in [18]. Also, Theorem 2.5 gives another proof of almost periodicity of Guinand's measure from [18].

3. PRELIMINARY PROPERTIES OF DISTRIBUTIONS WITH STRONGLY DISCRETE SUPPORT

Proposition 3.1. *Suppose $f \in S^*(\mathbb{R}^d)$ has a strongly discrete support Λ. Then for some $m < \infty$*

$$f = \sum_{\lambda \in \Lambda} \sum_{\|k\| \leq m} p_{\lambda,k} D^k \delta_\lambda, \quad k \in (\mathbb{N} \cup \{0\})^d.$$

If, in addition, the support satisfies the condition

(8) $\exists h < \infty, \ c > 0 \quad such \ that \quad |\lambda - \lambda'| > c\min\{1, |\lambda|^{-h}\} \quad \forall \lambda, \lambda' \in \Lambda, \ \lambda \neq \lambda',$

then

(9) $\exists n \in \mathbb{N}, C < \infty \quad such \ that \quad |p_{\lambda,k}| \leq C\max\{1, |\lambda|^n\} \quad \forall k, \forall \lambda \in \Lambda.$

Proof of Proposition 3.1. Let $\lambda \in \Lambda$ and $\varepsilon \in (0,1)$ be such that

$$\inf\{|\lambda - \lambda'| : \ \lambda' \in \Lambda, \ \lambda' \neq \lambda\} > \varepsilon.$$

Let φ be a function on $\mathbb{R}$ such that

(10) $\varphi(|x|) \in C^\infty(\mathbb{R}^d), \quad \varphi(|x|) = 0$ for $|x| > 1/2, \quad \varphi(|x|) = 1$ for $|x| \leq 1/3.$

Then

$$\varphi_{\lambda,k,\varepsilon}(x) = \frac{(x-\lambda)^k}{k!}\varphi(|x-\lambda|/\varepsilon) \in S(\mathbb{R}^d),$$

where, as usual, $k! = k_1! \cdots k_d!$. It is easily shown that

$$f(\varphi_{\lambda,k,\varepsilon}) = (-1)^{\|k\|}p_{\lambda,k}.$$

Let f satisfy (1) with some m, n. We get

$$|f(\varphi_{\lambda,k,\varepsilon})| \leq \sup_{|x-\lambda|<\varepsilon} \max\{1, |x|^n\} \sum_{\|\alpha+\beta\| \leq m} c(\alpha,\beta)\left|D^\alpha\varphi\left(\frac{|x-\lambda|}{\varepsilon}\right)D^\beta\left(\frac{(x-\lambda)^k}{k!}\right)\right|,$$

where $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ and $c(\alpha, \beta) < \infty$. Note that

$$|D^\alpha\varphi(|x-\lambda|/\varepsilon)| \leq \varepsilon^{-\|\alpha\|}c(\alpha)$$

and

$$D^\beta(x-\lambda)^k = \begin{cases} 0 \text{ if } k_j < \beta_j \text{ for at least one } j, \\ c(k,\beta)(x-\lambda)^{k-\beta} \text{ if } k_j \geq \beta_j \text{ for all } j. \end{cases}$$

Since $|x - \lambda| < 1$, we get

$$\max\{1, |x|^n\} \leq 2^n \max\{1, |\lambda|^n\}.$$

Taking into account that

$$\operatorname{supp}\varphi(|x-\lambda|/\varepsilon) \subset B(\lambda,\varepsilon),$$

we get

$$(11) \qquad |p_{\lambda,k}| \leq \sum_{\|\alpha+\beta\|\leq m,\,\beta_j\leq k_j\;\forall j} c(k,\alpha,\beta)\max\{1,|\lambda|^n\}\varepsilon^{\|k\|-\|\alpha+\beta\|}.$$

For $\|k\| > m$ we take $\varepsilon \to 0$ and obtain $p_{\lambda,k} = 0$.

Next, let (8) be true and (9) not be satisfied. Then there are $k, \|k\| \leq m$, and a sequence $\lambda_s \to \infty$ such that $|\lambda_s| > 2$ for all s and

$$(12) \qquad \log|p_{\lambda_s,k}|/\log|\lambda_s| \to \infty, \quad s \to \infty.$$

Put $\beta_s = c|\lambda_s|^{-h}$ and

$$\psi_{s,k}(x) = \frac{(x-\lambda_s)^k}{k!}\varphi\left(\frac{x-\lambda_s}{\beta_s}\right), \qquad \Psi_k(x) = \sum_{s=1}^{\infty}\frac{\psi_{s,k}(x)}{p_{\lambda_s,k}}.$$

By (12),

$$1/p_{\lambda_s,k} = o(1/|\lambda_s|^N), \quad |\lambda_s| \to \infty, \quad \text{for every } N < \infty.$$

Since

$$D^j(\psi_{s,k}(x)) = O(|\lambda_s|^{h\|j\|}), \;\; j \in (\mathbb{N}\cup\{0\})^d, \quad \text{and} \quad \operatorname{supp}\psi_{s,k} \cap \operatorname{supp}\psi_{s',k} = \emptyset, \;\; s \neq s',$$

we see that

$$D^j(\Psi_k(x)) = o(1/|x|^{N-h\|j\|}), \quad x \to \infty,$$

and $\Psi_k \in S(\mathbb{R}^d)$.

By (8), $\lambda \notin B(\lambda_s, c|\lambda_s|^{-h})$ for all $\lambda \in \Lambda \setminus \{\lambda_s\}$, therefore, $f(\Psi_k)$ is equal to

$$\sum_{\lambda\in\Lambda}\sum_{\|j\|\leq m}\sum_s (-1)^{\|j\|}p_{\lambda,j}p_{\lambda_s,k}^{-1}D^j(\psi_{s,k})(\lambda) = \sum_s\sum_{\|j\|\leq m}(-1)^{\|j\|}p_{\lambda_s,j}p_{\lambda_s,k}^{-1}D^j(\psi_{s,k})(\lambda_s).$$

Since $D^j(\psi_{s,k})(\lambda_s) = 0$ for $j \neq k$ and $D^k(\psi_{s,k})(\lambda_s) = 1$, we obtain the contradiction. ∎

Remark. It follows immediately from (11) that for all k such that $\|k\| = m$ we have $|p_{\lambda,k}| \leq C\max\{1,|\lambda|^n\}$. If Λ is uniformly discrete, then we take $\varepsilon \equiv \mathrm{const}$ and obtain the above inequality for all k. The last assertion was earlier proved by V. Palamodov [19].

Proposition 3.1 is sharp, in a sense.

Example of a "bad" distribution. Let $\Lambda \subset \mathbb{R}^d$ be an arbitrary strongly discrete set, for which (8) is not satisfied. Therefore there exist sequences $\{\lambda_s\}$, $\{\lambda_s'\} \subset \Lambda$ with the properties

$$\lambda_s \neq \lambda_s', \qquad |\lambda_s| > s, \qquad \frac{\log|\lambda_s - \lambda_s'|}{\log|\lambda_s|} \to -\infty, \quad s \to \infty.$$

Define the measure

$$(13) \qquad \mu = \sum_s p_s(\delta_{\lambda_s'} - \delta_{\lambda_s}), \qquad p_s = |\lambda_s - \lambda_s'|^{-1}.$$

For each $\varphi \in S(\mathbb{R}^d)$ we have

$$|\mu(\varphi)| \leq \sum_s |p_s||\varphi(\lambda_s') - \varphi(\lambda_s)| \leq \sum_s |p_s|\sup\{|\varphi'(x)| : x \in B(\lambda_s,|\lambda_s' - \lambda_s|)\}|\lambda_s' - \lambda_s|.$$

Since $\varphi'(x) = o(|x|^{-N})$ for every N as $|x| \to \infty$, we see that the latter series converges. Therefore $\mu \in S^*(\mathbb{R}^d)$. On the other hand,

$$\log |p_s| / \log |\lambda_s| \to \infty, \quad s \to \infty,$$

hence $\{p_s\}$ does not satisfy (9). Also, for each $T < \infty$ the condition $\rho_\mu(r) = O(r^T)$ is not satisfied.

Next, let $\tilde{\Lambda} \subset \mathbb{R}^d$ be a strongly discrete set such that $\tilde{\Lambda} \cap \Lambda = \emptyset$ and for each T the condition $n_{\tilde{\Lambda}}(r) = O(r^T)$ is not satisfied. Let ν be a finite positive measure such that $\operatorname{supp} \nu = \tilde{\Lambda}$. Clearly, $\nu \in S^*(\mathbb{R}^d)$. Therefore the measure $\mu + \nu$ is a tempered distribution, and for each T the conditions $n_{\tilde{\Lambda} \cup \Lambda}(r) = O(r^T)$ and $\rho_{\mu+\nu}(r) = O(r^T)$ are not satisfied.

4. Preliminary properties of distributions with strongly discrete spectrum

Proposition 4.1. *Suppose f is a linear functional on the space $S(\mathbb{R}^d)$ that satisfies (1) with some n, m, and has a strongly discrete spectrum Γ. Then $\operatorname{ord} \hat{f} \le \max\{0, n - d\}$, and the coefficients $q_{\gamma,j}$ in representation (3) satisfy the estimate*

$$|q_{\gamma,j}| \le C' \max\{1, |\gamma|^m\} \text{ for } \|j\| = n - d.$$

For the case of a uniformly discrete Γ we get

$$|q_{\gamma,j}| \le C \max\{1, |\gamma|^m\} \quad \forall j, \ \|j\| \le n - d.$$

Corollary 4.2. *Suppose f is a linear functional on the space $S(\mathbb{R}^d)$ that satisfies (1) with $n \le d$ and some m, and has a strongly discrete spectrum Γ. Then $\hat{f}$ is a measure,*

$$\hat{f} = \sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma, \quad |q_\gamma| \le C' \max\{1, |\gamma|\}^m.$$

Proof of Proposition 4.1. First note that $f \in S^*(\mathbb{R}^d)$, therefore, $\hat{f} \in S^*(\mathbb{R}^d)$ as well. Let $\gamma \in \Gamma$ and pick $\varepsilon \in (0, 1)$ such that

$$\inf\{|\gamma - \gamma'| : \ \gamma' \in \Gamma, \ \gamma' \ne \gamma\} > \varepsilon.$$

Let φ be the same as in the proof of Proposition 3.1. Put

$$\varphi_{\gamma,l,\varepsilon}(y) = \frac{(y - \gamma)^l}{l!} \varphi(|y - \gamma|/\varepsilon) \in S(\mathbb{R}^d).$$

We have

$$(14) \qquad (-1)^{\|l\|} q_{\gamma,l} = \sum_{\|j\| \le m'} q_{\gamma,j} D^j \delta_\gamma(\varphi_{\gamma,l,\varepsilon}(y)) = (\hat{f}, \varphi_{\gamma,l,\varepsilon}) = (f, \hat{\varphi}_{\gamma,l,\varepsilon}).$$

Note that

$$\hat{\varphi}_{\gamma,l,\varepsilon}(x) = e^{-2\pi i \langle x, \gamma \rangle} (l!)^{-1} (-2\pi i)^{-\|l\|} D^l(\widehat{\varphi(\cdot/\varepsilon)}) = c(l) e^{-2\pi i \langle x, \gamma \rangle} \varepsilon^{d+\|l\|} (D^l \hat{\varphi})(\varepsilon x).$$

Therefore,

$$D^k(\hat{\varphi}_{\gamma,l,\varepsilon}(x)) = \varepsilon^{d+\|l\|} \sum_{\alpha+\beta=k} c(\alpha, \beta) D^\alpha \left[e^{-2\pi i \langle x, \gamma \rangle} \right] D^\beta [(D^l \hat{\varphi})(\varepsilon x)]$$

$$= \sum_{\alpha+\beta=k} c(\alpha, \beta)(-2\pi i)^{\|\alpha\|} \gamma^\alpha e^{-2\pi i \langle x, \gamma \rangle} \varepsilon^{d+\|l\|+\|\beta\|} (D^{\beta+l} \hat{\varphi})(\varepsilon x).$$

Since $\hat{\varphi} \in S(\mathbb{R}^d)$, we get for every k, $\|k\| \leq m$, and every $M < \infty$

$$(15) \qquad |D^k(\hat{\varphi}_{\gamma,l,\varepsilon}(x))| \leq C(M)\varepsilon^{d+\|l\|} \max\{1, |\gamma|^m\}(\max\{1, |\varepsilon x|^M\})^{-1}.$$

Pick $M = n$. By (1), we have

$$|(f, \hat{\varphi}_{\gamma,l,\varepsilon})| \leq c(f) \sup_{\mathbb{R}^d}[\max\{1, |x|^n\} \max_{\|k\| \leq m} |D^k(\hat{\varphi}_{\gamma,l,\varepsilon}(x))|].$$

Hence,

$$|q_{\gamma,l}| = |(f, \hat{\varphi}_{\gamma,l,\varepsilon})| \leq C'(\hat{\varphi})\varepsilon^{d+\|l\|} \max\{1, |\gamma|^m\} \sup_{x \in \mathbb{R}^d}[\max\{1, |x|^n\}(\max\{1, |\varepsilon x|^n\})^{-1}].$$

Since the sup in the right-hand side of the inequality equals ε^{-n}, we obtain

$$|q_{\gamma,l}| \leq C' \max\{1, |\gamma|^m\}\varepsilon^{\|l\|+d-n}.$$

If $\|l\| > n - d$, we take $\varepsilon \to 0$ and get $q_{\gamma,l} = 0$, hence ord $\hat{f} \leq n - d$.
    For $\|l\| = n - d$ we get $|q_{\gamma,l}| \leq C' \max\{1, |\gamma|^m\}$.
    If Γ is uniformly discrete, we take $\varepsilon = \varepsilon_0 < \eta(\Gamma)/2$ for all $\gamma \in \Gamma$ and obtain the bound

$$|q_{\gamma,l}| \leq \varepsilon_0^{d-n}C' \max\{1, |\gamma|^m\} \quad \forall l, \ \|l\| \leq n - d. \qquad \blacksquare$$

M. Kolountzakis and J. Lagarias proved in [14] that the Fourier transform of every measure μ on the line $\mathbb{R}$ with support of bounded density, bounded masses $\mu(\{x\})$, and strongly discrete spectrum is also a measure $\hat{\mu} = \sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma$ with uniformly bounded q_γ. The following proposition generalizes this result for distributions from $S^*(\mathbb{R}^d)$.

Proposition 4.3. *Suppose $f \in S^*(\mathbb{R}^d)$ has the form (2) with some m and countable Λ, and a strongly discrete spectrum Γ. If*

$$\rho_f(r) = O(r^{d+H}), \quad r \to \infty, \quad H \geq 0,$$

then ord $\hat{f} \leq H$. If, in addition, H is integer, then for $\|j\| = H$ we get $|q_{\gamma,j}| \leq C' \max\{1, |\gamma|^m\}$. For the case of uniformly discrete Γ we get

$$|q_{\gamma,j}| \leq C \max\{1, |\gamma|^m\} \quad \forall j, \quad \|j\| \leq H.$$

Corollary 4.4. *If $f \in S^*(\mathbb{R}^d)$ has the form (2) with some m and countable Λ, strongly discrete spectrum Γ, and*

$$\rho_f(r) = O(r^d) \quad r \to \infty,$$

then $\hat{f}$ is a measure, and

$$\hat{f} = \sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma, \ |q_\gamma| \leq C' \max\{1, |\gamma|^m\}.$$

Corollary 4.5. *If $f \in S^*(\mathbb{R}^d)$ has the form (2) with some m, bounded $\kappa_f(\lambda)$, strongly discrete support Λ of bounded density, and strongly discrete spectrum Γ, then $\hat{f}$ is a measure, and*

$$\hat{f} = \sum_{\gamma \in \Gamma} q_\gamma \delta_\gamma, \ |q_\gamma| \leq C' \max\{1, |\gamma|^m\}.$$

Proof of Proposition 4.3. Let $\varphi_{\gamma,l,\varepsilon}$ be the same as in the proof of Proposition 4.1. By (14) and (2),

$$(-1)^{\|l\|}q_{\gamma,l} = (f, \hat{\varphi}_{\gamma,l,\varepsilon}) = \sum_{\lambda \in \Lambda} \sum_{\|k\| \leq m} p_{\lambda,k}(-1)^{\|k\|} D^k(\hat{\varphi}_{\gamma,l,\varepsilon}(\lambda)).$$

Using (15), we get

$$|q_{\gamma,l}| \leq C'(m,M)\varepsilon^{d+\|l\|} \max\{1, |\gamma|^m\} \sum_{\lambda \in \Lambda} \kappa_f(\lambda)(\max\{1, |\varepsilon\lambda|^M\})^{-1}.$$

We have

$$\sum_{\lambda \in \Lambda} \kappa_f(\lambda)(\max\{1, |\varepsilon\lambda|^M\})^{-1} = \rho_f(1/\varepsilon) + \varepsilon^{-M} \int_{1/\varepsilon}^{\infty} t^{-M} d\rho_f(t)$$

Pick $M > d + H$. Integrating by parts and using the estimate for $\rho_f(r)$, we see that the right-hand side is equal to

$$O(\varepsilon^{-d-H}) + \varepsilon^{-M} M \int_{1/\varepsilon}^{\infty} \rho_f(t)t^{-M-1}dt = O(\varepsilon^{-d-H}) \quad \text{as } \varepsilon \to \infty.$$

Finally,

$$|q_{\gamma,l}| \leq C' \max\{1, |\gamma|^m\}\varepsilon^{\|l\|-H}.$$

If $\|l\| > H$, we take $\varepsilon \to 0$ and get $q_{\gamma,l} = 0$, hence, $m' = \operatorname{ord} \hat{f} \leq H$.

If H is integer, we get $|q_{\gamma,l}| \leq C' \max\{1, |\gamma|^m\}$ for $\|l\| = H$.

If Γ is uniformly discrete, we take $\varepsilon = \varepsilon_0 < \eta(\Gamma)/2$ for all $\gamma \in \Gamma$ and obtain the bound

$$|q_{\gamma,l}| \leq \varepsilon_0^{-H} C' \max\{1, |\gamma|^m\} \quad \forall l, \|l\| \leq m'. \qquad \blacksquare$$

5. Almost periodic distributions

Recall that a continuous function g on $\mathbb{R}^d$ is almost periodic if for any $\varepsilon > 0$ the set of ε-almost periods of g

$$\{\tau \in \mathbb{R}^d : \sup_{x \in \mathbb{R}^d} |g(x+\tau) - g(x)| < \varepsilon\}$$

is a relatively dense set in $\mathbb{R}^d$.

Almost periodic functions are uniformly bounded on $\mathbb{R}^d$. The class of almost periodic functions is closed with respect to taking absolute values, linear combinations, and maximum, and minimum of a finite family of almost periodic functions. A limit of a uniformly (in $\mathbb{R}^d$) convergent sequence of almost periodic functions is also almost periodic. A typical example of an almost periodic function is an absolutely convergent exponential sum $\sum c_n \exp\{2\pi i\langle x, \omega_n \rangle\}$ with $\omega_n \in \mathbb{R}^d$ (see, for example, [2]).

A measure μ on $\mathbb{R}^d$ is called almost periodic if the function

$$(\psi \star \mu)(t) = \int_{\mathbb{R}^d} \psi(x-t)d\mu(x)$$

is almost periodic in $t \in \mathbb{R}^d$ for each continuous function ψ on $\mathbb{R}^d$ with compact support. A distribution $f \in S^*(\mathbb{R})$ is almost periodic if the function $(\psi \star f)(t) = f(\psi(\cdot - t))$ is almost periodic in $t \in \mathbb{R}^d$ for each $\psi \in S(\mathbb{R}^d)$ (see [7], [17], [18], [21]). Clearly, every almost periodic

distribution has a relatively dense support. But there are measures that are almost periodic tempered distributions, but are not almost periodic as measures (see [17]).

Proof of Theorem 2.5. ii) Pick $\psi \in S(\mathbb{R}^d)$. The inverse Fourier transform of the function $\psi(x - t)$ is $\hat{\psi}(-y)e^{2\pi i\langle y,t\rangle}$. Therefore,

$$(16) \qquad f(\psi(\cdot - t)) = \hat{f}(\hat{\psi}(-y)e^{2\pi i\langle y,t\rangle}) = \sum_{\gamma \in \Gamma} q_\gamma \hat{\psi}(-\gamma)e^{2\pi i\langle \gamma,t\rangle}.$$

Since $\hat{\psi}(-y) \in S(\mathbb{R}^d)$, we get $|\hat{\psi}(-\gamma)| \leq C(\psi,T)\max\{1,|\gamma|\}^{-T-1}$, therefore, the sum in (16) is majorized by the following expression

$$C(\psi,T)\left[\sum_{\gamma \in \Gamma, |\gamma| < 1}|q_\gamma| + \sum_{\gamma \in \Gamma, |\gamma| \geq 1}|q_\gamma||\gamma|^{-T-1}\right] = C(\psi,T)\left[\rho_{\hat{f}}(1) + \int_1^\infty r^{-T-1}d\rho_{\hat{f}}(r)\right].$$

Integrating by parts, we obtain that the integral converges. Then the sum in (16) converges absolutely, and $\psi \star f$ is almost periodic.

Next, iii) implies ii). Indeed, since $|q_\gamma| \leq C|\gamma|^{T_2}$ for $|\gamma| \geq 1$, we get, after integrating by parts,

$$\rho_{\hat{f}}(r) \leq C'\left[\rho_{\hat{f}}(1) + \int_1^r t^{T_2}dn_\Gamma(t)\right] = O(r^{T_1+T_2}), \quad r \to \infty.$$

Then i), Proposition 3.1, and the Lemma below implies iii).

Lemma 5.1. *If a set Γ satisfies (6), then $n_\Gamma(r) = O(r^{d(h+1)})$ as $r \to \infty$.*

Proof of the Lemma. Consider the annuli

$$A_s = \{y \in \mathbb{R}^d : s - 1 \leq |y| < s\}, \ s \in \mathbb{N}.$$

By (6),

$$B(\gamma, (c/2)s^{-h}) \cap B(\gamma', (c/2)s^{-h}) = \emptyset \quad \text{for } \gamma, \gamma' \in A_s \cap \Gamma, \ \gamma \neq \gamma'.$$

Hence for s such that $(c/2)s^{-h} < 1$ the sum of the volumes of the balls $B(\gamma, (c/2)s^{-h})$, $\gamma \in A_s \cap \Gamma$, does not exceed the volume of the annulus $A_{s-1} \cup A_s \cup A_{s+1}$. Therefore we have

$$\#(\Gamma \cap A_s) \leq \frac{(s+2)^d - (s-1)^d}{[(c/2)s^{-h}]^d} \leq Cs^{dh+d-1}, \quad C < \infty,$$

and

$$n_\Gamma(r) \leq \sum_{s < r+1}\#(\Gamma \cap A_s) = O(r^{d(h+1)}). \quad \blacksquare$$

Next, iv) and Corollary 4.2, or v) and Corollary 4.4, or vi) and Corollary 4.5 implies iii). $\blacksquare$

Proof of Theorem 2.6. i) Let f be an almost periodic tempered distribution with a strongly discrete spectrum Γ. By Proposition 3.1, the Fourier transform of f has the form (3). Suppose that $m' \neq 0$ and $q_{\gamma',j'} \neq 0$ for some $\gamma' \in \Gamma$ and $j' = (j_1', \ldots, j_d')$, $\|j'\| = m'$. Without loss of generality suppose that $j_1' \neq 0$. Let $j'' = (j_1' - 1, j_2', \ldots, j_d')$. Pick

$$\varepsilon < \min\{|\gamma' - \gamma| : \gamma \in \Gamma\}, \text{ and } \varphi_{\gamma',j'',\varepsilon}(y) = \frac{(y - \gamma')^{j''}}{j''!}\varphi((y - \gamma')/\varepsilon),$$

where φ is defined in (10). We have

$$(17) \qquad \hat{f}(e^{2\pi i \langle y,t \rangle} \varphi_{\gamma',j'',\varepsilon}(y)) = \sum_{\gamma \in \Gamma} \sum_{\|j\| \leq m'} (-1)^{\|j\|} q_{\gamma,j} D^j (e^{2\pi i \langle y,t \rangle} \varphi_{\gamma',j'',\varepsilon}(y))(\gamma).$$

Since

$$D^j(\varphi_{\gamma',j'',\varepsilon}(y))(\gamma) = \begin{cases} 0 \text{ if } \gamma \neq \gamma' \text{ or } j \neq j'', \\ 1 \text{ if } \gamma = \gamma' \text{ and } j = j'', \end{cases}$$

we see that expression (17) is equal to

$$(-1)^{\|j''\|} q_{\gamma',j''} e^{2\pi i \langle \gamma',t \rangle} + (-1)^{\|j'\|} q_{\gamma',j'} j_1' 2\pi i t_1 e^{2\pi i \langle \gamma',t \rangle}.$$

Here the first summand is bounded and the second one is unbounded in $t \in \mathbb{R}^d$. Hence the function

$$f(\hat{\varphi}_{\gamma',j'',\varepsilon}(x-t)) = \hat{f}(e^{2\pi i \langle y,t \rangle} \varphi_{\gamma',j'',\varepsilon}(y))$$

is unbounded and not almost periodic. We obtained a contradiction, therefore, $m' = 0$ and $\hat{f}$ is a measure.

ii) Let μ_1 be an atomic part of the measure $\hat{f}$. Then μ_1 has a countable support and $|\mu_1|(B(r)) = O(r^T)$ as $r \to \infty$. By Theorem 2.5, part ii), $\hat{\mu}_1(x)$ and, of course, the inverse Fourier transform $\check{\mu}_1(x) = \hat{\mu}_1(-x)$ are almost periodic distributions. Hence for each $\varphi \in S(\mathbb{R}^d)$ the function $g = (f - \check{\mu}_1) \star \varphi$ is almost periodic and its Fourier transform is the measure $\nu := \hat{\varphi}(\hat{f} - \mu_1)$ such that $\nu(\{y\}) = 0$ for all $y \in \mathbb{R}^d$. By Lemma 2 from [5],

$$\lim_{R \to \infty} \frac{1}{\omega_d R^d} \int_{B(R)} e^{-2\pi i \langle x,y \rangle} g(x) dx = 0,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Therefore, by the Uniqueness Theorem for almost periodic functions, $g(x) \equiv 0$ and $\hat{f} = \mu_1$.

To prove part iii) suppose that A does not satisfy (6). Consider two sequences $\{\lambda_s\}_1^\infty$, $\{\lambda_s'\}_1^\infty \subset A$ such that

$$\lambda_s \to \infty, \quad \log|\lambda_s' - \lambda_s|/\log|\lambda_s| \to -\infty, \quad s \to \infty, \quad \nu_s := \lambda_s' - \lambda_s \neq 0.$$

We can dilute the sequences such that $|\nu_1| < 1/3$,

$$|\nu_s| < \min\{|\nu_{s-1}|^3/\pi^3, |\nu_{s-1}|/(2s-2)\}, \quad |\lambda_s| > |\lambda_{s-1}| + 1, \quad s \geq 2.$$

We show that the inverse Fourier transform $\check{\mu}$ of the measure μ from (13) is not an almost periodic distribution.

Put $t_s = \nu_s/(2|\nu_s|^2)$. Clearly, we have for $j > s$

$$(18) \qquad 2\pi|t_s| < |\nu_{s+1}|^{-1/3} \leq |\nu_j|^{-1/3}, \quad |e^{2\pi i \langle \nu_j, t_s \rangle} - 1| \leq 2\pi|\langle \nu_j, t_s \rangle| < |\nu_j|^{2/3}, \quad j > s,$$

$$|e^{2\pi i \langle \lambda_s', t \rangle} - e^{2\pi i \langle \lambda_s, t \rangle}| = |e^{2\pi i \langle \nu_s, t_s \rangle} - 1| = 2.$$

Let

$$\psi(x) = \sum_s |\nu_s|^{1/2} \varphi(x - \lambda_s),$$

where φ is defined in (10). Since $|\nu_s| = o(1/|\lambda_s|^N)$ as $s \to \infty$ for every $N > 0$ and $\operatorname{supp} \varphi(x) \subset B(1/2)$, we see that $\psi \in S^*(\mathbb{R}^d)$. We have

$$(19) \qquad \check{\mu}(\hat{\psi}(y-t)) = \mu(\psi(x)e^{2\pi i \langle x,t \rangle}) = \sum_{j=1}^{\infty} |\nu_j|^{-1/2} [\varphi(\nu_j)e^{2\pi i \langle \lambda_j', t \rangle} - \varphi(0)e^{2\pi i \langle \lambda_j, t \rangle}].$$

Taking into account the equalities $\varphi(\nu_j) = \varphi(0) = 1$ and (18), we get that (19) for $t = t_s$ is bounded from below by the expression

$$(20) \qquad\qquad 2|\nu_s|^{-1/2} - 2\sum_{j=1}^{s-1} |\nu_j|^{-1/2} - \sum_{j=s+1}^{\infty} |\nu_j|^{1/6}.$$

It is easy to check that

$$|\nu_s|^{-1/2} > 2\sum_{j=1}^{s-1} |\nu_j|^{-1/2}, \qquad \sum_s |\nu_j|^{1/6} < \infty.$$

Therefore, (20) tends to ∞ as $s \to \infty$, and (19) is unbounded on the sequence t_s. Hence the function $\check{\mu}(\hat{\psi}(y-t))$ is not almost periodic, and $\check{\mu}$ is not an almost periodic distribution. $\blacksquare$

6. Proofs of theorems 2.1 and 2.3

If $\Lambda_1 = \Lambda_2 = \Lambda$, then the condition "$\Lambda - \Lambda$ is strongly discrete" implies that Λ is uniformly discrete and $\eta(\Lambda) > 0$. In the general case, we need some additional conditions for uniform discreteness of Λ_1, Λ_2 (you can see that if $\Lambda_1 = \{0\}$ then $\Lambda_1 - \Lambda_2$ is strongly discrete for every strongly discrete Λ_2).

Let Λ_1, Λ_2 be relatively dense sets, and let $\lambda_s \neq \lambda_s'$ be points from Λ_2 such that $\lambda_s - \lambda_s' \to 0$ as $s \to \infty$. There are $R < \infty$ and $x_s \in \Lambda_1$ such that $\lambda_s, \lambda_s' \in B(x_s, R)$. Therefore there exist infinitely many points of the set $\Lambda_1 - \Lambda_2$ in the ball $B(R)$, which is impossible. Hence $\eta(\Lambda_2) > 0$ and, similarly, $\eta(\Lambda_1) > 0$.

If f_1, f_2 satisfy the conditions of Theorem 2.1, then the almost periodicity of the distributions f_1, f_2 follows from Theorem 2.5, part i) or ii). In particular, Λ_1 and Λ_2 are relatively dense. If f_1, f_2 satisfy the conditions of Theorem 2.3, then, by Corollary 4.5, we get that $\hat{f}_j = \sum_{\gamma \in \Gamma_j} q_\gamma^{(j)} \delta_\gamma$ are measures and

$$q_\gamma^{(j)} = O(|\gamma|^m), \quad |\gamma| \to \infty,$$

for some $m < \infty$. Using part iii) of Theorem 2.5, we obtain that both f_j are almost periodic distributions as well. Hence the sets Λ_1, Λ_2 are uniformly discrete both under the conditions of Theorem 2.1 and under those of Theorem 2.3. By Proposition 3.1, $\operatorname{ord} f_j < \infty$, $j = 1, 2$.

We apply Proposition 4.3 to the measure $\tilde{\mu}(y) = \hat{f}_1(-y)$ with $m = 0$ and $H = T - d$ (we can assume that $T \geq d$). Since the Fourier transform of $\tilde{\mu}(y)$ is just the tempered distribution f_1, with uniformly discrete support, we get the representation (2) for f_1 with

$$\sup_{\lambda \in \Lambda_1} \sup_{\|k\| \leq \operatorname{ord} f_1} |p_{\lambda,k}| < \infty.$$

Put $M = \inf_{\lambda \in \Lambda_1} \kappa_{f_1}(\lambda)$, and pick a number $\varepsilon \in (0, \eta(\Lambda_1)/5)$ such that

$$\sum_{\|k\| \leq \operatorname{ord} f_1} |p_{\lambda,k}| < M/(2\varepsilon) \quad \forall \lambda \in \Lambda_1.$$

Let $\varphi_\varepsilon(x)$ be a nonnegative C^∞-function such that

$$\operatorname{supp}\varphi_\varepsilon \subset B(\varepsilon + \varepsilon^2), \quad \varphi_\varepsilon(x) = 1 \text{ for } |x| < \varepsilon.$$

Put

$$\varphi_{k,\varepsilon}(x) = \varphi_\varepsilon(x)x^k/k!, \quad k \in (\mathbb{N} \cup \{0\})^d.$$

Since f_1 is almost periodic, we see that the function $g_{k,\varepsilon} = f_1 \star \varphi_{k,\varepsilon}$ is almost periodic.

Fix $\lambda \in \Lambda_1$. By (4), there exists k' such that $|p_{\lambda,k'}| \geq M$. Put

$$A = \{k \in (\mathbb{N} \cup \{0\})^d : k_j \leq k'_j \ \forall j = 1, \ldots, d\}.$$

For $x \in B(\lambda, \varepsilon)$ we have

$$g_{k',\varepsilon}(x) = \left(\sum_{\|k\| \leq \operatorname{ord} f_1} p_{\lambda,k} D^k \delta_\lambda \right) \frac{(\lambda - x)^{k'}}{k'!} = \sum_{k \in A} p_{\lambda,k}(-1)^{\|k\|}(\lambda - x)^{k'-k}/(k'-k)!.$$

Therefore,

$$|g_{k',\varepsilon}(x)| \geq |p_{\lambda,k'}| - \varepsilon \sum_{k \in A, k \neq k'} |p_{\lambda,k}| > M/2.$$

Now set

$$h_\varepsilon(x) = \min\{1, 2M^{-1} \sup_{\|k\| \leq \operatorname{ord} f_1} |g_{k,\varepsilon}(x)|\}.$$

Clearly, $h_\varepsilon(x)$ is an almost periodic function and

$$h_\varepsilon(x) = 1 \text{ for } x \in \bigcup_{\lambda \in \Lambda_1} B(\lambda, \varepsilon), \qquad \operatorname{supp} h_\varepsilon \subset \bigcup_{\lambda \in \Lambda_1} B(\lambda, \varepsilon + \varepsilon^2), \qquad 0 \leq h_\varepsilon(x) \leq 1.$$

Let ψ be an arbitrary continuous function on $\mathbb{R}^d$ with support in the ball $B(\eta(\Lambda_1)/5)$. It is readily seen that the function $\psi \star h_\varepsilon$ is almost periodic as well. Since $\varepsilon < \eta(\Lambda_1)/5$, we see that for each fixed $t \in \mathbb{R}^d$ the support of the function $\psi(\cdot - t)$ intersects with at most one ball $B(\lambda, \varepsilon + \varepsilon^2)$. Therefore we get

$$\left| \frac{\psi \star h_\varepsilon(t)}{\omega_d \varepsilon^d} - \frac{1}{\omega_d \varepsilon^d} \int_{B(\lambda,\varepsilon)} h_\varepsilon(x)\psi(x - t)dx \right| \leq \left| \frac{1}{\omega_d \varepsilon^d} \int_{B(\lambda,\varepsilon+\varepsilon^2) \setminus B(\lambda,\varepsilon)} h_\varepsilon(x)\psi(x - t)dx \right|$$

$$(21) \hspace{4cm} \leq \frac{\omega_d(\varepsilon + \varepsilon^2)^d - \omega_d \varepsilon^d}{\omega_d \varepsilon^d} \sup_{\mathbb{R}^d} |\psi(x)|,$$

and

$$(22) \hspace{1cm} \left| \frac{1}{\omega_d \varepsilon^d} \int_{B(\lambda,\varepsilon)} h_\varepsilon(x)\psi(x - t)dx - \psi(\lambda - t) \right| \leq \sup_{x \in B(\lambda,\varepsilon)} |\psi(x - t) - \psi(\lambda - t)|.$$

It follows from (21) and (22) that the almost periodic functions $(\omega_d \varepsilon^d)^{-1}(\psi \star h_\varepsilon)$ converge uniformly as $\varepsilon \to 0$ to the function $\psi \star \delta_{\Lambda_1}$, where $\delta_{\Lambda_1} = \sum_{\lambda \in \Lambda_1} \delta_\lambda$. Hence the function $\psi \star \delta_{\Lambda_1}$ is almost periodic. It is readily seen that the same is true for every continuous function ψ on $\mathbb{R}^d$ with a compact support. A similar construction works for $\delta_{\Lambda_2} = \sum_{\lambda \in \Lambda_2} \delta_\lambda$.

Thus $\delta_{\Lambda_1}, \delta_{\Lambda_2}$ are almost periodic measures with a strongly discrete set of differences $\Lambda_1 - \Lambda_2$. To finish the proof we apply the following theorem.

Theorem 6.1 (see [4], Theorem 6). *Let μ_1, μ_2 be almost periodic measures on $\mathbb{R}^d$ with countable supports, and $\inf_{x \in \mathbb{R}^d} |\mu_1(x)| > 0$, $\inf_{x \in \mathbb{R}^p} |\mu_2(x)| > 0$. If the set of differences of points of $\operatorname{supp} \mu_1$ and $\operatorname{supp} \mu_2$ is strongly discrete, then the supports of both measures are finite unions of translates of a single full-rank lattice L.* ∎

References

[1] M. Baake and R. Moody, eds., Directions in Mathematical Quasicrystals, CRM Monograph series, **13**, AMS, Providence RI, 379p.

[2] C. Corduneanu, Almost Periodic Functions, Second English ed.) Chelsea, New-York, 1989 (Distributed by AMS and Oxford University Press).

[3] S. Yu. Favorov, Bohr and Besicovitch Almost Periodic Discrete Sets and Quasicrystals, Proc. Amer. Math. Soc. **140** (2012), 1761–1767.

[4] S. Yu. Favorov, Fourier Quasicrystals and Lagarias' Conjecture, Proc. Amer. Math. Soc. **144** (2016) , 3527-3536.

[5] S. Yu. Favorov, Some Properties of Measures with Discrete Support, Matematychni Studii **46**, No. 2 (2016), 189-195.

[6] S. Yu. Favorov, Large Fourier Quasicrystals and Wiener's Theorem, J. Fourier Anal. Appl., to appear.

[7] J. G. de Lamadrid, L. N. Argabright, Almost Periodic Measures. Memoirs of the AMS, No. 428, Providence RI, (1990), 218p.

[8] N. Lev, A. Olevskii, Measures with Uniformly Discrete Support and Spectrum. C. R. Acad. Sci., ser. 1 **351**, 599-603 (2013).

[9] N. Lev, A. Olevskii, Quasicrystals and Poisson's Summation Formula. Invent. Math. **200**, 585–606 (2015).

[10] N. Lev, A. Olevskii, Fourier Quasicrystals and Discreteness of the Diffraction Spectrum. Advances in Mathematics, **315**, 1-26 (2017).

[11] N. Lev, A. Olevskii, Quasicrystals with Discrete Support and Spectrum. Rev. Mat. Iberoam., **32**, no.4, 1341-1252 (2016).

[12] M. N. Kolountzakis. On the Structure of Multiple Translations Tilings by Polygonal Regions, Discr. Comp. Geom. **23** (2000), 4, 537–553.

[13] M. N. Kolountzakis. Fourier Pairs of Discrete Support with Little Structure, J. Fourier Anal. Appl. **22** (2016), 1, 1–5.

[14] M. N. Kolountzakis, J. C. Lagarias, Structure of Tilings of the Line by a Function, Duke Math. Journal, **82**, (1996), 653-678.

[15] J. C. Lagarias, Mathematical Quasicrystals and the Problem of Diffraction, in [1], 61-93.

[16] J. C. Lagarias, Geometric Models for Quasicrystals I. Delone Set of Finite Type, Discr. Comp. Geom. **21** (1999), 161–191.

[17] Y. Meyer, Quasicrystals, Almost Periodic Patterns, Mean–periodic Functions, and Irregular Sampling, African Diaspora Journal of Mathematics, **13** (1), 1–45 (2012).

[18] Y. Meyer, Guinand's Measure are Almost Periodic Distributions, Bull. of the Hellenic Math. Soc., **61**, 11–20 (2017).

[19] V. P. Palamodov, A Geometric Characterization of a Class of Poisson Type Distributions, J. Fourier Anal. Appl. **23** (2017), 5, 1227–1237.

[20] J. Patera, ed., Quasicrystals and Discrete Geometry, Fields Institute Monographs, AMS, Providence RI, 289 p.

[21] L. I. Ronkin Almost Periodic Distributions and Divisors in Tube Domains, Zap. Nauchn. Sem. POMI **247** (1997). P. 210–236 (Russian).

[22] W. Rudin Functional Analysis, McGraw-Hill Book Company, New York, St. Louis, Sun Francisco, (1973), 443p.

[23] V. S. Vladimirov Equations of Mathematical Physics, Marcel Dekker Inc., New York, 1971, 418p.

KARAZIN'S KHARKIV NATIONAL UNIVERSITY, SVOBODY SQ., 4, 61022, KHARKIV, UKRAINE
E-mail address: `sfavorov@gmail.com`