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A REMARK ON PROJECTIVE LIMITS OF FUNCTION SPACES

A. KAMPOUKOU AND V. NESTORIDIS

ABSTRACT. Let $\Omega \subset \mathbb{C}^d$ be an open set and $K_m, m = 1, 2, \dots$ an exhaustion of Ω by compact subsets of Ω. We set $\Omega_m = K_m^\circ$ and let $X_m(\Omega_m)$ be a topological space of holomorphic functions on Ω_m between $A^\infty(\Omega_m)$ and $H(\Omega_m)$. Then we show that the projective limit of the family $X_m(\Omega_m)$, $m = 1, 2, \dots$, under the restriction maps is homeomorphic and linearly isomorphic to the Fréchet space $H(\Omega)$, independently of the choice of the spaces $X_m(\Omega_m)$.

1. Introduction

Let Y be a space of holomorphic functions on a domain Ω in $\mathbb{C}^d, d = 1, 2, \dots$. For most of the spaces Y encountered the following holds. If for every bump V of Ω there exists a function f_V in Y which is not holomorphically extendable to V, then there exists a function f in Y not extendable to any V. Moreover, the set of such functions f in Y is dense and G_δ in Y (see [5], [6]). The term "bump" has been used in [6]. According to [6] if Ω is a domain, a "bump" of Ω is a pair (B_1, B_2) of two opens balls such that $B_1 \subset \overline{B_1} \subset B_2$, $B_2 \cap \Omega \neq \emptyset$ and $B_2 \cap \Omega^c \neq \emptyset$.

Examples of such spaces are the space $H(\Omega)$ of all holomorphic functions in Ω endowed with the topology of uniform convergence on compacta, the spaces $A^p(\Omega)$ and $H^p_\infty(\Omega)$, Bergman spaces, Hardy spaces etc., provided that Ω satisfies certain criteria. We notice that our space $A^p(\Omega)$ is not a Bergman space. For us, $A^p(\Omega)$ is the space of all holomorphic functions $f \in H(\Omega)$ such that each partial derivative of f of order at most p has a continuous extension on $\overline{\Omega}$ endowed with its natural topology.

Most of these spaces, endowed with their natural topology, contain $A^\infty(\Omega)$ and they are contained in $H(\Omega)$: $A^\infty(\Omega) \subset Y \subset H(\Omega)$. Furthermore, the two inclusion maps $i_1 \colon A^\infty(\Omega) \to Y$, $i_2 \colon Y \to H(\Omega)$ with $i_1(f) = f$ and $i_2(w) = w$ are continuous. We denote by $M(\Omega)$ the set of such topological function spaces on Ω.

Let K_m, $m = 1, 2, \dots$, be an exhaustion by compact subsets of the open set $\Omega \subset \mathbb{C}^d$ and $\Omega_m = K_m^\circ$. Let $X_m(\Omega_m)$ be any topological function space belonging to $M(\Omega_m)$. Let X denote the projective (inverse) limit of the family $X_m(\Omega_m), m = 1, 2, \dots$ where $T_{m,n} : X_m(\Omega_m) \to X_n(\Omega_n), n \leq m$ is the restriction map $f \to f|_{\Omega_n}$.

Then we show that X is a topological vector space homeomorphic and linearly isomorphic to the Fréchet space $H(\Omega)$ of all holomorphic functions on Ω endowed with the topology of uniform convergence on all compact subsets of Ω, independently of the choice of the topological spaces $X_m(\Omega_m)$ in $M(\Omega_m)$. This holds even if the topological spaces $X_m(\Omega_m)$ are neither complete, nor vector spaces.

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2. About projective limits

We need the following particular case of projective limits (see [3]).

Definition 2.1. Let $(Y_n, \mathcal{T}_n), n = 1, 2, \dots$ be a sequence of topological spaces. For $n < m$, let $T_{m,n} \colon Y_m \to Y_n$ be a continuous map and let $T_{n,n}$ be the identity map from Y_n to itself. We assume that $T_{m,s} = T_{n,s} \circ T_{m,n}$ for all $s \le n \le m$.

Then the projective (or inverse) limit of this family is defined to be the following subset Y of the cartesian product $\prod_{n=1}^{\infty} Y_n$ where

$$Y = \left\{ (g_1, g_2, \dots) \in \prod_{n=1}^{\infty} Y_n \colon T_{m,n}(g_m) = g_n \text{ for all } n \le m \right\}.$$

The topology $\mathcal{T}_Y$ on Y which we consider is the relative topology from $\prod_{n=1}^{\infty} Y_n$ endowed with the product topology.

Proposition 2.2. [3] *Under the above notation and assumptions, a net $g^i = (g_1^i, g_2^i \dots) \in Y$, $i \in I$ converges to $g = (g_1, g_2, \dots) \in Y$ in the topology $\mathcal{T}_Y$ of Y if and only if $g_k^i \to g_k$ in the topology T_k of Y_k for all $k = 1, 2, \dots$.*

3. Some function spaces

Let Ω be an open subset of $\mathbb{C}^d$, $d \ge 1$. We denote by $H(\Omega)$ the space of holomorphic functions on Ω endowed with the topology of the uniform convergence on compacta. This is a Fréchet space with seminorms $\|f\|_m = \sup_{z \in K_m} |f(z)|, m = 1, 2, \dots$ where K_m is an exhaustion of Ω by compact subsets of Ω; for instance we can take (see [7], [4])

$$K_m = \left\{ z \in \Omega : \|z\| \le m \text{ and } \operatorname{dist}(z, \Omega^c) \ge \frac{1}{m} \right\}.$$

In what follows, we assume that G is a bounded open subset of $\mathbb{C}^d$, although the spaces that we will consider can also be defined for an unbounded open set G, with some modifications in the definition of their natural topology.

The space $A^\infty(G)$ is defined as the set of holomorphic functions f on G such that each mixed partial derivative of f extends continuously on the closure $\overline{G}$ of G. The natural topology of $A^\infty(G)$ is defined by the seminorms

$$\sup_{z \in G} \left| \frac{\partial^{a_1 + \dots + a_d} f}{\partial {z_1}^{a_1} \dots \partial {z_d}^{a_d}}(z) \right|, a_1, \dots, a_d \in \{0, 1, 2, \dots\}, \text{ where } z = (z_1, \dots, z_d).$$

This is also a Fréchet space .

Next we are interested in spaces $Y = X(G)$ containing $A^\infty(G)$ and contained in $H(G)$, i.e. $A^\infty(G) \subset Y \subset H(G)$. We also require that the inclusion maps are continuous; that is, if a net $f^i \in A^\infty(G)$, $i \in I$, converges in the topology of $A^\infty(G)$ to a function $f \in A^\infty(G)$, then $f^i \to f$ in the topology of $X(G)$, as well; and if a net $w^i \in X(G)$, $i \in I$, converges to $w \in X(G)$ in the topology of $X(G)$ then $w^i \to w$ in the topology of $H(G)$. We denote by $M(G)$ the set of all such topological spaces.

Such spaces are the spaces $A^p(G)$ consisting of all functions $f \in H(G)$ for which every mixed partial derivative of order less than or equal to p extends continuously on $\overline{G}$ and the spaces $H_\infty^p(G)$ consisting of all functions $f \in H(G)$ for which every mixed partial derivative of order less than or equal to p is bounded on $\overline{G}$. The above two spaces are Banach spaces, provided that $p < +\infty$ and Fréchet spaces if $p = +\infty$. Their topologies are defined by the seminorms

$$\sup_{z\in G}\left|\frac{\partial^{a_1+\cdots+a_d} f}{\partial {z_1}^{a_1}\ldots\partial {z_d}^{a_d}}(z)\right|, a_1+\cdots+a_d \le p, \text{ where } z=(z_1,\ldots,z_d).$$

Another example of such a space is the p-Bergman space, $0 < p < +\infty$, consisting of all holomorphic functions f on G, such that $\int_G |f|^p d\nu < \infty$, where $d\nu$ denotes the Lebesgue measure on $G \subset \mathbb{C}^d$ (see [2]).

Other examples are Hardy spaces on the disc or the ball or on more general domains (see [1], [8], [9]) , as well as subclasses of the Nevanlinna class, BMOA, VMOA, Dirichlet spaces on the disc.

All the previous spaces are complete spaces. However, we can also consider non complete spaces containing $A^\infty(G)$ and contained in $H(G)$ such that the two injections are continuous. For instance, consider the set $Y = A^\infty(G)$ with the relative topology induced by $H(G)$; that is, for $f_n, f \in A^\infty(G)$ the convergence in the new space is the uniform convergence on compacta $f_n \to f$. This space contains $A^\infty(G)$, is contained in $H(G)$ and the injections $i_1 : A^\infty(G) \to Y$, $i_2\colon Y \to H(G)$ with $i_1(f) = f$ and $i_2(g) = g$ are continuous where $A^\infty(G)$ and $H(G)$ are endowed with their natural topologies and Y with the relative topology induced by $H(G)$. However, this space is not complete in general. For instance, if $(\overline{G})^\circ = G$ or if G is a bounded, open subset of $\mathbb{C}$, then $A^\infty(G)$ with the relative topology induced by $H(G)$ is not complete.

It is also possible that a topological space belongs to $M(G)$ although it is not even a vector space. To give such an example we consider the set

$$Y = A^\infty(G) \cup \{f\colon G \to \mathbb{C} \text{ holomorphic such that } |f(z)| < 1, \text{ for all } z \in G\},$$

endowed with the topology of uniform convergence on G. Then $A^\infty(G) \subset Y \subset H(G)$ and if $f_n, f \in A^\infty(G)$ are such that $f_n \to f$ in the topology of $A^\infty(G)$, it follows trivially that $f_n \to f$ in the topology of Y. Also if $\phi_n, \phi \in Y$ are such that $\phi_n \to \phi$ in the topology of Y, then obviously the convergence is uniform on each compact subset of G.

4. The result

We start with the following definition.

Definition 4.1. Let G be a bounded open subset of $\mathbb{C}^d$. We denote by $M(G)$ the set of all topological spaces $X(G)$ satisfying $A^\infty(G) \subset X(G) \subset H(G)$ and such that the two injections $i_1\colon A^\infty(G) \to X(G)$, $i_1(f) = f$ and $i_2\colon X(G) \to H(G)$, $i_2(g) = g$ are continuous; that is, if a net $f^i \in A^\infty(G)$, $i \in I$, converges to $f \in A^\infty(G)$, in the topology of $A^\infty(G)$, then $f^i \to f$ in the topology of $X(G)$, as well. Moreover, if a net $w^i \in X(G)$ converges to $w \in X(G)$, in the topology of $X(G)$, then $w_i \to w$ uniformly on compacta of G.

Let Ω be an open subset of $\mathbb{C}^d$, possibly unbounded. Let $K_m, m = 1, 2, \ldots$ be an exhausting family of compact subsets of Ω; that is, $K_m \subset K^\circ_{m+1}$ and $\cup_{m=1}^\infty K_m = \Omega$ (see [7], [4]). We set $\Omega_m = K^\circ_m$; then $\Omega_m \subset \overline{\Omega_m} \subset \Omega_{m+1}$. For each $m = 1, 2, \ldots$ let $X_m(\Omega_m)$ be a space in $M(\Omega_m)$.

For $n \le m$ we consider the restriction map $T_{m,n}\colon X_m(\Omega_m) \to X_n(\Omega_n)$ given by $T_{m,n}(f) = f|_{\Omega_n}$, for every $f \in X_m(\Omega_m)$. These maps are well defined and continuous. Indeed, $X_m(\Omega_m) \subset H(\Omega_m)$ and $\overline{\Omega_n} \subset \Omega_m$, provided that $n < m$; thus, for every $f \in X_m(\Omega_m)$, it follows that $T_{m,n}(f) = f|_{\Omega_n} \in A^\infty(\Omega_n) \subset X_n(\Omega_n)$. The maps $T_{m,n}$ are clearly well-defined and continuous if $n = m$ (because $T_{m,n}$ is the identity map on $X_m(\Omega_m)$). Also if a net $f^i \in X_m(\Omega_m)$ converges to $f \in X_m(\Omega_m)$, in the topology of $X_m(\Omega_m)$, then by assumption, it converges uniformly on compacta of Ω_m. The Weierstrass theorem yields uniform convergence on compacta of every mixed partial derivative. Since $\overline{\Omega_n}$ is a compact subset of Ω_m, it follows that $f^i|_{\Omega_n} \to f|_{\Omega_n}$ in the topology of $A^\infty(\Omega_n)$. By assumption, this implies $f^i \to f$ in the topology of $X_n(\Omega_n)$. Therefore, $T_{m,n} : X_m(\Omega_m) \to X_n(\Omega_n)$ is continuous. One can also very easily verify that $T_{m,s} = T_{n,s} \circ T_{m,n}$, for all $s \le n \le m$. Therefore, we can define the projective limit X, which is a topological space (see [3]).

Theorem 4.2. *Under the above assumptions and notation the topological space X is homeomorphic and linearly isomorphic to $H(\Omega)$.*

Proof. Let $f \in H(\Omega)$; then $f|_{\Omega_m} \in A^\infty(\Omega_m) \subset X_m(\Omega_m)$. Thus, if we set $g_m = f|_{\Omega_m}$ and $g = (g_1, g_2, \dots) = g(f)$, then $g \in \prod_{m=1}^\infty X_m(\Omega_m)$. For $n \le m$ one can easily verify that $T_{m,n}(g_m) = g_n$. Therefore, $g \in X$. In this way we have defined a map $\Phi \colon H(\Omega) \to X$, where $\Phi(f) = g(f)$.

The map Φ is one to one. Indeed, if $g(f) = g(w)$ it follows that $f|_{\Omega_m} = w|_{\Omega_m}$ for all $m = 1, 2, \dots$. Since $\cup_m \Omega_m = \Omega$, it follows that $f(z) = w(z)$ for all $z \in \Omega$; thus $f = w$. This proves that the map Φ is one to one.

The map Φ is also onto. Indeed, if $g = (g_1, g_2, \dots) \in X$, then for every $z \in \Omega$ there exists a least natural number $n(z)$ so that $z \in \Omega_m$, for all $m \ge n(z)$, because $\Omega_m \subset \Omega_{m+1}$ and $\cup_m \Omega_m = \Omega$. Furthermore, since $T_{m,n(z)}(g_m) = g_{n(z)}$, it follows that $g_m(z) = g_{n(z)}(z)$, for all $m \ge n(z)$. We set $f(z) = g_{n(z)}(z)$. We will show that this function is holomorphic on Ω. Indeed, let $z_0 \in \Omega$; then $z_0 \in \Omega_{n(z_0)}$. Since $\Omega_{n(z_0)}$ is open, there exists a ball $B(z_0, r)$ contained in $\Omega_{n(z_0)}$. For each $z \in B(z_0, r)$, we have $n(z) \le n(z_0)$ and $f(z) = g_{n(z)}(z) = g_{n(z_0)}(z)$, but $g_{n(z_0)} \in X_{n(z_0)}(\Omega_{n(z_0)}) \subset H(\Omega_{n(z_0)})$. It follows that f is holomorphic in $B(z_0, r)$. Since z_0 is arbitrary in Ω, it follows that $f \in H(\Omega)$. It is immediate that $\Phi(f) = g$. Since g is arbitrary in X, it follows that the map Φ is onto.

Further, Φ is continuous. Indeed, if a net $f^i \in H(\Omega)$, $i \in I$ converges to $f \in H(\Omega)$, uniformly on compacta, by Weierstrass theorem the same holds for every mixed partial derivative. Since $\overline{\Omega_m}$ is a compact subset of Ω it follows that $f^i|_{\Omega_m} \to f|_{\Omega_m}$ in the topology of $A^\infty(\Omega_m)$. By assumption this implies the convergence $f^i|_{\Omega_m} \to f|_{\Omega_m}$ in the topology of $X_m(\Omega_m)$. According to Proposition 2.2, it follows that $\Phi(f^i) \to \Phi(f)$ in the topology of X. Thus, the map Φ is continuous.

Finally, we will show that Φ^{-1} is also continuous. Suppose that $\Phi(f^i) = (f^i|_{\Omega_1}, f^i|_{\Omega_2}, \dots) \to \Phi(f) = (f|_{\Omega_1}, f|_{\Omega_2}, \dots)$ in the topology of X. According to Proposition 2.2, this implies that $f^i|_{\Omega_m} \to f|_{\Omega_m}$ for all $m = 1, 2, \dots$ in the topology of $X_m(\Omega_m)$, which, by assumption, implies the uniform convergence on compacta of Ω_m. We will show that $f^i \to f$ uniformly on each compact set $K \subset \Omega$. Let K be such a compact set. Since $K \subset \Omega = \cup_{m=1}^\infty \Omega_m$ and the open sets Ω_m satisfy $\Omega_m \subset \Omega_{m+1}$, by compactness of K it follows that there exists m_0 such that $K \subset \Omega_{m_0}$, but $f^i|_{\Omega_{m_0}} \to f|_{\Omega_{m_0}}$ uniformly on each compact subset of Ω_{m_0}; in particular on K. Thus, $f^i \to f$ uniformly on each compact set $K \subset \Omega$ and therefore, $f^i \to f$ in the topology of $H(\Omega)$. Thus, Φ^{-1} is continuous, and Φ is a homeomorphism. This completes the proof. $\square$

Remark 4.3. It is well known and easy to verify that a denumerable projective limit of Fréchet spaces is also a Fréchet space. In our case X is homeomorphic to the Fréchet space $H(\Omega)$, although the spaces $X_m(\Omega_m)$ may not be Fréchet spaces. Furthermore, independently of the fact that the spaces $X_m(\Omega_m)$ are topological vector spaces nor do they have linear structure, one can easily verify that X is always a topological vector space and that the map Φ is a linear isomorphism.

Remark 4.4. Let us consider the compact set

$$K_t = \left\{ z \in \Omega \colon |z| \leq t, \operatorname{dist}(z, \Omega^c) \geq \frac{1}{t} \right\}, \, t \geq t_0,$$

for a convenient choice of $t_0 > 0$ such that $K_{t_0}^\circ \neq \emptyset$. Then for $t < s$ it follows that $K_t \subseteq K_s^\circ$ and $\cup_{t \geq t_0} K_t = \Omega$. We set $\Omega_t = K_t^\circ$ and let $X_t(\Omega_t)$ be any element of $M(\Omega_t)$. One can prove that the projective limit of the family $X_t(\Omega_t)$ as $t \to +\infty$, where $T_{s,t}$ is the restriction map $f \to f|_{\Omega_t}$, $t < s$, is homeomorphic and linearly isomorphic to $H(\Omega)$. The proof is similar to that of Theorem 4.2.

Remark 4.5. As one of the referees suggested, the result of theorem 4.2 can be adapted in order to hold for harmonic functions on domains of $\mathbb{R}^d$.

Let G be an open and bounded subset of $\mathbb{R}^d$, $d \geq 2$. We denote by $h(G)$ the set of harmonic functions on G endowed with the topology of the uniform convergence on compacta and by $a^\infty(G)$ the set of harmonic functions u on G such that each mixed partial derivative of u extends continuously on the closure $\overline{G}$ of G, endowed with the seminorms $\sup_{z \in G} \left| \frac{\partial^{a_1 + \cdots + a_d} u}{\partial {x_1}^{a_1} \ldots \partial {x_d}^{a_d}}(x) \right|$, $a_1, \ldots, a_d \in \{0, 1, 2, \ldots\}$, where $x = (x_1, \ldots, x_d)$.

We denote by $m(G)$ the set of all topological spaces $X(G)$ satisfying $a^\infty(G) \subset X(G) \subset h(G)$ and such that the two injections $i_1 \colon a^\infty(G) \to X(G)$, $i_1(u) = u$ and $i_2 \colon X(G) \to h(G)$, $i_2(w) = w$ are continuous; that is, if a net $u^i \in a^\infty(G)$, $i \in I$, converges to $u \in a^\infty(G)$, in the topology of $a^\infty(G)$, then $u^i \to u$ in the topology of $X(G)$, as well. Moreover, if a net $w^i \in X(G)$ converges to $w \in X(G)$, in the topology of $X(G)$, then $w_i \to w$ uniformly on compacta of G.

Let Ω be an open subset of $\mathbb{R}^d$ and let K_m be an exhaustive sequence of compact subsets of Ω. We set $\Omega_m = K_m^\circ$. Let $X_m(\Omega_m) \in m(\Omega_m)$. Let also $T_{m,n}$ be the restriction maps as above. Then the projective limit X of the sequence $(X_m(\Omega_m), T_{m,n})$, is homeomorhic and linearly isomorphic to $h(\Omega)$.

The proof is similar to that of Theorem 4.2.

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Department of Mathematics, National and Kapodistrian University of Athens, Panepistemiopolis, 15784 Athens, Greece.
E-mail address: kampouk@math.uoa.gr

Department of Mathematics, National and Kapodistrian University of Athens, Panepistemiopolis, 15784 Athens, Greece.
E-mail address: vnestor@math.uoa.gr