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Mixed binary/ternary covering codes

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Abstract

In this paper we investigate optimal mixed binary/ternary covering codes with b binary coordinates, t ternary coordinates and covering radius 1. We show that there exist optimal codes that are not equivalent to the direct product of F_2^b, F_3^{t-4} and the perfect ternary Hamming code of length 4.

Keywords: covering radius, mixed covering codes, football pool problem

1 Introduction

Covering problems arise naturally from variety of practical needs. A general description of covering problem can be done in the following way. Assume we have certain number of questions Q_1, $Q_2,\ldots,$ Q_n. For each i, $1 \le i \le n$ the set A_i consists of all feasible answers of question Q_i. For given positive integer R we want to compose a set ζ of n-tuples $(a_1, a_2, ..., a_n)$ where $a_i \in A_i$ for each i, $1 \le i \le n$ having the following property. For any n-tuple $\mathbf{y} = (y_1, y_2, ..., y_n)$ of feasible answers there exists an element $\mathbf{x} = (x_1, x_2, ..., x_n)$ from ζ such that $\mathbf{y}$ and $\mathbf{x}$ differ in at most R coordinates. The set ζ is referred to as a covering code and a covering code of minimum cardinality is called optimal.

The most important application of covering codes is connected to error-correcting codes. These are codes designed to correct errors when a message (usually binary) is sent through a noisy channel. The covering radius of a code is the minimum positive integer R such that any vector lies within Hamming distance R from a codeword. The Hamming distance $d(x, y)$ between two vectors $\mathbf{x}$ and $\mathbf{y}$ is the number of coordinates in which they differ. The covering radius is important characteristic of the code. For example, it gives the maximum number of errors that can be corrected.

However, probably the first application of covering codes is connected with football pools. Assume we have certain number (usually 13 or 14) of football matches and for each one of them we have three alternatives: mark it by 1 if we think the home team will win; mark it by **x** if we think that the two teams will draw and mark it by 2 if we think that the away team will win. When we mark all the games we create a prediction. If, after all the games have finished and the results are known, we have a prediction whit correct guess to every single game, we win first prize. If we have only one incorrect guess then we win second prize and so on. Since each prediction is paid for, we wish to design a set of as little as possible predictions that ensures first, second, third or fourth prize. It is clear that to win first prize we need to use all feasible predictions, so we need 3^n predictions where n is the number of games.

In terms of coding theory the above problem is described as: for a positive integers n and $R = 1,2$ or 3 find and a ternary code ζ of length n and minimum cardinality, such that for any ternary vector **y** there exists a codeword $x \in \zeta$ such that d(**x; y**) $\leq R$, i.e. **y** and **x** differ in at most R coordinates. The described problem is known as the football pool problem. Assume for some of the games we think we know the exact outcome, for some b of the games we foresee one of the three outcomes as impossible and for t of the games we are not able to rule out any of the outcomes. Then, for those b of the games we have to choose between two guesses. The problem now reduces naturally to mixed binary/ternary covering codes with b binary and t ternary coordinates.

Minimizing the number of codewords in a mixed binary/ternary code of given covering radius is widely studied problem in coding theory [4], [5], [6], [7]. For $b + t \leq 14$ and $R = 1,2,3$ denote by $K(b,t,R)$ the minimum cardinality of a code with b binary, t ternary coordinates and covering radius R. For convenience let $K(b,t,1) = K(b,t)$. The problem of finding $K(b,t,R)$ is far from trivial. The exact values are known only for small values of b and t. To limit $K(b,t,R)$ we need to find a lower bound (usually obtained by combinatorial arguments) and an upper bound (obtained by explicit construction). When the two bounds coincide we have the exact value of $K(b,t,R)$. After knowing $K(b,t,R)$ the problem of interest is to find up to equivalence all optimal codes, i.e. codes with b binary, t ternary coordinates, covering radius R and cardinality $K(b,t,R)$. Two codes are equivalent if one can be obtained from the other by a permutation of the coordinate positions and a permutation of the elements of each coordinate. For good overview and interesting results on the football pool problem and mixed covering codes the reader is referred to [2], [3], [4], [5], [7].

In this paper we discuss the case $R = 1$. Let **x** be a codeword from a code ζ with b binary and t ternary coordinates and covering radius 1. It is easy to see that the number of vectors **y** such that d(**x**; **y**) ≤ 1 is equal to $2t + b + 1$. Therefore $|\zeta|.(2t + b + 1) \geq 2^b.3^t$, implying that

$$K(b,t) \geq \frac{2^b.3^t}{2t + b + 1} \tag{1}$$

We describe now an easy construction for obtaining upper bounds on $K(b+1,t,R)$ and $K(b,t+1,R)$ terms of $K(b,t,R)$.

Let ζ be a code of b binary, t ternary coordinates and covering radius R. Then $F_2 \times \zeta$ ($F_3 \times \zeta$ respectively) is a code of $b+1$ binary and t ternary (b binary and $t+1$ ternary, respectively) coordinates and covering radius R. Thus,

$$K(b+1,t,R) \leq 2K(b,t,R) \text{ and } K(b,t+1,R) \leq 3K(b,t,R).$$

The above construction applied repeatedly leads to

$$K(b+i,t+j,R) \leq 2^i.3^j K(b,t,R) \tag{2}$$

where i and j are positive integers.

Remark. Note that instead of the direct product $F_2 \times \zeta$ we can use $\{\{0\} \times \zeta_1\} \cup \{\{1\} \times \zeta_2\}$ where ζ_1 and ζ_2 are equivalent to ζ. Denote such a construction by $F_2 \odot \zeta$ and by analogy define $F_2^i \times F_3^j \odot \zeta$ for some nonnegative integers i and j.

An up to date tables of known results for mixed binary/ternary covering codes are maintained at: www.sztaki.hu/~keri/codes/, [6]. Some of the known results for $R = 1$ are shown in Table 1.

b	*t*	*K(b,t)*
0	4	9
1	4	18
0	5	27
2	4	36
1	5	45-54
3	4	60-72
0	6	71-73
4	4	107-128

Table 1. Exact values and bounds on $K(b,t)$

It is seen from the above table that the exact value of $K(b,t)$ is not known for as small values of b and t as $b=1$ and $t=5$. Note that $K(0,4)=9$ implies that (1) turns into equality for $b=0$ and $t=4$. We infer that the balls of radius 1 with centers the codewords do not intersect and cover the whole vector space F_3^4. Such codes are called perfect. It is well known that up to equivalence there exists unique perfect ternary code of length 4 and covering radius 1. It is given by:

$$\xi = \{0000, 0111, 0222, 1012, 1120, 1201, 2021, 2210, 2102\}:$$

This particular code is a member of the important family of Hamming codes. All codes of the form $F_2^b \times F_3^{b-4} \odot \xi$ are referred to as *appended* Hamming codes. Further, note that all upper bounds for $K(1,4)$, $K(0,5)$, $K(2,4)$, $K(1,5)$ and $K(3,4)$ are attained for appended Hamming codes. The lower bound $K(2,4) \geq 36$ implying $K(2,4)=36$ is proven in [2]. For each of the above cases it is important to know whether all optimal (or those giving the best known upper bounds for $(b,t)=(1,5)$ or $(3,4)$) codes are appended Hamming codes.

2 Results

In this section for each of the cases $(b,t)=(1,4);(0,5);(2,4);(1,5)$ and $(3,4)$ we answer whether all optimal codes are appended Hamming codes.

Proposition 1. An optimal covering code with $b=1$ and $t=4$ is appended Hamming code.

Proof. Suppose C is an optimal covering code with $b=1$ and $t=4$ and covering radius 1. Since $K(1,4)=18$ we have $|C|=18$ and assume the first coordinate is the binary one. Let for $i=0,1; j=0,1,2$ and $q \in \{2,3,4,5\}$, P_{ij}^q be the set of all vectors with first coordinate i and q-th coordinate j without the first and the q-th coordinates; C_{ij}^q be the set of all codewords with first coordinate i and q-th coordinate j without the first and and the q-th coordinates. Let $c_{ij}^q = \left|C_{ij}^q\right|$ and for simplicity $P_{ij}^2 = P_{ij}$, $C_{ij}^2 = C_{ij}$ and $c_{ij}^2 = c_{ij}$ Note that a codeword from C_{ij} covers 7 vectors in P_{ij} and one vector in each P_{rs} where $r \in \{0,1\}$, $s \in \{0,1,2\}$ and $d(ij, rs) = 1$. Without loss of generality assume

$$c_{00} = \min\{c_{ij} \mid i \in \{0,1\}, j \in \{0,1,2\}\}.$$

Since $\left|P_{00}\right| = 27$ we have $7c_{00} + c_{01} + c_{02} + c_{10} \geq 27$ and therefore

$$18 = |C| = \sum_{i \in \{0,1\}, j \in \{0,1,2\}} c_{ij} \geq c_{00} + 27 - 7c_{00} + c_{11} + c_{12} \geq 27 - 4c_{00} :$$

The above inequality implies $4c_{00} \geq 9$ and therefore $c_{00} \geq 3$. Since

$$18 = |C| = \sum_{i \in \{0,1\}, j \in \{0,1,2\}} c_{ij} \geq 6c_{00} = 18$$

we infer that $c_{ij} = 3$ for all $i \in \{0,1\}, j \in \{0,1,2\}$. Hence $c_{ij}^q = 3$ for all $i \in \{0,1\}$, $j \in \{0,1,2\}$ and $q \in \{2,3,4,5\}$.

Let N be the number of covered vectors in P_{00} from the three codewords from C_{00}. Since $N \leq 21$ and $c_{01} + c_{02} + c_{01} \geq 27 - N$ we have $18 = \left|C\right| \geq 9 + 27 - N$, implying $18 \leq N \leq 21$. It is easy to see that up to equivalence there exist three sets of three codewords each such $18 \leq N \leq 21$. These are:

$$S_1 = \{000;\ 001;\ 222\};\ S_2 = \{000;\ 011;\ 222\};\ S_3 = \{000;\ 111;\ 222\}:$$

In the first case $N = 18$ and the following 9 vectors 110; 111; 112; 121; 211; 012; 120; 102; 210 are not covered by the codewords in C_{00} and therefore

$$C_{01} \cup C_{02} \cup C_{10} = \{110;\ 111;\ 112;\ 121;\ 211;\ 012;\ 120;\ 102;\ 210\}.$$

Now $c_{01}^3 = c_{02}^3 = 3$ implies that in $C_{01} \cup C_{02}$ there exist one vector with first coordinate 0, three vectors with first coordinate 1 and two vectors with first coordinate 2. We infer that $012;\ 210;\ 211 \in (C_{01} \cup C_{02})$. It follows that all vectors from C_{10} have first coordinate 1. Then C_{10} is not equivalent to either of S_1, S_2 or S_3, a contradiction.

In the second case we have that the following 8 vectors 110; 101; 112; 121; 102; 120; 201; 210 are not covered by the codewords in C_{00} and therefore

$$\{110;\ 101;\ 112;\ 121;\ 102;\ 120;\ 201;\ 210\} \subset (C_{01} \cup C_{02} \cup C_{10}).$$

Thus, there is one unknown vector in $C_{01} \cup C_{02} \cup C_{10}$. Now $c_{01}^3 = c_{02}^3 = 3$ implies that in $C_{01} \cup C_{02}$ there exist one vector with first coordinate 0, three vectors with first coordinate 1 and two vectors with first coordinate 2. We infer that the unknown vector from $C_{01} \cup C_{02} \cup C_{10}$ has first coordinate 0 and all vectors from C_{10} have first coordinate 1. Then C_{10} is not equivalent to either of S_1, S_2 or S_3, a contradiction.

In the third case note that for any two codewords **x** and **y** with first coordinate i we have $d(x, y) \geq 3$. Indeed, if $d(x, y) \leq 3$ then there exist $j \in \{0,1,2\}$ and $q \in \{2,3,4,5\}$ such that C_{ij}^q is not equivalent to S_3, a contradiction. We infer that the distance between any two vectors with equal first coordinate is at least three. It is easy to see that such a set is equivalent to the Hamming code ξ. This completes the proof. ⋄

It has been shown in [1] that an optimal covering code for $t = 5$ is appended Hamming code.

Proposition 2. There exist optimal codes for $b = 2$ and $t = 4$ that are not appended Hamming codes.

Proof. For a code C of 2 binary and 4 ternary coordinates assume the first two coordinates are the binary ones. For $i, j \in \{0,1\}$ let C_{ij} be the set of vectors obtained by the codewords with first and second coordinates equal to i and j respectively without these two coordinates.

Note that if C is an appended Hamming code then all sets C_{ij} are equivalent to ξ. Consider the code C having:

$$C_{00} = \{0000;\ 0111;\ 0222;\ 1012;\ 1120;\ 1201;\ 2012;\ 2120;\ 2201\}$$
$$C_{01} = \{0021;\ 0210;\ 0102;\ 1021;\ 1210;\ 1102;\ 2000;\ 2111;\ 2222\}$$
$$C_{10} = \{0012;\ 0120;\ 0201;\ 1000;\ 1111;\ 1222;\ 2021;\ 2210;\ 2102\}$$
$$C_{11} = \{0012;\ 0120;\ 0201;\ 1000;\ 1111;\ 1222;\ 2012;\ 2120;\ 2201\}.$$

It is easy to check that C has covering radius 1 and C_{00} is not equivalent to ξ. Therefore C is not appended Hamming.

In what follows we describe two inequivalent optimal codes that are not appended Hamming codes. Let $A = \{000,111,222\}$, $B = \{012,120,201\}$, $C = \{021,210,102\}$. It is easy to check that the vectors from the set A cover exactly the vectors $F_3^3 \setminus (B \cup C)$. The corresponding claims for B and C are also true.

As above, let C_{ijk} for $i, j \in \{0,1\}$ and $k \in \{0,1,2\}$ be the set of vectors obtained from the codewords having first three coordinates i, j, k without these three coordinates. The code, described above can be represented in a table form as:

	0	1	2
00	A	B	B
01	C	C	A
10	B	A	C
11	B	A	B

It is easy to see that the condition for such a table to give a covering is the following: for all $i, j \in \{0,1\}$ and $k \in \{0,1,2\}$ the union of C_{ijk} and $\bigcup_{d(ijk,pqr)=1} C_{pqr}$ equals $\{A, B, C\}$.

One more example for optimal code that is not appended Hamming code is given below:

	0	1	2
00	A	A	A
01	B	B	C
10	C	C	B
11	A	A	A

Using the approach from Proposition 2 it is easy to find codes, giving the upper bound for $K(1,5)$ and $K(3,4)$ that are not appended Hamming codes. The details are left to the reader.

References

[1] E. Kolev, *Codes over GF(3) of length 5, 27 codewords and covering radius 1*, J. Combin. Designs, 1 (1993), 265-275.

[2] E. Kolev, *Mixed covering codes with two binary and four ternary coordinates*, Applied algebra, algebraic algorithms and error-correcting codes, Lecture Notes in Computer Science, Vol. 948, Springer-Verlag, Berlin (1995), 312-322.

[3] R. Bertolo, P. R. J. Osterg ard and W. D. Weakley, *An updated table of binary/ternary mixed covering codes*, J. Combin. Des., 12 (2004), 157-176.

[4] W. Haas, *Lower bounds for the football pool problem for 7 and 8 matches*, Electron. J. Combin, 14 (2007), http://www.combinatorics.org/ojs/index.php/eljc/article/view/v14i1r27/pdf.

[5] H. J. L. Kamps and J. H. van Lint, *The football pool problem for 5 matches*, J. Combin. Theory, 3 (1967), 315-325.

[6] G. Keri, Tables for covering codes, http://www.sztaki.hu/~keri/codes.

[7] G. Keri and P. R. J. Ostergard, *Further results on the covering radius of small codes*, Discrete Math., 307 (2007), 69-77.

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