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ON QUATERNIONIC EQUATIONS

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Abstract. After reviewing some basic facts about quaternions and quaternionic equations, we discuss our work on the existence of quaternion solutions q of the Higher Order Sylvester Equation $aq^n + q^n b = c$ where $n = 1, 2, \ldots$ and a, b, c are also quaternions. In the cases when solutions exist an explicit formula for them is provided using both the Hamilton and matrix representation of quaternions. Our results on recursive solution of bilateral polynomial quaternionic equations of a very general type that includes the Higher Order Sylvester Equation and the Algebraic Riccati Equation using Mathematica are also discussed. The number of solutions is counted with the use of Groebner bases.

1. Quaternion essentials

A set F equipped with two operations "$+$" and "$\cdot$" is called a skew-field (or a division ring) if the pair $(F, +)$ is an abelian group, the pair $(F \setminus \{\mathbf{0}\}, \cdot)$ is a non-abelian group, the operations $+$ and $\cdot$ are connected through distributivity, and the additive and multiplicative identities $\mathbf{0}$ and $\mathbf{1}$ respectively are distinct. If the pair $(F \setminus \{\mathbf{0}\}, \cdot)$ is an abelian group then F is a *field*. For example, $\mathbb{R}$ with the usual addition and multiplication of real numbers is a field. Similarly $\mathbb{R}^2$ is a field with the usual vector addition in $\mathbb{R}^2$ and multiplication

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

$$(a, b) = a + ib \ , \ (c, d) = c + id \ , \ i^2 = -1.$$

With this multiplication $\mathbb{R}^2$ is denoted by $\mathbb{C}$.

In an effort to extend the properties of a field to $\mathbb{R}^n$, where $n > 2$, in 1843 William Rowan Hamilton identified

$$(a, b, c, d) \in \mathbb{R}^4$$

with the *quaternion*

$$q = (a, b, c, d) = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

with the (in general) noncommutative *Hamilton product*

$$q_1 q_2 = (a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (x, y, z, w) = q_3$$

computed through

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}.$$

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A simple example of non-commutativity is

$$\mathbf{ij} = \mathbf{k}, \mathbf{ji} = -\mathbf{k}.$$

With the above multiplication and usual addition $\mathbb{R}^4$ is denoted by $\mathbb{H}$ and is an example of a four-dimensional associative *normed division algebra* or *skew-field* (due to the loss of commutativity in multiplication) over the real numbers. The following notation is used for quaternions q, their conjugate $\bar{q}$, norm or modulus $|q|$ and inverse q^{-1}:

$$q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \ , \ \bar{q} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2} \ , \ q^{-1} = \frac{\bar{q}}{|q|^2} \ , \ |q| \neq 0$$

$|q| = 1$: unit quaternions, identified with the four-dimensional sphere S^3

$q = 0 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$: imaginary quaternions.

For $n > 4$, actually the next step is $n = 8$, associativity is lost as well, and we obtain the set $\mathbb{O}$ of *octonions* or *Cayley numbers* associated with the *exceptional Lie groups*. Octonions were discovered in 1843 by John T. Graves, and independently by Arthur Cayley who first published an article on them in 1845. They can be viewed as pairs of quaternions with multiplication

$$(q_1, q_2)(q_3, q_4) = (q_1 q_3 - \bar{q}_4 q_2, q_4 q_1 + q_2 \bar{q}_3).$$

2. Applications

The main application of quaternions is in the description of $3D$-rotations in computer graphics and aviation, i.e., in the description of the elements of the Lie group $SO(3)$. Specifically: if

$$u = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = (b, c, d) \in \mathbb{R}^3$$

$|u| = 1$ unit imaginary quaternion

$\theta \in \mathbb{R}$: an angle

$$t = \cos\theta\mathbf{1} + b\sin\theta\mathbf{i} + c\sin\theta\mathbf{j} + d\sin\theta\mathbf{k}$$

then, the *conjugation map*

$$q \mapsto t^{-1} q t$$

describes the rotation of the vector

$$q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = (q_1, q_2, q_3) \in \mathbb{R}^3$$

by an angle 2θ around the axis u passing through the origin, clockwise facing in the direction of u.

Using quaternions, instead of *Euler angles*, to calculate rotations in aircraft and spacecraft computers we avoid running into dangerous situations such as "gimbal lock" (as in the famous Apollo 11 Moon mission incident [6]).

3. Matrix representation of quaternions

In 1858 Cayley showed that the quaternion

$$q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

can be identified with the (2×2) matrix

$$q = \begin{pmatrix} a+id & -b-ic \\ b-ic & a-id \end{pmatrix} = \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}$$

or

$$q = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

In both of these representations

$$|q|^2 = \det q\,,\ \bar{q}: \text{ the Hermitian adjoint of } q\,,\ q^{-1}: \text{ is the matrix inverse of } q.$$

As a result of the above matrix representations, unit quaternions can be identified with the Lie group $SU(2,\mathbb{R})$ of (2×2) unitary matrices.

For example, using the first matrix representation:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

while using the second representation

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

As a consequence of the matrix determinant property $\det(AB) = \det A \det B$ we have the *two-square identity* discovered by Gauss in 1820:

$$\text{For } z_1 = a_1 + ib_1 = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \in \mathbb{C} \text{ and } z_2 = a_2 + ib_2 = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \in \mathbb{C}$$

$$z_1 z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) = \begin{pmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + a_2b_1) \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{pmatrix}$$

$$\det(z_1)\det(z_2) = \det(z_1z_2) \implies$$

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2$$

and the *four square identity*, discovered by Euler 100 years before quaternions :

$$\text{For } q_1 = \begin{pmatrix} a_1 + ib_1 & c_1 + id_1 \\ -c_1 + id_1 & a_1 - ib_1 \end{pmatrix} \in \mathbb{H} \text{ and } q_2 = \begin{pmatrix} a_2 + ib_2 & c_2 + id_2 \\ -c_2 + id_2 & a_2 - ib_2 \end{pmatrix} \in \mathbb{H}$$

$$\det(q_1)\det(q_2) = \det(q_1q_2) \implies$$

$$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2)$$

$$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + a_2b_1 + c_1d_2 - d_1c_2)^2$$

$$+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2.$$

4. Quaternionic equations

In recent years there has been an increasing interest in the extension and the formulation in the context of quaternions of problems and results usually obtained within real and complex numbers [12, 15]. Of particular interest has been the study of quaternionic equations extending classical algebraic equations such as the *Sylvester Equation* [18, 19]

$$AX + XB = C$$

and the *Algebraic Riccati Equation* [16, 15]

$$AX + XB - XRX - C = 0$$

of infinite horizon control theory, where in the classical set-up A, B, C, R, X are real $(n \times n)$ matrices.

Along with other low degree quaternion polynomial equations, quaternion versions of the above equations have been studied by many authors, see for example [7]-[11],[20]. The fundamental papers on the subject of quaternionic equations are [5, 13, 14]. A quaternion version of Gauss's *Fundamental theorem of algebra* was proved in [5] and states that quaternion polynomials of degree n of the type

$$f(x) = a_0 x a_1 x \cdots x a_n + \phi(x)$$

where $a_i \neq 0$ for $i = 0, 1, ..., n$ and $\phi(x)$ is a sum of a finite number of similar monomials $b_0 x b_1 x \cdots x b_k$ where $k < n$, have at least one root. The proof does not apply to polynomials of degree n with more than one term of degree n, such as the Sylvester equation but it does apply to the algebraic Riccati equation

$$aq + qb - c = 0 \ , \ aq + qb - qrq - c = 0.$$

We remark that quaternionic equations are richer in solutions than complex ones. For example, as shown by Hamilton, the equation $q^2 + 1 = 0$ has infinitely many quaternion solutions

$$q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \ , \ b, c, d \in \mathbb{R} \ , \ b^2 + c^2 + d^2 = 1$$

instead of the, just two, complex solutions $q = \pm i$.

5. Some recent results on quaternionic equations

Interested in quaternion polynomial equations in [1] we looked at the *Higher Order Sylvester Quaternion Equation*

$$aq^n + q^n b = c$$

and obtained explicit formulas for the solution(s). We approached the problem in the following three ways:

(i) Combined the paper of Niven [14] on finding roots of quaternions and the paper of Janovska [8] on the solution of the first order Sylvester Quaternion Equation $ap + pb = c$ through the transformation $p = q^n$.

(ii) Studied $ap+pb = c$ using a matrix representation of quaternions and Gaussian elimination for linear systems of equations and then solved $aq^n + q^n b = c$ using the theory for finding roots of matrices.

(iii) Numerically solved $aq^n + q^n b = c$ by reducing it to a system of polynomials and then used the Newton-Raphson scheme for nonlinear systems of equations.

The third approach led to the study of general bilateral quaternionic equations of the form

$$\sum_{k=0}^{n}\left(a_kq^kb_k+c_kq^kd_k\right)=0$$

through reduction to a system of four polynomial equations in four unknowns. The number of solutions was studied by finding a Gröbner basis for the ideal generated by the family of these four polynomials and considering the cardinality of its variety. The results can be found in [2]. For an extension of our approach to the solution of octonionic equations we refer to [3].

6. Solution of $aq^n + q^nb = c$ using Hamilton's formulation

The details of the material presented in this section can be found in [1]. So, letting $p = q^n$ the Higher Order Sylvester Quaternion Equation $aq^n + q^nb = c$ becomes $ap + pb = c$. As shown in [8], if $ab \neq \mathbf{0}$ then there exists a unique quaternion solution p if and only if $|a| \neq |b|$ and/or Re $a \neq -$Re b. The solution is given by the formula

$$\begin{aligned}p &= \left(2\mathrm{Re}\, b + a + |b|^2a^{-1}\right)^{-1}\left(c + a^{-1}c\bar{b}\right)\\ &= p_1\mathbf{1} + p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k}\end{aligned}$$

with explicit formulas available for $p_i, i = 1, 2, 3, 4$.

For $q = p^{1/n} \in \mathbb{H}$ there are two cases :

(1) $\mathrm{Im}\, p = p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k} \neq 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \implies$ exactly n distinct quaternion solutions q.

(2) $\mathrm{Im}\, p = p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, i.e., $p = p_1 \in \mathbb{R}$. Two subcases:

  (a) $n = 2$ and $p = p_1 > 0 \implies$ only two quaternion solutions q , namely $q = \pm\sqrt{p_1}$ (the usual square roots of a real number).

  (b) $n \neq 2$ and/or $p = p_1 < 0 \implies$ infinitely many quaternion solutions q.

In each case, the quaternion solutions q are obtained as follows: In Case (1): for $K = 0, 1, ..., n-1$,

$$\begin{aligned}q &= p^{\frac{1}{n}}\\ &= \frac{1}{r^{n-1}}\frac{\sin(\theta+\frac{2\pi K}{n})}{\sin n\theta}p + r(\cos(\theta+\frac{2\pi K}{n}) - \cot n\theta\,\sin(\theta+\frac{2\pi K}{n})),\end{aligned}$$

$$r = |p|^{\frac{1}{n}} = (p_1^2+p_2^2+p_3^2+p_4^2)^{\frac{1}{2n}} > 0,$$

$$\theta = \begin{cases}\frac{1}{n}\arctan\left(\frac{\sqrt{p_2^2+p_3^2+p_4^2}}{p_1}\right) & \text{if } p_1 > 0,\\ \frac{1}{n}\left(\arctan\left(\frac{\sqrt{p_2^2+p_3^2+p_4^2}}{p_1}\right)+\pi\right) & \text{if } p_1 < 0\,.\end{cases}.$$

In Case (2): for each $K = 0, 1, ..., n-1$

$$q = p^{\frac{1}{n}} = r\cos\frac{s\pi + 2\pi K}{n}\mathbf{1} + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}$$

$$y, z, w \in \mathbb{R} : y^2 + z^2 + w^2 = r^2\sin^2\frac{s\pi + 2\pi K}{n}$$

$$s = \begin{cases} 0 & \text{if } p = p_1 > 0 \\ \\ 1 & \text{if } p = p_1 < 0 \ . \end{cases}$$

If $s = 0$ and $n = 2$ then

$$y^2 + z^2 + w^2 = r^2 \sin^2 \frac{s\pi + 2\pi K}{n} \implies$$
$$y^2 + z^2 + w^2 = 0 \implies y = z = w = 0$$
$$q = p^{\frac{1}{2}} = r \cos \pi K \mathbf{1} = (-1)^K r \ , \quad K = 0, 1.$$

Example 1.

For the quaternionic equation

$$aq^3 + q^3 b = c$$

where

$$q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$$
$$a = 1 + 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \ , \ b = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \ , \ c = -1 + 6\mathbf{i} + 0\mathbf{j} + \mathbf{k}$$

with

$$ab = 12 - 12\mathbf{i} - 6\mathbf{j} + 0\mathbf{k} \neq 0$$

we find

$$p = 0.835165 + 1.28571\mathbf{i} - 1.65934\mathbf{j} - 0.923077k \implies \operatorname{Im} p \neq 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$
$$r = 1.34636 \ , \ \theta = 0.407176$$

thus there are three solutions q_1, q_2, q_3 given by:

$$q_1 = 1.23628 + 0.298941\mathbf{i} - 0.385813\mathbf{j} - 0.214625\mathbf{k}$$
$$q_2 = -1.07989 + 0.450817\mathbf{i} - 0.581824\mathbf{j} - 0.323664\mathbf{k}$$
$$q_3 = -0.156393 - 0.749759\mathbf{i} + 0.967637\mathbf{j} + 0.538288\mathbf{k}.$$

7. Solution of $aq^n + q^n b = c$ using matrix representation

In matrix representation

$$p = q^n \ , \ p = \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}$$
$$a = \begin{pmatrix} A & -B \\ \bar{B} & \bar{A} \end{pmatrix} , b = \begin{pmatrix} C & -D \\ \bar{D} & \bar{C} \end{pmatrix} , c = \begin{pmatrix} E & -Z \\ \bar{Z} & \bar{E} \end{pmatrix}$$
$$aq^n + q^n b = c \Leftrightarrow ap + pb = c \Leftrightarrow$$
$$\begin{pmatrix} A + C & 0 & -\bar{D} & -B \\ -D & -B & -A - \bar{C} & 0 \\ \bar{B} & \bar{D} & 0 & \bar{A} + C \\ 0 & \bar{A} + \bar{C} & -\bar{B} & -D \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \\ w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} E \\ -Z \\ \bar{Z} \\ \bar{E} \end{pmatrix} .$$

The above system has a unique solution if and only if the determinant of the coefficient matrix M is nonzero, and in this case,

$$\begin{pmatrix} z \\ \bar{z} \\ w \\ \bar{w} \end{pmatrix} = M^{-1} \begin{pmatrix} E \\ -Z \\ \bar{Z} \\ \bar{E} \end{pmatrix} .$$

We consider, from the matrix point of view, the question of finding $q = p^{1/n}$. The matrix

$$p = \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} \text{ has eigenvalues } \lambda = \operatorname{Re} z \pm ih$$

$$h = \sqrt{\det p - (\operatorname{Re} z)^2} = \sqrt{(\operatorname{Im} z)^2 + |w|^2}.$$

If $h \neq 0$, meaning $\operatorname{Im} z \neq 0$ and/or $w \neq 0$, then the eigenvalues are distinct and p can be diagonalized

$$p = PDP^{-1} \implies q = p^{1/n} = PD^{1/n}P^{-1} \ (n \text{ solutions})$$

where, assuming $w \neq 0$,

$$P = \begin{pmatrix} -w & -w \\ i(h - \operatorname{Im} z) & -i(h + \operatorname{Im} z) \end{pmatrix}$$

$$D = \begin{pmatrix} \operatorname{Re} z + ih & 0 \\ 0 & \operatorname{Re} z - ih \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -\frac{h+\operatorname{Im} z}{2hw} & -\frac{i}{2h} \\ \frac{\operatorname{Im} z - h}{2hw} & \frac{i}{2h} \end{pmatrix}$$

$$D^{1/n} = \begin{pmatrix} (\operatorname{Re} z + ih)^{1/n} & 0 \\ 0 & (\operatorname{Re} z - ih)^{1/n} \end{pmatrix}.$$

If $h \neq 0$ and $w = 0$ then

$$p = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \text{ eigenvalues } \lambda = z \text{ , } \lambda = \bar{z}$$

$$q = p^{\frac{1}{n}} = \begin{pmatrix} z^{\frac{1}{n}} & 0 \\ 0 & \bar{z}^{\frac{1}{n}} \end{pmatrix} \text{ (} n \text{ solutions).}$$

If $h = 0$ then $w = 0$ and $z \in \mathbb{R}$,

$$p = z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies q = z^{\frac{1}{n}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\frac{1}{n}}.$$

If $n = 2$ and $p = p_1 > 0$, i.e., if $z = p_1$ then

$$p = p_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies$$

$$q = \pm\sqrt{p_1}\, \mathbf{I}^{1/2} = \pm\sqrt{p_1} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ (two solutions)}$$

while, if $n = 2$ and $p = p_1 = z < 0$ then

$$q = |p_1|^{1/2}(-\mathbf{I})^{1/2} \text{(infinitely many solutions)}$$

$$(-\mathbf{I})^{1/2} = \begin{pmatrix} ib & c + id \\ -c + id & -ib \end{pmatrix} \ b, c, d \in \mathbb{R} \text{ , } b^2 + c^2 + d^2 = 1.$$

Similarly, if $n \neq 2$ then there are infinitely many solutions

$$q = p^{1/n} = \begin{pmatrix} r \cos \frac{s\pi + 2\pi K}{n} + i\psi & \zeta + i\omega \\ -\zeta + i\omega & r \cos \frac{s\pi + 2\pi K}{n} - i\psi \end{pmatrix}$$

$$r = |p_1|^{\frac{1}{n}} > 0$$

where $\psi, \zeta, \omega \in \mathbb{R}$ are, for each choice of $K \in \{0, 1, ..., n-1\}$, any real numbers satisfying

$$\psi^2 + \zeta^2 + \omega^2 = r^2 \sin^2 \frac{s\pi + 2\pi K}{n}$$

$$s = \begin{cases} 0 & \text{if } p = p_1 > 0 \\ \\ 1 & \text{if } p = p_1 < 0 \, . \end{cases}$$

Example 2.

As in Example 1, let us consider the equation

$$aq^3 + q^3 b = c$$

where

$$q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = \begin{pmatrix} x_1 + ix_4 & -x_2 - ix_3 \\ x_2 - ix_3 & x_1 - ix_4 \end{pmatrix}$$

$$q^3 = p = \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}$$

$$a = 1 + 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} = \begin{pmatrix} 1+i & -3+4i \\ 3+4i & 1-i \end{pmatrix}$$

$$b = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2i & 2-2i \\ -2-2i & -2i \end{pmatrix}$$

$$c = -1 + 6\mathbf{i} + 0\mathbf{j} + \mathbf{k} = \begin{pmatrix} -1+i & -6 \\ 6 & -1-i \end{pmatrix}.$$

Solving the system

$$\begin{pmatrix} 1+3i & 0 & 2+2i & -3+4i \\ 2-2i & -3+4i & -1+i & 0 \\ 3+4i & -2-2i & 0 & 1+i \\ 0 & 1-3i & -3-4i & 2-2i \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \\ w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} -1+i \\ -6 \\ 6 \\ -1-i \end{pmatrix}$$

we find

$$\det M = -273 \, , \; , h = 2.29315$$

$$z = 0.835165 - 0.923077i \; , \; w = 1.28571 - 1.65934i$$

$$p = \begin{pmatrix} 0.835165 - 0.923077i & -1.28571 + 1.65934i \\ 1.28571 + 1.65934i & 0.835165 + 0.923077i \end{pmatrix}$$

$$P = \begin{pmatrix} -1.28571 + 1.65934i & -1.28571 + 1.65934i \\ 3.21623i & -1.37007i \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -0.0871635 - 0.112493i & -0.218041i \\ -0.204615 - 0.264076i & 0.218041i \end{pmatrix}$$

$$D = \begin{pmatrix} 0.835165 + 2.29315i & 0 \\ 0 & 0.835165 - 2.29315i \end{pmatrix}.$$

Using the three different cubic roots of D

$$D^{1/3} = \begin{pmatrix} (0.835165 + 2.29315i)^{1/3} & 0 \\ 0 & (0.835165 - 2.29315i)^{1/3} \end{pmatrix}$$

we obtain the roots q_1, q_2, q_3 as in the Hamilton formulation. We have:

$$(0.835165 + 2.29315i)^{1/3} =$$

$$-1.07989 + 0.80406i, -0.156393 - 1.33724i, 1.23628 + 0.53318i$$

$$(0.835165 - 2.29315i)^{1/3}$$
$$= -1.07989 - 0.80406i, -0.156393 + 1.33724i, 1.23628 - 0.53318i.$$

Using the pair
$$-1.07989 + 0.80406i\,,\ -1.07989 - 0.80406i$$
we find
$$q_2 = PD^{1/3}P^{-1} = \begin{pmatrix} -1.07989 - 0.323662i & -0.450816 + 0.581824i \\ 0.450815 + 0.581824i & -1.07989 + 0.323665i \end{pmatrix}$$
$$= -1.07989 + 0.450817\mathbf{i} - 0.581824\mathbf{j} - 0.323664\mathbf{k}.$$

Likewise we find q_1 andq_3 as in Example 1.

8. Numerical Solution

Every bilateral quaternion polynomial equation of the form
$$\sum_{n=0}^{k}(a_n q^n b_n + c_n q^n d_n) = 0$$
where $k \in \{0, 1, ...\}$,
$$q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = \begin{pmatrix} x_1 + ix_2 & -x_3 - ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix}$$
$$a_n = \begin{pmatrix} a_1(n) + ia_2(n) & -a_3(n) - ia_4(n) \\ a_3(n) - ia_4(n) & a_1(n) - ia_2(n) \end{pmatrix}$$
$$b_n = \begin{pmatrix} b_1(n) + ib_2(n) & -b_3(n) - ib_4(n) \\ b_3(n) - ib_4(n) & b_1(n) - ib_2(n) \end{pmatrix}$$
$$c_n = \begin{pmatrix} c_1(n) + ic_2(n) & -c_3(n) - ic_4(n) \\ c_3(n) - ic_4(n) & c_1(n) - ic_2(n) \end{pmatrix}$$
$$d_n = \begin{pmatrix} d_1(n) + id_2(n) & -d_3(n) - id_4(n) \\ d_3(n) - id_4(n) & d_1(n) - id_2(n) \end{pmatrix}$$
can be reduced to a system of four nonlinear equations
$$f_i(x_1, x_2, x_3, x_4) = 0\ ,\ i = 1, 2, 3, 4$$
or in vector notation
$$F(X) = 0\ ,\ X = (x_1, x_2, x_3, x_4)^T\ ,\ F = (f_1, f_2, f_3, f_4)^T.$$
Traditionally, such systems are solved recursively using the Newton-Raphson method for nonlinear systems [4]. The method produces a sequence
$$X^{(m+1)} = X^{(m)} + H^{(m)}\ ,\ m = 0, 1, 2, ...$$
where the correction term H is computed by solving using Gaussian elimination the *Jacobian system*
$$F'(X^{(m)})H^{(m)} = -F(X^{(m)})$$
assuming that the (4×4) Jacobian matrix
$$F'(X) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le 4}$$
is nonsingular at least in a neighborhood of the solution. Convergence of the method depends on making a good initial guess $X(0)$.

Example 3.

As in Examples 1 and 2, we consider the equation

$$aq^3 + q^3b = c \ , \ q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = \begin{pmatrix} x_1 + ix_2 & -x_3 - ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix}$$

$$a = 1 + \mathbf{i} + 3\mathbf{j} - 4\mathbf{k} = \begin{pmatrix} 1+i & -3+4i \\ 3+4i & 1-i \end{pmatrix}$$

$$b = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2i & 2-2i \\ -2-2i & -2i \end{pmatrix}$$

$$c = -1 + 1\mathbf{i} + 6\mathbf{j} + 0\mathbf{k} = \begin{pmatrix} -1+i & -6 \\ 6 & -1-i \end{pmatrix}.$$

In this case,

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4) &= 1 + x_1^3 - 9x_1^2x_2 - 3x_1x_2^2 + 3x_2^3 - 3x_1^2x_3 \\ &+x_2^2x_3 - 3x_1x_3^2 + 3x_2x_3^2 + x_3^3 + 6x_1^2x_4 \\ &-2x_2^2x_4 - 2x_3^2x_4 - 3x_1x_4^2 + 3x_2x_4^2 + x_3x_4^2 - 2x_4^3 = 0 \end{aligned}$$

$$\begin{aligned} f_2(x_1,x_2,x_3,x_4) &= -1 + 3x_1^3 + 3x_1^2x_2 - 9x_1x_2^2 - x_2^3 + 18x_1^2x_3 \\ &-6x_2^2x_3 - 9x_1x_3^2 - x_2x_3^2 - 6x_3^3 + 15x_1^2x_4 \\ &-5x_2^2x_4 - 5x_3^2x_4 - 9x_1x_4^2 - x_2x_4^2 - 6x_3x_4^2 - 5x_4^3 = 0 \end{aligned}$$

$$\begin{aligned} f_3(x_1,x_2,x_3,x_4) &= 6 - x_1^3 + 18x_1^2x_2 + 3x_1x_2^2 - 6 \\ &x_2^3 - 3x_1^2x_3 + x_2^2x_3 + 3x_1x_3^2 - 6x_2x_3^2 + x_3^3 \\ &-3x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + 3x_1x_4^2 - 6x_2x_4^2 + x_3x_4^2 + x_4^3 = 0 \end{aligned}$$

$$\begin{aligned} f_4(x_1,x_2,x_3,x_4) &= 2x_1^3 + 15x_1^2x_2 - 6x_1x_2^2 \\ &-5x_2^3 + 3x_1^2x_3 - x_2^2x_3 - 6x_1x_3^2 - 5x_2x_3^2 - x_3^3 \\ &-3x_1^2x_4 + x_2^2x_4 + x_3^2x_4 - 6x_1x_4^2 - 5x_2x_4^2 - x_3x_4^2 + x_4^3 = 0 \end{aligned}$$

$$\det F'(X) = 2457(-3x_1^4 - 2x_1^2(x_2^2 + x_3^2 + x_4^2) + (x_2^2 + x_3^2 + x_4^2)^2)^2.$$

The solution of nonlinear systems of equations with the use of Mathematica version 9 is done through the command FindRoot which uses a damped or under-relaxed Newton-Raphson method to alleviate poor initial approximations $X(0)$ to the solution X of the system. For an initial guess $X(0) = (1,0,0,0)^T$, after 6 iterations, using the FindRoot command the method converged to a six decimal point accuracy to

$$\begin{aligned} X &= (1.23628, -0.214625, 0.298941, -0.385813)^T \\ &= 1.23628 - 0.214625\mathbf{i} + 0.298941\mathbf{j} - 0.385813\mathbf{k}. \end{aligned}$$

For an initial guess $X(0) = (-1,0,0,0)^T$, after 8 iterations, the method converged to

$$\begin{aligned} X &= (-1.07989, -0.323664, 0.450817, -0.581824)^T \\ &= -1.07989 - 0.323664\mathbf{i} + 0.450817\mathbf{j} - 0.581824\mathbf{k}. \end{aligned}$$

For an initial guess $X(0) = (0,0,1,0)^T$, after 13 iterations, the method converged to

$$X = (-0.156393, 0.538288, -0.749759, 0.967637)^T$$

$$= -0.156393 + 0.538288\mathbf{i} - 0.749759\mathbf{j} + 0.967637\mathbf{k}$$

in agreement with the results of Example 1.

9. Number of roots

Regarding the number of quaternion roots, Groebner bases (named by Bruno Buchberger in 1965 after his PhD advisor Wolfgang Gröbner) [17] can be used. For f_i, $i = 1, 2, 3, 4$ as in Example 3 the command

$$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{x_1, x_2, x_3, x_4\}]$$

produces the Groebner basis

$$g_1(x_1, x_2, x_3, x_4) = -40812436757196811351 - 87282629347203853672 2x_4^3$$
$$-259508388244212637420 8x_4^6 + 398243737994104863078 4x_4^9$$
$$g_2(x_1, x_2, x_3, x_4) = 151x_3 + 117x_4$$
$$g_3(x_1, x_2, x_3, x_4) = 151x_2 - 84x_4$$
$$g_4(x_1, x_2, x_3, x_4) = 306079851124102221x_1 + 1494541173065598080x_4$$
$$+8317576546288866584x_4^4 - 10940762032805078656x_4^7.$$

There are 9 **complex** roots x_4 of the first member of the reduced Groebner basis above (by Gauss's Fundamental Theorem). For each one of them we find a unique answer for x_1, x_2, x_3. Thus a total of 9 complex solutions (x_1, x_2, x_3, x_4).

The command

$$NSolve[g_1(x_1, x_2, x_3, x_4) = 0, x_4]$$

produces the roots

$$x_4 = -0.581824, x_4 = -0.483819 - 0.837998i, x_4 = -0.483819 + 0.837998i$$
$$x_4 = -0.385813, x_4 = 0.192907 - 0.334124i, x_4 = 0.192907 + 0.334124i$$
$$x_4 = 0.290912 - 0.503874i, x_4 = 0.290912 + 0.503874i, x_4 = 0.967637.$$

There are three real x_4 roots of the first member of the reduced Groebner basis. For each one of them we find a unique answer for x_1, x_2, x_3. Thus a total of 3 quaternion roots. for more details on this approach we refer to [2].

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