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Matrix theorems and interchange for lattice group-valued series in the filter convergence setting

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Abstract

We investigate some properties of unconditional convergence of series taking values in lattice groups. We give some matrix and Schur-type theorems in the filter convergence context for lattice group-valued measures, and deduce an interchange theorem for series in the lattice group setting. Furthermore we pose some open problems.

1. Introduction

In the literature there have been several studies about Schur-type and different kinds of matrix theorems and related topics, in connection with limit theorems for measures with values in various types of structures, like for example normed spaces, topological and lattice groups. A survey on these subjects can be found, for example, in [4, 10, 15, 22]. In [5] some basic matrix theorems were proved in the Riesz space context, with respect to the so-called $(*)$-convergence, whose nature is topological. However, there are some Riesz spaces, in which order and (D)-convergence are not generated by any topology, and so they do not coincide with $(*)$-convergence, for example the space $L^0(X, \mathcal{M}, \mu)$ of all μ-measurable real-valued functions with identification up to μ-null sets, where μ is a σ-additive and σ-finite non-negative extended real-valued measure. In this space, order and (D)-convergence coincide with almost everywhere convergence and $(*)$-convergence is equal to convergence in measure (see also [23]).

Some very important subjects which are associated with matrix theorems and widely investigated in the literature are the Schur-type and limit theorems. Among the studies on these topics in the setting of lattice groups and/or filter convergence, we quote [13, 14, 16]. Moreover, there have been some recent investigations and developments about matrix and Schur-type theorems for series taking values in abstract spaces, in connection with some interchange theorems, by requiring convergence of

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sequences of series on each element of a family of subsets of $\mathbb{N}$, satisfying suitable properties. Among the related references, we quote [1, 2, 3, 6, 20, 21, 22] for the normed space and topological group context with respect to usual or statistical convergence, and [11] for the lattice group setting with respect to filter convergence.

In this paper, for technical reasons, we use filter (D)-convergence, since this tool allows us to replace a "series" of regulators with a single regulator by using the well-known Fremlin theorem. Moreover, in connection with (D)-convergence, we deal with unconditional convergence of series, giving a characterization and relating it with σ-additivity of suitable lattice group-valued measures. We extend some earlier results proved in [1, 2, 3, 11] and, under suitable conditions on the involved filters of $\mathbb{N}$, we prove some matrix and Schur-type theorems, by requiring only filter (D)-convergence of suitable sequences of series rather than (D)-convergence in the usual sense (as it was done in [11]). Furthermore, we deduce an interchange theorem of series in this context. We use a sliding hump argument, the Fremlin lemma and the Maeda-Ogasawara-Vulikh representation theorem, relating some properties of convergence of lattice group-valued series with the corresponding ones of real-valued series. Finally, we pose some open problems.

2. Preliminaries

In this paper we deal with the lattice group setting and together with the order convergence we consider the (D)-convergence, which is a variant of the so-called ε-technique. Note that, if $\varepsilon > 0$ and A is any subset of the real line having supremum $s \in \mathbb{R}$, then there exists $a \in A$ such that $s-\varepsilon < a$. In general, this is not true in lattice groups. For instance, let R be the space of all real functions defined on $[0,1]$ with the pointwise ordering. Put $A = \{f_n \colon n \in \mathbb{N}\}$, where $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, is defined by $f_n(x) := 1 - x^n$, $n \in \mathbb{N}$, $x \in [0,1]$. Then the lattice supremum of A in R is the function f defined by $f(x) = 0$ if $x = 1$ and $f(x) = 1$ if $0 \le x < 1$. If ε is considered as a constant function, then there are no elements $h \in A$ with $f(x)-\varepsilon = g(x) \le h(x)$ for every $x \in [0,1]$. So, we use the double sequence technique, taking into account that, if $(a_{t,l})_{t,l}$ is a bounded double sequence of real numbers such that $(a_{t,l})_l$ is an order sequence for each $t \in \mathbb{N}$, then for every $t \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\varphi(t) \in \mathbb{N}$ with $a_{t,l} \le \varepsilon$ for each $l \ge \varphi(t)$. Since the inequality $a_{t,\varphi(t)} \le \varepsilon$ holds for every $t \in \mathbb{N}$, we get also $\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \le \varepsilon$. So, in the lattice group context, it is natural to replace ε with $\bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$. We will follow this idea in defining (D)-convergence, which was presented by D. H. Fremlin. In [17, Theorem A], using this tool, he gave a direct proof of the fact that, given two σ-Dedekind complete vector lattices R_1 and R_2, a sublattice $R_0 \subset R_1$ and a positive linear order continuous function $\phi : R_0 \to R_2$, and if R_0^σ is the least sublattice of R_1 containing R_0 and closed with respect to countable suprema, then ϕ admits an extension $\phi^\sigma : R_0^\sigma \to R_2$ if and only if R_2 is weakly σ-distributive. Moreover, in [17, Lemma 1C], he presented a tool by means of which it is possible to deal with a single (D)-sequence rather than a sequence of regulators. Here the role of (D)-convergence is important because it allows us to do this procedure without requiring further assumptions on R_2. The (D)-convergence is used in the literature in many topics, for instance in the theory of Bochner and Kurzweil-Henstock integrals and in several kinds of limit theorems for finitely and countably additive measures (see also [10, 15, 19] and their bibliographies).

Now we recall the following concepts and results.

Definitions 2.1 (a) An abelian partially ordered group $R = (R, +, \leq)$ is called a *lattice group* iff it is a lattice and $a + c \leq b + c$ whenever a, b, $c \in R$ and $a \leq b$.

(b) A lattice group R is said to be (σ)-*Dedekind complete* iff every nonempty (countable) subset of R, bounded from above, has supremum in R.

(c) A sequence $(\sigma_p)_p$ of positive elements of a lattice group R is called an *order sequence* or (O)-*sequence* iff it is decreasing and $\bigwedge_p \sigma_p = 0$.

(d) A bounded double sequence $(a_{t,l})_{t,l}$ in R is called a (D)-*sequence* or a *regulator* iff for each $t \in \mathbb{N}$ the sequence $(a_{t,l})_l$ is an (O)-sequence.

(e) A lattice group R is said to be *weakly* σ-*distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$ for every (D)-sequence $(a_{t,l})_{t,l}$.

(f) A Dedekind complete lattice group is said to be *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there exists a countable subset $A^* \subset A$, such that $\bigvee A = \bigvee A^*$.

We now recall the following theorem, which links (O)- and (D)-sequences in lattice groups.

Theorem 2.2 (see also [9, Theorem 2.3]) *Let R be a super Dedekind complete and weakly σ-distributive lattice group. Then for every (D)-sequence $(a_{t,l})_{t,l}$ in R there are an (O)-sequence $(\tau_p)_p$ in R and a sequence $(\varphi_p)_p$ in $\mathbb{N}^{\mathbb{N}}$, with*

$$\bigvee_{t=1}^{\infty} a_{t,\varphi_p(t)} \leq \tau_p \quad \textit{for each } p \in \mathbb{N}. \tag{2.1}$$

We now recall the Maeda-Ogasawara-Vulikh representation theorem (see also [7, 23]).

Theorem 2.3 *Given a σ-Dedekind complete lattice group R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded isomorphically as a subgroup of $C_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^\Omega : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if we denote by $\widehat{a}$ an element of $C_\infty(\Omega)$ which corresponds to $a \in R$ under the above isomorphism, then for any family $(a_\lambda)_{\lambda \in \Lambda}$ of elements of R with $R \ni a_0 = \bigvee_\lambda a_\lambda$ (where the supremum is taken with respect to R), then $\widehat{a_0} = \bigvee_\lambda \widehat{a_\lambda}$ with respect to $C_\infty(\Omega)$, and we get $\widehat{a_0}(\omega) = \sup_\lambda [\widehat{a_\lambda}(\omega)]$ in the complement of a meager subset of Ω. The same is true for $\bigwedge_\lambda a_\lambda$.*

The following result (Fremlin lemma, see [17, Lemma 1C], [19, Theorem 3.2.3]) allows us to replace a countable family or a "series" of (D)-sequences with a single regulator.

Lemma 2.1 *Let R be any σ-Dedekind complete lattice group and $(a_{t,l}^{(n)})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence*

$(a_{t,l})_{t,l}$ *in* R *with*

$$u \wedge \Big(\sum_{n=1}^{q} \Big(\bigvee_{t=1}^{\infty} a^{(n)}_{t,\varphi(t+n)} \Big) \Big) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \textit{for every } q \in \mathbb{N} \quad \textit{and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now recall some basic properties of filters, which will be useful in the sequel.

Definitions 2.4 (a) Let Q be a countable set and $\mathcal{F}$ be a filter of Q. We say that $\mathcal{F}$ is *free* iff it contains the filter $\mathcal{F}_{\text{cofin}}$ of all cofinite subsets of Q.

(b) Let $\mathcal{F}$ be a free filter of Q. A subset of Q is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q.

(c) A free filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$ (see also [6, 14]).

(d) Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

(e) A free filter $\mathcal{F}$ of Q is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

(f) If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{A \cap I \colon A \in \mathcal{F}\}$. It is not difficult to see that $\mathcal{F}(I)$ is a filter of I (see also [10]).

Remarks 2.5 (a) Note that the concept of block-respecting filter is the analog of that of Ramsey ultrafilter (see also [18, pp. 257–259]).

(b) Observe that, if $\mathcal{F}$ is a block-respecting filter of $\mathbb{N}$, then $\mathcal{F}(I)$ is a block-respecting filter of I for every $I \in \mathcal{F}^*$ (see also [10]).

(c) Every filter having a countable base is both block-respecting and diagonal (see also [6]).

(d) Let $D = (D_n)_n$ be a disjoint partition of $\mathbb{N}$ into infinite subsets. For each sequence $C = (C_n)_n$ of finite subsets $C_n \subset D_n$ and every $q \in \mathbb{N}$, set $B_{q,C} := \bigcup_{n=q}^{\infty} (D_n \setminus C_n)$

The filter generated by the sets of type $B_{q,C}$ is non-diagonal and block-respecting. Furthermore, the filter of all subsets of $\mathbb{N}$ having asymptotic density one is diagonal and not block-respecting (see also [6, 10]).

(e) Some relations between diagonal and/or block-respecting filters and other properties of filters can be found in [6] (see also [10, 16]).

From now on, assume that R is a super Dedekind complete weakly σ-distributive lattice group. Some examples are $\mathbb{R}^{\mathbb{N}}$ and $L^0(X, \mathcal{M}, \mu)$, where μ is a positive, σ-additive and σ-finite extended real-valued measure (see also [15, 23]).

Definitions 2.6 (a) Let $\mathcal{F}$ be a free filter of $\mathbb{N}$. A sequence $(x_n)_n$ in R *$(O\mathcal{F})$-converges to* $x \in R$ iff there is an (O)-sequence $(\sigma_p)_p$ with

$$\{n \in \mathbb{N} : |x_n - x| \leq \sigma_p\} \in \mathcal{F} \quad \text{whenever } p \in \mathbb{N}. \tag{2.2}$$

(b) A sequence $(x_n)_n$ in R *$(D\mathcal{F})$-converges to* $x \in R$ iff there is a (D)-sequence $(a_{t,l})_{t,l}$ with

$$\Big\{ n \in \mathbb{N} : |x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big\} \in \mathcal{F} \quad \text{for any } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

(c) We say that $(x_n)_n$ *(O)-converges* (resp. *(D)-converges*) to $x \in R$ iff it $(O\mathcal{F}_{\text{cofin}})$- (resp. $(D\mathcal{F}_{\text{cofin}})$-)converges to x.

Remark 2.1 Observe that, since R is super Dedekind complete and weakly σ-distributive, $(O\mathcal{F})$- and $(D\mathcal{F})$-convergence are equivalent (see also [14, Theorem 2.3]).

When $R = \mathbb{R}$, instead of $(O\mathcal{F})$- or $(D\mathcal{F})$-convergence we will write simply $(\mathcal{F})$-convergence. Moreover, define

$$(O\mathcal{F})\sum_{j=1}^{\infty} x_j := (O\mathcal{F})\lim_n \sum_{j=1}^{n} x_j, \quad (D\mathcal{F})\sum_{j=1}^{\infty} x_j := (D\mathcal{F})\lim_n \sum_{j=1}^{n} x_j.$$

We now recall the following results.

Proposition 2.1 (see also [8, Proposition 2.4]) *Let $\mathcal{F}$ be any free filter, and $(x_n)_n$ be a sequence in $\mathbb{R}$, such that $(\mathcal{F})\lim\limits_n x_n = x \in \mathbb{R}$. Then there exists a subsequence $(x_{n_q})_q$ of $(x_n)_n$, with $\lim\limits_q x_{n_q} = x$ in the usual sense.*

Lemma 2.2 (see also [14, Lemma 2.2]) *Let $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$, $(a_{i,j})_{i,j}$ be a double sequence in R with $(O\mathcal{F})\lim\limits_i a_{i,j} = 0$ for every $j \in \mathbb{N}$ with respect to a single (O)-sequence $(\sigma_p)_p$. Then for every $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$ with $J \subset I$ and $(O)\lim\limits_{i\in J} a_{i,j} = 0$ for each j with respect to $(\sigma_p)_p$.*

Now we recall the concept of unconditionally convergent series in the lattice group context (see also [11]). Let $\mathcal{I}_{\text{fin}}$ be the family of all finite subsets of $\mathbb{N}$.

Definition 2.1 A series $\sum\limits_{j=1}^{\infty} a_j$ in R is said to be *unconditionally convergent* iff there is a regulator $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a set $A_0 \in \mathcal{I}_{\text{fin}}$ with

$$\Big|\sum_{j\in A_1} a_j - \sum_{j\in A_2} a_j\Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{2.3}$$

whenever $A_1, A_2 \in \mathcal{I}_{\text{fin}}$, $A_1, A_2 \supset A_0$.

We have the following characterization (see also [10, Proposition IV.1.34]).

Proposition 2.2 *A series $\sum\limits_{j=1}^{\infty} a_j$ is unconditionally convergent if and only if the limit*

$$l_A := \sum_{j\in A} a_j = (D)\lim_q \sum_{j\in A\cap[1,q]} a_j$$

exists in R uniformly with respect to $A \subset \mathbb{N}$ and with respect to a single regulator $(a_{t,l})_{t,l}$ independent of A, that is for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a positive integer $\overline{q}$ with

$$\Big|\sum_{j\in A\cap[1,q]} a_j - l_A\Big| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for each } q \ge \overline{q} \text{ and } A \subset \mathbb{N}.$$

Remark 2.2 Observe that, since Dedekind completeness of R implies sequential completeness of R with respect to (D)-convergence, and also uniformly with respect to a parameter (see also [15]), by the Cauchy criterion we get that a series $\sum_{j=1}^{\infty} a_j$ is unconditionally convergent if and only if it is (D)-*Cauchy* uniformly with respect to $A \subset \mathbb{N}$, that is there is a regulator $(a_{t,l})_{t,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{q} \in \mathbb{N}$ with

$$\Big| \sum_{j \in A \cap [q,q+r]} a_j \Big| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{whenever } q \geq \overline{q},\ r \in \mathbb{N} \text{ and } A \subset \mathbb{N}.$$

We now recall the notion of σ-additive lattice group-valued measure (see also [10]).

Definition 2.2 Let G be any nonempty set and $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra. A finitely additive measure $m : \Sigma \to \mathbb{R}$ is said to be *σ-additive* on Σ iff $(D) \lim_n v(m) \Big(\bigcup_{l=n}^{\infty} H_l \Big) = 0$ for each disjoint sequence $(H_n)_n$ in Σ, where $v(m)(E) := \bigvee \{ |m(A)|,\ A \in \Sigma,\ A \subset E \}$, $E \in \Sigma$, denotes the *semivariation* of m on Σ.

Proposition 2.3 (see also [10, Proposition IV.1.35]) *Let $\sum_{j=1}^{\infty} a_j$ be an unconditionally convergent series, and set $m(A) := \sum_{j \in A} a_j$ for each $A \subset \mathbb{N}$. Then m is σ-additive on $\mathcal{P}(\mathbb{N})$.*

Now we give a condition on a family $\mathcal{W}$ of subsets of $\mathbb{N}$ in order that a series, filter convergent on each element of $\mathcal{W}$, is convergent on every subset $A \subset \mathbb{N}$ uniformly with respect to A, extending [3, Lemma 2.2]. For other similar conditions existing in the literature, see for instance [1, 3].

Definition 2.3 A family $\mathcal{W}$ of subsets of $\mathbb{N}$ containing $\mathcal{I}_{\text{fin}}$ is said to *satisfy property (M)* iff for each sequence $(F_k)_k$ in $\mathcal{I}_{\text{fin}}$, such that $\max F_k < \min F_{k+1}$ for each $k \in \mathbb{N}$, there exist a set $B \in \mathcal{W}$ and a finite set $D \subset \mathbb{N}$, such that

$$\bigcup_{k \in \mathbb{N} \setminus D} F_k \subset B \subset \bigcup_{k \in \mathbb{N}} F_k. \tag{2.4}$$

Example 2.1 Let c_0 be the space of all real sequences convergent to 0 and $\mathbb{E}$ be the set of all even natural numbers. For each $n \in \mathbb{N}$, put $E_n := [1, 2n] \cap \mathbb{E}$, $O_n := [1, 2n] \cap (\mathbb{N} \setminus \mathbb{E})$. For every $A \subset \mathbb{N}$ and $n \in \mathbb{N}$, set $w_n(A) := \sharp(A \cap E_n) - \sharp(A \cap O_n)$. We claim that the family

$$\mathcal{W} := \mathcal{I}_{\text{fin}} \cup \{ A \subset \mathbb{N} :\ \text{the sequence } (w_n(A))_n \text{ does not belong to } c_0 \}$$

fulfils property (M). Indeed, following [3, Example 3.1], let $(F_k)_k$ be as in Definition 2.3, pick $D = \{1\}$ and take $C := \bigcup_{k=2}^{\infty} F_k$. If $C \in \mathcal{W}$, then C satisfies (2.4). If $C \notin \mathcal{W}$, then let n_1 be the first element of F_1, and set $B := C \cup \{n_1\}$. It is not difficult to check that, for each $n \in \mathbb{N}$ such that $2n \geq n_1$, we get $w_n(B) = w_n(C) + 1$ if $n_1 \in \mathbb{E}$ and $w_n(B) = w_n(C) - 1$ if $n_1 \in \mathbb{N} \setminus \mathbb{E}$. In both cases we have $\lim_n w_n(B) \neq 0$. Thus, $B \in \mathcal{W}$ and B satisfies (2.4), getting the claim.

3. The main results

Before giving our main theorems, we present a technical lemma, which links filter and ordinary convergence of series.

Lemma 3.1 *Let $\mathcal{F}$ be any free filter of $\mathbb{N}$, $\mathcal{W}$ be a family of subsets of $\mathbb{N}$, satisfying property (M), $(a_n)_n$ be a sequence of real numbers, and assume that the series $(\mathcal{F}) \sum_{n \in B} a_n$ $(\mathcal{F})$-converges in $\mathbb{R}$ (that is, the sequence $\Big(\sum_{i \in B \cap [1,n]} a_i \Big)_n$ $(\mathcal{F})$-converges in $\mathbb{R}$) for each $B \in \mathcal{W}$.*

Then the series $\sum_{n \in A} a_n$ converges uniformly with respect to $A \subset \mathbb{N}$ in the usual sense.

Proof: By virtue of the Cauchy criterion, it is enough to prove that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $\Big| \sum_{n \in A \cap [q,r]} a_n \Big| \leq \varepsilon$ for each $r > q \geq n_0$ and $A \subset \mathbb{N}$. If we deny the thesis, then we find a positive real number ε and a sequence $(F_k)_k$ in $\mathcal{I}_{\text{fin}}$, with $\max F_k < \min F_{k+1}$ and $\Big| \sum_{n \in F_k} a_n \Big| > \varepsilon$ for each k. Now, proceeding analogously as in [3, Lemma 2.2], set $G_k = \{n \in F_k : a_n \geq 0\}$ and $H_k = \{n \in F_k : a_n < 0\}$. Without loss of generality, we can suppose that the set $K := \Big\{ k \in \mathbb{N} : \sum_{n \in G_k} a_n > \varepsilon/2 \Big\}$ is infinite. Now, let us consider the sequence G_k, $k \in K$. Since $\mathcal{W}$ satisfies property (M), there are an element $B \in \mathcal{W}$ and a finite set $D \subset K$ such that $\bigcup_{k \in K \setminus D} G_k \subset B \subset \bigcup_{k \in K} G_k$. Since, by hypothesis, $(\mathcal{F}) \sum_{n \in B} a_n$ exists in $\mathbb{R}$, then, by virtue of Proposition 2.1, there is an infinite subset $A \subset \mathbb{N}$ such that the sequence $\Big(\sum_{i \in B \cap [1,n]} a_i \Big)_{n \in A}$ is convergent in $\mathbb{R}$. Observe that for every $q \in A$ there is $k \in K \setminus D$ with $q < \min F_k$ and, if $r \in A$, $r > \max F_k$, then $\sum_{n \in [q,r] \cap B} a_n \geq \sum_{n \in G_k} a_n > \varepsilon/2$, and thus the series $\Big(\sum_{i \in B \cap [1,n]} a_i \Big)_{n \in A}$ is not Cauchy, which contradicts the hypothesis. This ends the proof. □

We now turn to our main matrix theorem, which extends to the filter convergence setting [1, Theorems 2.3 and Lemma 3.4], [3, Lemma 2.2] [11, Theorem 3.2], [13, Theorem 3.1].

Theorem 3.1 *Let R be a super Dedekind complete and weakly σ-distributive lattice group, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $\mathcal{W}$ satisfy property (M) and $(a_{i,j})_{i,j}$ be a double sequence in R, such that:*

3.1.1) $(D)\lim\limits_i a_{i,j} = 0$ *for every* $j \in \mathbb{N}$*;*

3.1.2) the series $(D\mathcal{F})\sum\limits_{j\in B} a_{i,j}$ $(D\mathcal{F})$*-converges (that is, the sequence* $\Big(\sum\limits_{j\in B\cap[1,n]} a_{i,j}\Big)_n$ $(D\mathcal{F})$*-converges) for each* $B \in \mathcal{W}$ *and* $i \in \mathbb{N}$ *with respect to a single regulator, independent of* B *and* i*;*

3.1.3) the family $\Big(\sum\limits_{j\in B} a_{i,j}\Big)_{i\in\mathbb{N},\, B\in\mathcal{I}_{\text{fin}}}$ *is equibounded;*

3.1.4) for every infinite subset $B \in \mathcal{W}$ *the sequence* $\Big((D\mathcal{F})\sum\limits_{j\in B} a_{i,j}\Big)_i$ $(D\mathcal{F})$*-converges to* 0 *with respect to a regulator* $(z_{t,l})_{t,l}$ *independent of* B*.*

Then we get:

3.1.5) the series $\sum\limits_{j=1}^{\infty} a_{i,j}$ *is unconditionally convergent for any* $i \in \mathbb{N}$*;*

3.1.6) there is a (D)*-sequence* $(d_{t,l})_{t,l}$ *such that for any subset* $A \subset \mathbb{N}$ *the sequence* $\Big(\sum\limits_{j\in A} a_{i,j}\Big)_i$ $(D\mathcal{F})$*-converges to* 0 *with respect to* $(d_{t,l})_{t,l}$*;*

3.1.7) if $\mathcal{F}$ *is also diagonal, then 3.1.5) and 3.1.6) follow directly from 3.1.j) , j=2,3,4.*

Proof: By super Dedekind completeness and weak σ-distributivity of R, 3.1.2) and Theorem 2.2, there is an (O)-sequence $(\sigma_p)_p$ such that the series $(O\mathcal{F})\sum\limits_{j\in B} a_{i,j}$ $(O\mathcal{F})$-converges with respect to $(\sigma_p)_p$ for each $B \in \mathcal{W}$ and $i \in \mathbb{N}$. Let us denote by $S^{(B,i)}$ its sum, and let Ω be as in the Maeda-Ogasawara-Vulikh representation theorem 2.3. Then there exists a meager set $N \subset \Omega$ such that the sequence $(\sigma_p(\omega))_p$ is decreasing and such that $\inf\limits_p \sigma_p(\omega) = \lim\limits_p \sigma_p(\omega) = 0$ for every $\omega \in \Omega \setminus N$. For each $B \in \mathcal{W}$, $i \in \mathbb{N}$ and $\omega \in \Omega \setminus N$ we get:

$$\begin{aligned}
\mathcal{F} \ni\ & \Big\{n \in \mathbb{N} : \Big|S^{(B,i)} - \sum_{j\in B\cap[1,n]} a_{i,j}\Big| \le \sigma_p\Big\} = \\
=\ & \Big\{n \in \mathbb{N} : \Big|S^{(B,i)}(\omega') - \sum_{j\in B\cap[1,n]} a_{i,j}(\omega')\Big| \le \sigma_p(\omega') \quad \text{for every } \omega' \in \Omega \setminus N\Big\} \qquad (3.1)\\
\subset\ & \Big\{n \in \mathbb{N} : \Big|S^{(B,i)}(\omega) - \sum_{j\in B\cap[1,n]} a_{i,j}(\omega)\Big| \le \sigma_p(\omega)\Big\},
\end{aligned}$$

and hence

$$\Big\{n \in \mathbb{N} : \Big|\sum_{j\in B\cap[1,n]} a_{i,j}(\omega) - S^{(B,i)}(\omega)\Big| \le \sigma_p(\omega)\Big\} \in \mathcal{F}.$$

So the series $(\mathcal{F})\sum\limits_{j\in B} a_{i,j}(\omega)$ $(\mathcal{F})$-converges in $\mathbb{R}$ for each $B \in \mathcal{W}$, $i \in \mathbb{N}$ and $\omega \in \Omega\setminus N$. From this and Lemma 3.1 it follows that the series $\sum\limits_{j\in A} a_{i,j}(\omega)$ converges and hence it is Cauchy in the usual sense uniformly with respect to $A \subset \mathbb{N}$, for every $i \in \mathbb{N}$ and

$\omega \in \Omega \setminus N$. By Theorem 2.3 and taking into account 3.1.3), for every $i \in \mathbb{N}$ there is a meager set $N_i \subset \Omega$, without loss of generality $N_i \supset N$, such that for every $i \in \mathbb{N}$ and $\omega \in \Omega \setminus N_i$ we get:

$$\begin{aligned} 0 &= \inf_{s\in\mathbb{N}}\Big(\sup_{n\geq s}\Big(\sup_{t\in\mathbb{N}}\Big(\sup_{A\subset\mathbb{N}}\Big|\sum_{j\in A\cap[n,n+t]}[a_{i,j}(\omega)]\Big|\Big)\Big)\Big) = \sup_{s\in\mathbb{N}}\Big(\inf_{n\geq s}\Big(\sup_{t\in\mathbb{N}}\Big(\sup_{A\subset\mathbb{N}}\Big|\sum_{j\in A\cap[n,n+t]}[a_{i,j}(\omega) \\ &= \Big[\bigwedge_{s\in\mathbb{N}}\Big(\bigvee_{n\geq s}\Big(\bigvee_{t\in\mathbb{N}}\Big(\bigvee_{A\subset\mathbb{N}}\Big|\sum_{j\in A\cap[n,n+t]}a_{i,j}\Big|\Big)\Big)\Big)\Big](\omega) = \Big[\bigvee_{s\in\mathbb{N}}\Big(\bigwedge_{n\geq s}\Big(\bigvee_{t\in\mathbb{N}}\Big(\bigvee_{A\subset\mathbb{N}}\Big|\sum_{j\in A\cap[n,n+t]}a_{i,j}\Big|\Big) \end{aligned}$$

Note that (3.2) holds for every $\omega \in \Omega \setminus \Big(\bigcup_{i=1}^{\infty} N_i\Big)$, and $\bigcup_{i=1}^{\infty} N_i$ is a meager subset of Ω. From (3.2) it follows that for each $i \in \mathbb{N}$ the series $\sum_{j\in A} a_{i,j}$ is (D)-Cauchy uniformly with respect to $A \subset \mathbb{N}$. By Remark 2.2, the series $\sum_{j\in A} a_{i,j}$ (D)-converges uniformly with respect to $A \subset \mathbb{N}$, with a corresponding regulator $(\alpha^{(i)}_{t,l})_{t,l}$. From this and Proposition 2.2 it follows that the series $\sum_{j=1}^{\infty} a_{i,j}$ is unconditionally convergent for every $i \in \mathbb{N}$, that is 3.1.5).

We now prove 3.1.6). Let $u = \bigvee_{i\in\mathbb{N},\, B\in\mathcal{I}_{\text{fin}}}\Big(\sum_{j\in B} a_{i,j}\Big)$. Since

$$\Big|\sum_{j\in A\cap[1,q]} a_{i,j}\Big| \leq u \tag{3.3}$$

for each i, $q \in \mathbb{N}$ and $A \subset \mathbb{N}$, and

$$\Big|\sum_{j\in A} a_{i,j}\Big| = (O)\lim_{q}\Big|\sum_{j\in A\cap[1,q]} a_{i,j}\Big| = (D)\lim_{q}\Big|\sum_{j\in A\cap[1,q]} a_{i,j}\Big| \tag{3.4}$$

for every $i \in \mathbb{N}$ and $A \subset \mathbb{N}$, from (3.3) and (3.4) we deduce that $\Big|\sum_{j\in A} a_{i,j}\Big| \leq u$ for any $i \in \mathbb{N}$ and $A \subset \mathbb{N}$, and hence, taking into account 3.1.3), we get $u = \bigvee_{i\in\mathbb{N},\, A\subset\mathbb{N}}\Big(\sum_{j\in A} a_{i,j}\Big)$. By the Fremlin lemma 2.1 there is a regulator $(b_{t,l})_{t,l}$ such that, for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $q \in \mathbb{N}$,

$$u \wedge \Big(\sum_{i=1}^{q}\Big(\bigvee_{t=1}^{\infty}\alpha^{(i)}_{t,\varphi(t+i)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)},$$

where $\alpha^{(i)}_{t,l}$, i, t, $l \in \mathbb{N}$, is as above, and hence for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$ there is $\overline{h} \in \mathbb{N}$ with

$$\Big|\sum_{j\in C} a_{i,j}\Big| \leq u \wedge \Big(\bigvee_{t=1}^{\infty}\alpha^{(i)}_{t,\varphi(t+i)}\Big) \leq \bigvee_{q=1}^{\infty}\Big(u \wedge \Big(\sum_{i=1}^{q}\Big(\bigvee_{t=1}^{\infty}\alpha^{(i)}_{t,\varphi(t+i)}\Big)\Big)\Big) \leq \bigvee_{i=1}^{\infty} b_{t,\varphi(t)} \tag{3.5}$$

whenever $C \subset]\overline{h}, +\infty[$.

Now, by virtue of 3.1.1), for every $j \in \mathbb{N}$ there exists a (D)-sequence $(\beta_{t,l}^{(j)})_{t,l}$, such that the sequence $(a_{i,j})_i$ (D)-converges to 0 with respect to $(\beta_{t,l}^{(j)})_{t,l}$. By Lemma 2.1, taking into account 3.1.3) and proceeding analogously as in (3.5), we find a regulator $(c_{t,l})_{t,l}$, such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$ there is $\overline{h} \in \mathbb{N}$ with

$$\Big|\sum_{j \in D} a_{q,j}\Big| \leq \bigvee_{i=1}^{\infty} c_{t,\varphi(t)} \tag{3.6}$$

whenever $q \geq \overline{h}$ and $D \subset [1, s]$. For any t, $l \in \mathbb{N}$ put $e_{t,l} := 6\,(z_{t,l} + b_{t,l} + c_{t,l})$ and $d_{t,l} := 2\,e_{t,l}$.

Now we prove that for any subset $A \subset \mathbb{N}$ the sequence $\Big(\sum\limits_{j \in A} a_{i,j}\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to the (D)-sequence $(d_{t,l})_{t,l}$, that is 3.1.6). Without loss of generality, we can suppose that A is infinite (see also [3, 11]). If we deny the thesis, then we find an element $\varphi \in \mathbb{N}^{\mathbb{N}}$, with

$$I^* = \Big\{ i \in \mathbb{N} : \Big|\sum_{j \in A} a_{i,j}\Big| \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \Big\} \notin \mathcal{F}.$$

Therefore, $F \cap (\mathbb{N} \setminus I^*) \neq \emptyset$ for every $F \in \mathcal{F}$. So, the set

$$I := \mathbb{N} \setminus I^* = \Big\{ i \in \mathbb{N} : \Big|\sum_{j \in A} a_{i,j}\Big| \not\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \Big\} \tag{3.7}$$

is $\mathcal{F}$-stationary. Observe that I is infinite, because $\mathcal{F}$ is a free filter. Thus there is a strictly increasing sequence $(k_r)_r$ in $\mathbb{N}$, such that $I = \{k_r \colon r \in \mathbb{N}\}$ and

$$\Big|\sum_{j \in A} a_{k_r,j}\Big| \not\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \quad \text{for every } r \in \mathbb{N}. \tag{3.8}$$

At the first step, we find an integer $k_1 > 1$ with $\Big|\sum\limits_{j \in A} a_{k_1,j}\Big| \not\leq \bigvee\limits_{t=1}^{\infty} d_{t,\varphi(t)}$. There is $l_1 > 1$ with

$$\Big|\sum_{j \in C} a_{k_1,j}\Big| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \tag{3.9}$$

whenever $C \subset]l_1, +\infty[$. Then we get

$$\Big|\sum_{j \in A \cap [1,l_1]} a_{k_1,j}\Big| \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}. \tag{3.10}$$

Indeed, if (3.10) does not hold, then

$$\Big|\sum_{j\in A} a_{k_1,j}\Big| \le \Big|\sum_{j\in A\cap[1,l_1]} a_{k_1,j}\Big| + \Big|\sum_{j\in A\cap]l_1,+\infty[} a_{k_1,j}\Big| \le \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

getting a contradiction with (3.8). Thus (3.10) is proved. Moreover, by (3.6) there is $r_1 > k_1$ with

$$\Big|\sum_{j\in D} a_{q,j}\Big| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \tag{3.11}$$

for each $q \ge r_1$ and $D \subset [1, l_1]$.

At the second step, we find two natural numbers $k_2 > i_2$ with

$$\Big|\sum_{j\in A} a_{k_2,j}\Big| \not\le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}, \tag{3.12}$$

and $l_2 > l_1$ with $\Big|\sum_{j\in C} a_{k_2,j}\Big| \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for every $C \subset]l_2, +\infty[$. We get

$$\Big|\sum_{j\in A\cap]l_1,l_2]} a_{k_2,j}\Big| \not\le \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} : \tag{3.13}$$

otherwise we have

$$\begin{aligned}\Big|\sum_{j\in A} a_{k_2,j}\Big| &\le \Big|\sum_{j\in A\cap]l_1,l_2]} a_{k_2,j}\Big| + \Big|\sum_{j\in A\cap[1,l_1]} a_{k_2,j}\Big| + \Big|\sum_{j\in A\cap]l_2,+\infty[} a_{k_2,j}\Big| \le \qquad (3.14)\\ &\le \bigvee_{t=1}^{\infty} e_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},\end{aligned}$$

getting a contradiction with (3.12). Thus (3.13) is satisfied.

By induction, we find two strictly increasing sequences $(k_r)_r$, $(l_r)_r$ in $\mathbb{N}$, with $l_r < k_r < l_{r+1}$ for each $r \in \mathbb{N}$, and:

I) $\Big|\sum_{j\in D} a_{k_r,j}\Big| \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$ for every $D \subset [1, l_{r-1}]$;

II) $\Big|\sum_{j\in C} a_{k_r,j}\Big| \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for each $C \subset]l_r, +\infty[$;

III) $\Big|\sum_{j\in F_r} a_{k_r,j}\Big| \not\le \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}$, where $F_r = A\cap]l_{r-1}, l_r]$, for any $r \ge 2$. Without loss of generality, we can suppose that $I \cap [l_r, l_{r+1}[\neq \emptyset$ for every r. Since $\mathcal{F}$ is block-respecting, there is a set $S := \{k_{s_1}, k_{s_2}, \ldots\} \in \mathcal{F}^*$, $S \subset I$, with $l_h \le k_{s_h} < l_{h+1}$ for every $h \in \mathbb{N}$. As $S \in \mathcal{F}^*$, then either $S_1 := \{k_{s_1}, k_{s_3}, k_{s_5}, \ldots\} \in \mathcal{F}^*$ or $S_2 :=$

$\{k_{s_2}, k_{s_4}, k_{s_6}, \ldots\} \in \mathcal{F}^*$. Without loss of generality, we assume that $S_1 \in \mathcal{F}^*$ (see also [6, 14]). From I) and II), for every $h \in \mathbb{N}$ we have

$$\Big|\sum_{j\in D} a_{k_{s_{2h-1}},j}\Big| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \quad \text{for every } D \subset [1, l_{2h-2}], \tag{3.15}$$

$$\Big|\sum_{j\in C} a_{k_{s_{2h-1}},j}\Big| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \text{for each } C \subset]l_{2h}, +\infty[.$$

From this we obtain

$$\Big|\sum_{j\in E_h} a_{k_{s_{2h-1}},j}\Big| \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}, \tag{3.16}$$

where $E_h = A \cap]l_{2h-2}, l_{2h}]$, for any $h \in \mathbb{N}$: otherwise, arguing analogously as in (3.14), we get

$$\Big|\sum_{j\in A} a_{k_{s_{2h-1}},j}\Big| \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

which contradicts (3.8).

By property (M), in correspondence with the sequence E_h, $h \in \mathbb{N}$, we find a finite set $D \subset \mathbb{N}$ and a set $W \in \mathcal{W}$, with $\bigcup\limits_{h\in\mathbb{N}\setminus D} E_h \subset W \subset \bigcup\limits_{h\in\mathbb{N}} E_h$. By 3.1.4), the sequence $\Big((D\mathcal{F})\sum\limits_{j\in W} a_{i,j}\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to the regulator $(z_{t,l})_{t,l}$. As seen in the proof of 3.1.5), the quantities $\sum\limits_{j\in W} a_{i,j}$, $i \in \mathbb{N}$, exist in R, and hence they are equal to $(D\mathcal{F})\sum\limits_{j\in W} a_{i,j}$. Thus the sequence $\Big(\sum\limits_{j\in W} a_{i,j}\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to the regulator $(z_{t,l})_{t,l}$, and so there is a set $F \in \mathcal{F}$ with

$$\Big|\sum_{j\in W} a_{i,j}\Big| \leq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)} \tag{3.17}$$

for each $i \in F$. Since $S_1 \in \mathcal{F}^*$, the family $\mathcal{F}(S_1) := \{E \cap S_1\colon E \in \mathcal{F}\}$ is a filter of S_1 (see also [10]). Thus, the set $F \cap S_1$ is infinite, and hence the set $F_* := F \cap \{k_{s_{2h-1}} : h \in \mathbb{N} \setminus D\}$ is infinite too, since D is finite. Pick $i_* \in F_*$, $i_* = k_{s_{2h_*-1}}$. Since $h_* \in \mathbb{N} \setminus D$, we get $E_{h_*} \subset W$, and hence $E_{h_*} = A \cap]l_{2h_*-2}, l_{2h_*}] \subset W \cap]l_{2h_*-2}, l_{2h_*}]$. Moreover, since

$$W \subset \bigcup_{h\in\mathbb{N}} E_h = A \cap \Bigg(\bigcup_{h\in\mathbb{N}}]l_{2h-2}, l_{2h}]\Bigg),$$

we have also $W \cap]l_{2h_*-2}, l_{2h_*}] \subset A \cap]l_{2h_*-2}, l_{2h_*}]$, and thus

$$W \cap]l_{2h_*-2}, l_{2h_*}] = A \cap]l_{2h_*-2}, l_{2h_*}] = E_{h_*}. \tag{3.18}$$

By (3.18) we get $W = (W \cap [1, l_{2h_*-2}]) \cup E_{h_*} \cup (W \cap]l_{2h_*}, +\infty[)$, and hence

$$\begin{aligned}\Big|\sum_{j \in E_{h_*}} a_{i_*,j}\Big| &\leq \Big|\sum_{j \in W} a_{i_*,j}\Big| + \Big|\sum_{j \in W \cap [1, l_{2h_*-2}]} a_{i_*,j}\Big| + \Big|\sum_{j \in W \cap]l_{2h_*}, +\infty[} a_{i_*,j}\Big| \\ &\leq \bigvee_{t=1}^{\infty} z_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)},\end{aligned}$$

which contradicts (3.16). Thus 3.1.6) is proved.

We now turn to 3.1.7). First of all, note that 3.1.5) holds for every block-respecting filter of $\mathbb{N}$ without using 3.1.1). Now we prove 3.1.6). By 3.1.4) and Theorem 2.2, in correspondence of the regulator $(z_{t,l})_{t,l}$ it is possible to find an (O)-sequence $(\tau_p)_p$, satisfying (2.1). By Lemma 2.2, for every $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, $J \subset I$, with $(O) \lim_i a_{i,j} = 0$ for every $j \in \mathbb{N}$ with respect to $(\tau_p)_p$. Set $c_{t,l} := \tau_l$, t, $l \in \mathbb{N}$. For every $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, $J \subset I$, such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$ there exists an integer $\overline{h} \in J$ with

$$\Big|\sum_{j \in D} a_{q,j}\Big| \leq \tau_{\varphi(1)} \leq \bigvee_{i=1}^{\infty} c_{t,\varphi(t)} \tag{3.19}$$

for any $q \geq \overline{h}$ and $D \subset [1, s]$. Let $(b_{t,l})_{t,l}$ be as in the proof of 3.1.6), and for every t, $l \in \mathbb{N}$ set $e_{t,l} := 6\,(z_{t,l} + b_{t,l} + c_{t,l})$, $d_{t,l} := 2\,e_{t,l}$. We claim that the (D)-sequence $(d_{t,l})_{t,l}$ satisfies 3.1.7).

Choose an infinite set $A \subset \mathbb{N}$. Proceeding by contradiction, similarly as in the proof of 3.1.6), we find an element $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that

$$I^* = \Big\{ i \in \mathbb{N} : \Big|\sum_{j \in A} a_{i,j}\Big| \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \Big\} \notin \mathcal{F}. \tag{3.20}$$

In correspondence with the $\mathcal{F}$-stationary set $I := \mathbb{N} \setminus I^*$ we find a set $J \in \mathcal{F}^*$, $J \subset I$, satisfying (3.19). Since $J \in \mathcal{F}^*$ and $\mathcal{F}$ is block- respecting, then, by Remark 2.5 (b), the filter $\mathcal{F}(J)$ of J is block-respecting too. Proceeding analogously as in the proof of 3.1.6), using the double sequence $(a_{i,j})_{i \in J, j \in \mathbb{N}}$ and the filter $\mathcal{F}(J)$, we obtain that the sequence $\Big(\sum_{j \in A} a_{i,j}\Big)_{i \in J}$ $(D\mathcal{F}(J))$-converges to 0 with respect to $(d_{t,l})_{t,l}$. But since J is $\mathcal{F}$-stationary, from (3.20) it is not difficult to deduce that the sequence $\Big(\sum_{j \in A} a_{i,j}\Big)_{i \in J}$ does not $(D\mathcal{F}(J))$-converge to 0 with respect to $(d_{t,l})_{t,l}$, obtaining a contradiction. This proves 3.1.7). □

A consequence of Theorem 3.1 is the following Schur-type theorem, which extends [14, Theorem 3.1 and Corollary 3.1.2].

Theorem 3.2 *Let R and $\mathcal{W}$ be as in Theorem 3.1, $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$ and $(a_{i,j})_{i,j}$ satisfy conditions 3.1.j), j=2,3,4. Then we get:*

3.2.1) $(D\mathcal{F}) \lim_i \Big(\sum_{j=1}^{\infty} |a_{i,j}|\Big) = 0;$

3.2.2) $(D\mathcal{F})\lim\limits_i\Big(\bigvee\limits_{A\subset\mathbb{N}}\Big(\sum\limits_{j\in A} a_{i,j}\Big)\Big)=0$;

3.2.3) if $m_i:\mathcal{P}(\mathbb{N})\to R$, $i\in\mathbb{N}$, *are defined by*

$$m_i(A):=\sum_{j\in A} a_{i,j},\quad A\in\mathcal{P}(\mathbb{N}),\tag{3.21}$$

then for every $\mathcal{F}$*-stationary set* $I\subset\mathbb{N}$ *there is an* $\mathcal{F}$*-stationary set* $J\subset I$ *with*

$$(D)\lim_n\Big(\bigvee_{i\in J} v(m_i)[n,+\infty[\Big)=0.$$

Proof: As we saw in the proof of 3.1.5), from 3.1.2) and Proposition 2.2 it follows that for each $i\in\mathbb{N}$ the series $\sum\limits_{j=1}^{\infty} a_{i,j}$ is unconditionally convergent. For $i\in\mathbb{N}$, let m_i be as in (3.21). By 3.1.6), the family $(m_i(A))$, $i\in\mathbb{N}$, $A\subset\mathbb{N}$, $(RD\mathcal{F})$-converges to 0. Moreover, by unconditional convergence and Proposition 2.3, the m_i's are σ-additive, and by hypothesis they are equibounded. From this and [14, Lemma 3.1] we get 3.2.1); 3.2.2) follows easily from 3.2.1), and 3.2.3) is a consequence of 3.2.2) and [14, Corollary 3.1.2]. □

Remark 3.1 Observe that from 3.2.1) it follows that $(D\mathcal{F})\lim\limits_i\Big(\bigvee\limits_{j\in\mathbb{N}}|a_{i,j}|\Big)=0$. So, Theorem 3.2 extends the basic matrix theorems given in [1, Theorem 2.3], [2, Theorem 4] and [13, Theorem 3.1].

Corollary 3.1 *Let* R, $\mathcal{F}$ *and* $\mathcal{W}$ *be as in Theorem 3.2, and* $(x_{i,j})_{i,j}$ *be a double sequence in* R*, such that:*

3.1.1) the limit $x_{0,j}:=(D\mathcal{F})\lim\limits_i x_{i,j}$ *exists in* R *for every* $j\in\mathbb{N}$*.*

3.1.2) the series $(D\mathcal{F})\sum\limits_{j\in B} x_{i,j}$ $(D\mathcal{F})$*-converges for each* $B\in\mathcal{W}$ *and* $i\geq 0$ *with respect to a single regulator, independent of* B *and* i*;*

3.1.3) the family $\Big(\sum\limits_{j\in B} x_{i,j}\Big)_{i\geq 0,B\in\mathcal{I}_{\text{fin}}}$ *is equibounded.*

Furthermore, suppose that

3.1.4) for every infinite subset $B\in\mathcal{W}$ *the sequence* $\Big((D\mathcal{F})\sum\limits_{j\in B}(x_{i,j}-x_{0,j})\Big)_i$ $(D\mathcal{F})$*-converges to* 0 *with respect to a regulator* $(z_{t,l})_{t,l}$ *independent of* B*.*

Then we get

3.1.5) $(D\mathcal{F})\lim\limits_i\sum\limits_{j=1}^{\infty} x_{i,j}=\sum\limits_{j=1}^{\infty} x_j$ *and*

3.1.6) $(D\mathcal{F})\lim\limits_i\sum\limits_{j\in A} x_{i,j}=\sum\limits_{j\in A} x_j$ *uniformly with respect to* $A\subset\mathbb{N}$*.*

Proof: The assertion follows by applying Theorem 3.2 to the double sequence $a_{i,j} = x_{i,j} - x_j$, i, $j \in \mathbb{N}$. $\square$

As a consequence of Corollary 3.1, we prove an interchange theorem for lattice group-valued series in the setting of filter convergence, extending [1, Theorem 3.6], [20, Theorem 2.5], [21, Theorem 1] and [22, Theorem 8.5.1].

Theorem 3.3 *Let* R, $\mathcal{F}$ *and* $\mathcal{W}$ *be as in Theorem 3.2,* $(x_j)_j$ *be a sequence in* R *and* $(x_{i,j})_{i,j}$ *be a double sequence in* R*, such that:*

3.3.1) the series $(D\mathcal{F})\sum\limits_{i=1}^{\infty} x_{i,j}$ $(D\mathcal{F})$*-converges in* R *to* x_j *for every* $j \in \mathbb{N}$*;*

3.3.2) the series $(D\mathcal{F})\sum\limits_{j\in B} x_{i,j}$ *and* $(D\mathcal{F})\sum\limits_{j\in B} x_j$ $(D\mathcal{F})$*-converge for each* $B \in \mathcal{W}$ *and* $i \in \mathbb{N}$ *with respect to a single regulator, independent of* B *and* i*;*

3.3.3) the families $\Big(\sum\limits_{i=1}^{q}\Big(\sum\limits_{j\in B} x_{q,j}\Big)\Big)_{q\in\mathbb{N}, B\in\mathcal{I}_{\rm fin}}$ *and* $\Big(\sum\limits_{j\in B} x_j\Big)_{B\in\mathcal{I}_{\rm fin}}$ *are equibounded;*

3.3.4) $(D\mathcal{F})\Big(\sum\limits_{i=1}^{\infty}\Big(\sum\limits_{j\in B} x_{i,j}\Big)\Big) = \sum\limits_{j\in B} x_j$ *for every infinite subset* $B \in \mathcal{W}$*, where the convergence of the series is intended with respect to a single regulator independent of* B*.*

Then we have

3.3.5) $(D\mathcal{F})\sum\limits_{i=1}^{\infty}\Big(\sum\limits_{j=1}^{\infty} x_{i,j}\Big) = \sum\limits_{j=1}^{\infty}\Big((D\mathcal{F})\sum\limits_{i=1}^{\infty} x_{i,j}\Big)$*;*

3.3.6) $(D\mathcal{F})\sum\limits_{i=1}^{\infty}\Big(\sum\limits_{j\in A} x_{i,j}\Big) = \sum\limits_{j\in A}\Big((D\mathcal{F})\sum\limits_{i=1}^{\infty} x_{i,j}\Big)$ *uniformly with respect to* $A \subset \mathbb{N}$*.*

Proof: We prove only 3.3.6), since the proof of 3.3.5) is analogous. For each q, $j \in \mathbb{N}$, set $y_{q,j} := \sum\limits_{i=1}^{q} x_{i,j}$. It is not difficult to check that $(y_{q,j})_{q,j}$ satisfies the hypotheses of Corollary 3.1. So we get, uniformly with respect to $A \subset \mathbb{N}$,

$$(D\mathcal{F})\lim_q\Big(\sum_{j\in A} y_{q,j}\Big) = \sum_{j\in A} x_j = \sum_{j\in A}\Big((D\mathcal{F})\sum_{i=1}^{\infty} x_{i,j}\Big),$$

and hence

$$(D\mathcal{F})\sum_{i=1}^{\infty}\Big(\sum_{j\in A} x_{i,j}\Big) = (D\mathcal{F})\lim_q\Big(\sum_{i=1}^{q}\Big(\sum_{j\in A} x_{i,j}\Big)\Big) =$$

$$= (D\mathcal{F})\lim_q\Big(\sum_{j\in A}\Big(\sum_{i=1}^{q} x_{i,j}\Big)\Big) = (D\mathcal{F})\lim_q\Big(\sum_{j\in A} y_{q,j}\Big) = \sum_{j\in A}\Big((D\mathcal{F})\sum_{i=1}^{\infty} x_{i,j}\Big). \qquad \square$$

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Open problems: (a) Prove similar theorems by replacing property (M) with some other conditions.

(b) Investigate some matrix and limit theorems with respect to other classes of filters.

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