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Nonlinear double resonant periodic problems with the scalar p-Laplacian

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Abstract

We study a nonlinear elliptic problem driven by the p-Laplacian and with a double resonance at $\pm\infty$. Following a variational approach based on the nonsmooth critical point theory and Morse theoretic techniques,we have multiplicity with at least three nontrivial solutions, when the double resonance occurs with respect to two successive eigenvalues of the negative p-Laplacian $\widehat{\lambda}_m < \widehat{\lambda}_{m+1}, m \neq 0$.

Keywords: Spectrum of the scalar p-Laplacian, double resonance, critical groups reduced homology sequence, Morse theory, multiplicity theorem.

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1 Introduction

In this paper we study the following nonlinear periodic problem:

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = f(t, u(t)) \text{ a.e. on } T = [0, b], \\ u(0) = u(b),\ u'(0) = u'(b),\ 1<p<\infty. \end{cases} \quad (1)$$

Scalar periodic problems with double resonance were investigated primarily for semilinear equations. In this direction we mention the works of Fabry–Fonda [7], Gasiński–Papageorgiou [10], Omari–Zanolin [13] and Su–Zhao [16]. Fabry–Fonda [7], Gasiński–Papageorgiou [10] and Su–Zhao [16], use Landesman–Lazer type conditions, while Omari-Zanolin [13] use certain nonresonance conditions involving the quotient $\frac{2F(t,x)}{x^2}$ where $F(t, x) = \int_0^x f(t, s)ds$ is the potential function corresponding to $f(t, x)$. In Fabry–Fonda [7] and Omari–Zanolin [13] the approach is degree theoretic, while in Gasiński–Papageorgiou [10] and Su–Zhao [16] the authors use variational methods coupled with techniques from Morse theory. From the aforementioned works Fabry–Fonda [7] and Omari–Zanolin

[13] prove only existence theorems, while Gasiński–Papageorgiou [10] and Su–Zhao [16] have multiplicity results. Gasiński–Papageorgiou [10] produce four solutions, while Su–Zhao [16] obtain two solutions. For equations driven by the periodic scalar p-Laplacian, to the best of our knowledge, there is only the work of Kyritsi–Papageorgiou [11], where the authors prove an existence theorem using conditions similar to those employed by Omari–Zanolin [13].

Combining variational methods based on the critical point theory with Morse theoretic techniques, we show that we have existence when the double resonance occurs at any spectral interval and we have multiplicity with at least three nontrivial solutions, when the double resonance occurs at any spectral interval distinct from the "principal" one $[\hat{\lambda}_0 = 0, \hat{\lambda}_1]$.

2 Multiplicity Theorem

The hypotheses on the reaction term $f(t,x)$ are:

H $f\colon T \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function s.t. for a.a. $t \in T$ $f(t,0) = 0$ and

(i) $|f(t,x)| \le a(t)(1 + |x|^{p-1})$ for a.a. $t \in T$, all $x \in \mathbb{R}$, with $a \in L^1(T)_+$;

(ii) there exist an integer $m \ge 1$ s.t.

$$\widehat{\lambda}_m \le \liminf_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \le \widehat{\lambda}_{m+1}$$

uniformly for a.a. $t \in T$, and

$$\lim_{|x|\to\infty} [f(t,x)x - pF(t,x)] = +\infty \text{ uniformly for a.a. } t \in T;$$

(iii) there exists a function $\theta \in L^1(T)$, $\theta(t) \le 0$ a.e. on T, $\theta \ne 0$ such that

$$\limsup_{x\to 0} \frac{pF(t,x)}{|x|^p} \le \theta(t) \text{ uniformly for a.a. } t \in T,$$

where $F(t,x) = \int_0^x f(t,s)ds$;

(iv) for every $r > 0$, there exists $\xi_r > 0$ s.t. $f(t,x) + \xi_r |x|^{p-2}x \ge 0$ for a.a. $t \in T$, all $x \in [-r, r]$.

We set $G_\pm(t,x) = \int_0^x g_\pm(t,s)ds$ and consider the C^1-functionals $\psi_\pm : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_0^b G_\pm(t,u(t))dt \text{ for all } u \in W^{1,p}_{\text{per}}(0,b).$$

Also, let $\varphi : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|u'\|_p^p - \int_0^b F(t,u(t))dt \text{ for all } u \in W^{1,p}_{\text{per}}(0,b).$$

We know that $\varphi \in C^1(W^{1,p}_{\text{per}}(0,b))$.

PROPOSITION 2.1. **If** *hypotheses H hold,* **then** $\psi_\pm$ *satisfy the C–condition.*

PROOF: We do the proof for ψ_+, the proof for ψ_- being similar.
We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$|\psi_+(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{2}$$

$$\text{and} \quad (1 + \|u_n\|)\psi'_+(u_n) \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b)^* \text{ as } n \to \infty. \tag{3}$$

From (3) we have

$$| < A(u_n), h > +\varepsilon \int_0^b |u_n|^{p-2}u_n h dt - \int_0^b g_+(t, u_n) h dt| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{4}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.
In (4) we choose $h = -u_n^- \in W^{1,p}_{\text{per}}(0,b)$. Then

$$\|(u_n^-)'\|_p^p + \varepsilon \|u_n^-\|_p^p \leq \varepsilon_n \text{ for all } n \geq 1,$$

$$\Rightarrow \quad u_n^- \to 0 \in W^{1,p}_{\text{per}}(0,b). \tag{5}$$

__Claim:__ $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded.
We proceed by contradiction. So, suppose that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}, n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we way assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T). \tag{6}$$

For (4) and (5) we have

$$| < A(y_n), h > +\varepsilon \int_0^b |y_n|^{p-2} y_n h dt - \int_0^b \frac{g_+(t, u_n^+)}{\|u_n^+\|^{p-1}} h dt| \leq \varepsilon'_n \|h\| \text{ with } \varepsilon'_n \to 0^+. \tag{7}$$

Choose $h = y_n - y \in W^{1,p}_{\text{per}}(0,b)$, pass to the limit as $n \to \infty$ and use (6). We obtain

$$\lim_{n\to\infty} < A(y_n), y_n - y >= 0,$$

$$\Rightarrow \quad y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b), \text{ hence } \|y\| = 1, y \geq 0. \tag{8}$$

Note that because of H($\underline{i}$) and (6), we have that $\{\frac{g_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable. So, by virtue of the Dunford-Pettis theorem, we may assume that

$$\frac{g_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\theta}_+ \text{ in } L^1(T). \tag{9}$$

Using hypothesis H($\underline{ii}$) and reasoning as in the proof of Proposition 14 of Aizicovici–Papageorgiou–Staicu [1], we show that

$$\widehat{\theta}_+ = (\xi + \varepsilon) y^{p-1} \text{ with } \widehat{\lambda}_m \leq \xi(t) \leq \widehat{\lambda}_{m+1} \text{ a.e. on T.} \tag{10}$$

So, if we return to (7), pass to the limit as $n \to \infty$ and use (8), (9) and (10), then

$$\begin{aligned} & < A(y), h >= \int_0^b \xi y^{p-1} h dt \text{ for all } h \in W^{1,p}_{\text{per}}(0,b), \\ \Rightarrow \quad & A(y) = \xi y^{p-1}, \\ \Rightarrow \quad & -(|y'(t)|^{p-2} y'(t))' = \xi(t) y(t)^{p-1} \text{ a.e. on T}, \\ & y(0) = y(b), y'(0) = y'(b). \end{aligned} \tag{11}$$

Recall that $\widehat{\lambda}_m \leq \xi(t) \leq \widehat{\lambda}_{m+1}$ a.e. on T. If $\xi \neq \widehat{\lambda}_m$ and $\xi \neq \widehat{\lambda}_{m+1}$, we have that $y = 0$, which contradicts (8). If $\xi(t) = \widehat{\lambda}_m$ a.e. on T or $\xi(t) = \widehat{\lambda}_{m+1}$ a.e. on T then from (11) and since $m \geq 1$, we infer that y must be nodal, which, contradicts (8). Therefore $\{u_n^+\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded. This proves the Claim.

From (5) and the Claim we infer that $\{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded. So, we may assume that $u_n \xrightarrow{w} u$ in $W^{1,p}_{\text{per}}(0,b)$ and $u_n \to u$ in $C(T)$. Hence, if in (4) we let $h = u_n - u$ and pass to the limit as $n \to \infty$, then

$$\begin{aligned} & \lim_{n \to \infty} < A(u_n), u_n - u >= 0, \\ \Rightarrow \quad & u_n \to u \text{ in } W^{1,p}_{\text{per}}(0,b) \end{aligned}$$

This proves that ψ_+ satisfies the *C–condition*. Similarly for ψ_-.

□

PROPOSITION 2.2. **If** *hypotheses H hold,* **then** φ *satisfy the C–condition.*

PROOF: We consider a sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$|\varphi(u_n)| \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1 \tag{12}$$

$$\text{and} \quad (1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b)^* \text{ as } n \to \infty. \tag{13}$$

From (13) we have

$$| < A(u_n, h) > - \int_0^b f(t, u_n) h dt| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{14}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.
In (14) we choose $h = u_n$ and obtain

$$-\|u_n'\|_p^p + \int_0^b f(t, u_n) u_n dt \leq \varepsilon_n \text{ for all } n \geq 1. \tag{15}$$

On the other hand from (12), we have

$$\|u_n'\|_p^p - \int_0^b pF(t, u_n) dt \leq pM_2 \text{ for all } n \geq 1. \tag{16}$$

Adding (15) and (16), we obtain

$$\int_0^b [f(t,u_n)u_n - pF(t,u_n)]dt \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1. \tag{17}$$

Claim: $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded.

We argue indirectly. So, suppose that $\|u_n\| \to \infty$ and set $y_n = \frac{u_n}{\|u_n\|}$ $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T) \text{ as } n \to \infty. \tag{18}$$

From (14) we have

$$| < A(y_n), h > - \int_0^b \frac{f(t,u_n)}{\|u_n\|^{p-1}} h dt| \leq \frac{\varepsilon_n \|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \text{ for all } n \geq 1. \tag{19}$$

It is clear from hypothesis H($\underline{i}$) that $\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable. Hence, if we set $h = y_n - y$ and pass to the limit as $n \to \infty$, then

$$\begin{aligned} &\lim_{n\to\infty} < A(y_n), y_n - y >= 0, \\ \Rightarrow\quad & y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b), \text{ hence } \|y\| = 1 \ . \end{aligned} \tag{20}$$

Since $\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable, by the Dunford–Pettis theorem, we may assume that

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\theta} \text{ in } L^1(T) \tag{21}$$

with $\widehat{\theta} = \xi|y|^{p-2}y$, $\widehat{\lambda}_m \leq \xi(t) \leq \widehat{\lambda}_{m+1}$ a.e. on T (see the proof of Proposition 2.1). Passing to the limit as $n \to \infty$ in (19) and using (20) and (21), we obtain

$$\begin{aligned} & < A(y), h >= \int_0^b \xi|y|^{p-2}yhdt \text{ for all } h \in W^{1,p}_{\text{per}}(0,b), \\ \Rightarrow\quad & A(y) = \xi|y|^{p-2}y, \\ \Rightarrow\quad & -(|y'(t)|^{p-2}y'(t))' = \xi(t)|y(t)|^{p-2}y(t) \text{ a.e. on T}, \\ & y(0) = y(b), y'(0) = y'(b). \end{aligned} \tag{22}$$

We know that $\widehat{\lambda}_m \leq \xi(t) \leq \widehat{\lambda}_{m+1}$ a.e. on T. First suppose that $\xi \neq \widehat{\lambda}_m$ and $\xi \neq \widehat{\lambda}_{m+1}$. Then from (22) we have that $y = 0$, which contradicts (20). So, we assume that $\xi(t) = \widehat{\lambda}_m$ a.e. on T or $\xi(t) = \widehat{\lambda}_{m+1}$ a.e. on T. Then we have $y(t) \neq 0$ for a.a. $t \in T$ (see Binding–Rynne [3]). Therefore $|u_n(t)| \to \infty$ for a.a. $t \in T$ and this by virtue of hypothesis H($\underline{ii}$) implies

$$\begin{aligned} & f(t,u_n(t))u_n(t) - pF(t,u_n(t)) \to +\infty \text{ for a.a } t \in T, \\ \Rightarrow\quad & \textstyle\int_0^b [f(t,u_n(t))u_n(t) - pF(t,u_n(t))]dt \to +\infty \text{ (by Fatou's lemma)} \end{aligned} \tag{23}$$

Comparing (17) and (23), we reach a contradiction. This proves the Claim.

By virtue of the Claim, we may assume that $u_n \xrightarrow{w} u$ in $W^{1,p}_{\text{per}}(0,b)$ and $u_n \to u$ in $C(T)$. Using $h = u_n - u$ in (14) and passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} < A(u_n), u_n - u >= 0, \\ \Rightarrow\quad & u_n \to u \text{ in } W^{1,p}_{\text{per}}(0,b) \end{aligned}$$

This proves the proposition.

□

PROPOSITION 2.3. **If** *hypotheses H hold,* **then** $u = 0$ *is a local minimizer of* $\psi_\pm$ *and of* φ.

PROOF: We do the proof for ψ_+, the proofs for ψ_- and φ being similar. By virtue of hypothesis H($\underline{iii}$), given $\widehat{\varepsilon} > 0$, we can find $\widehat{\delta} = \widehat{\delta}(\widehat{\varepsilon}) > 0$ s.t.

$$F(x,t) \le \frac{1}{p}(\theta(t) + \widehat{\varepsilon})|x|^p \text{ for a.a. } t \in T, \text{ all } |x| \le \widehat{\delta}. \tag{24}$$

Let $u \in \widehat{C}^1(T)$ with $||u||_{C^1(T)} \le \widehat{\delta}$. Then

$$\begin{aligned} \psi_+(u) &= \frac{1}{p}||u'||_p^p + \frac{\varepsilon}{p}||u||_p^p - \int_0^b G_+(t,u)dt \\ &\ge \frac{1}{p}||u'||_p^p - \int_0^b F(t,u^+)dt \\ &\ge \frac{1}{p}||u'||_p^p - \frac{1}{p}\int_0^b \theta|u|^p dt - \frac{\widehat{\varepsilon}}{p}||u||^n \text{ (see (24))} \\ &\ge \frac{\xi_0 - \widehat{\varepsilon}}{p}||u||^p \end{aligned} \tag{25}$$

Choosing $\widehat{\varepsilon} \in (0, \xi_0)$ we infer that

$$\begin{aligned} & \psi_+(u) \ge 0 \text{ for all } u \in \widehat{C}^1(T) \text{ with } ||u||_{C^1(T)} \le \widehat{\delta}, \\ \Rightarrow\quad & u = 0 \text{ is a local } \widehat{C}^1(T)\text{-minimizer of } \psi_+, \\ \Rightarrow\quad & u = 0 \text{ is a local } W^{1,p}_{\text{per}}(0,b)\text{-minimizer of } \psi_+ \\ & \text{(see Proposition 3.3 of Kyritsi–Papageorgiou [12]).} \end{aligned}$$

Similarly for the functionals ψ_- and φ. □

We may assume that $u = 0$ is an isolated critical point of ψ_-. Indeed, otherwise we can find $\{u_n\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)\setminus\{0\}$ such that $u_n \to 0$ in $W^{1,p}_{\text{per}}(0,b)$ and

$$\begin{aligned} & \psi'_+(u_n) = 0 \text{ for all } n \ge 1, \\ \Rightarrow\quad & A(u_n) + \varepsilon|u_n|^{p-2}u_n = N_{g_+}(u_n) \text{ for all } n \ge 1, \\ & \quad\text{where } N_{g_+}(u)(\cdot) = g_+(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}_{\text{per}}(0,b). \end{aligned} \tag{26}$$

Acting on (26) with $-u_n^- \in W^{1,p}_{\text{per}}(0,b)$, we obtain $u_n \geq 0$ for all $n \geq 1$ and so (26) becomes

$$\begin{aligned} & A(u_n) = N_f(u_n) \text{ for all } n \geq 1, \\ & \qquad \text{where } N_f(u)(\cdot) = f(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}_{\text{per}}(0,b), \\ \Rightarrow \quad & u_n \in C^1(T) \text{ is a solution of (1) for all } n \geq 1. \end{aligned}$$

Hence we have produced a whole sequence of distinct nontrivial (and in fact positive) solutions of (1) and so we are done.

Reasoning as in Aizicovici–Papageorgiou–Staicu [1] (see the proof of Proposition 29), we can find $\rho_+ \in (0,1)$ small s.t.

$$\psi_+(0) = 0 < inf[\psi_+(u) : ||u|| = \rho_+] = \widehat{\eta}_+. \tag{27}$$

In a similar way, we show that we can find $\rho_- \in (0,1)$ small s.t.

$$\psi_-(0) = 0 < inf[\psi_-(u) : ||u|| = \rho_-] = \widehat{\eta}_-. \tag{28}$$

Now we are ready to produce two constant sign solutions for problem (1).

□

PROPOSITION 2.4. If *hypotheses H hold,*
then *problem (1) has at least two constant sign solutions*

$$u_0 \in int\widehat{C}_+ \text{ and } v_0 \in -int\widehat{C}_+.$$

PROOF: Let $\xi \in \mathbb{R}$, $\xi > 0$. Then

$$\psi_+(\xi) = -\int_0^b F(t,\xi)dt$$

From hypothesis H($\underline{ii}$) it follows that

$$\widehat{\lambda}_m \leq \liminf_{|\xi|\to\infty} \frac{pF(t,\xi)}{|\xi|^p} \leq \limsup_{|\xi|\to\infty} \frac{pF(t,\xi)}{|\xi|^p} \leq \widehat{\lambda}_{m+1} \text{ uniformly for a.a. } t \in T.$$

(see, for example, Aizicovici–Papageorgiou–Staicu [1], Remark 26). Therefore

$$\psi_+(\xi) \to -\infty \text{ as } \xi \to +\infty. \tag{29}$$

From (27), (29) and Proposition 2.1, it follows that we can apply the mountain pass theorem, and obtain $u_0 \in W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\psi_+(0) = 0 < \widehat{\eta}_+ \leq \psi_+(u_0) \tag{30}$$

$$\psi_+'(u_0) = 0. \tag{31}$$

From (30) we have $u_0 \neq 0$. From (31) we have

$$A(u_0) + \varepsilon|u_0|^{p-2}u_0 = N_{g_+}(u_0). \tag{32}$$

Acting on (32) with $-u_0^- \in W^{1,p}_{\text{per}}(0,b)$, we show that $u_0 \geq 0$. So (32) becomes

$$\begin{aligned} & A(u_0) = N_f(u_0), \\ \Rightarrow\ & -(|u_0'(t)|^{p-2}u_0'(t))' = f(t,u_0(t)) \text{ a.e. on } T, \\ & u_0(0) = u_0(b), u_0'(0) = u_0'(b), \\ \Rightarrow\ & u_0 \in \widehat{C}_+ \setminus \{0\} \text{ solves problem (1).} \end{aligned} \tag{33}$$

Let $r = ||u_0||_\infty$. Then by virtue of hypothesis H($\underline{iv}$), we can find $\xi_r > 0$ s.t.

$$\begin{aligned} & f(t,u_0(t)) + \xi_r u_0(t)^{p-1} \geq 0 \text{ a.e. on } T \\ \Rightarrow\ & (|u_0'(t)|^{p-2}u_0'(t))' \leq \xi_r u_0(t)^{p-1} \text{a.e. on } T \text{ (see (33))}, \\ \Rightarrow\ & u_0 \in int\widehat{C}_+ \text{ (see Vazquez [17]).} \end{aligned}$$

Similarly, working this time with ψ_- and using (28), we obtain a second constant sign solution $v_0 \in -int\widehat{C}_+$.

□

Next using Morse theory, we will produce a third nontrivial solution for problem (1). To this end, first we prove the following auxiliary result which will be helpful in computing certain critical groups at infinity. Our result extends Lemma 2.4 of Perera–Schechter [15] which is formulated in Hilbert spaces, with stronger hypotheses and for functionals satisfying the *PS–condition.*

LEMMA 2.5. If *X is a Banach, $(\tau,u) \to h_\tau(u)$ belongs in $C^1([0,1]\times X)$ and is bounded, there is $R > 0$ s.t. for all $\tau \in [0,1]$ $K_{h_\tau} \subseteq \overline{B}_R$, the maps $u \to \partial_\tau h_\tau(u)$ and $u \to h_\tau'(u)$ are both locally Lipschitz, h_0 and h_1 satisfy the C–condition and there exist $\beta \in \mathbb{R}$ and $\delta > 0$ s.t.*

$$h_\tau(u) \leq \beta \Rightarrow (1+\|u\|)\|h_\tau'(u)\|_* \geq \delta \text{ for all } \tau \in [0,1],$$

then *$C_k(h_0,\infty) = C_k(h_1,\infty)$ for all $k \geq 0$.*

PROOF: Since by hypothesis $h \in C^1([0,1]\times X)$, we know that its admits a peudogradient vector field $\widehat{v} = (v_0, v) : [0,1] \times (X \setminus \overline{B}_R) \to [0,1]\times X$. Moreover, taking into account the construction of the pseudogradient vector field, we know that we can have $v_0 = \partial_\tau h_\tau$. Also by definition $(\tau,u) \to v_\tau(u)$ is locally Lipschitz and in fact for every $\tau \in [0,1]$, $v_\tau(\cdot)$ is a pseudogradient vector field for the function $h_\tau(\cdot)$ (see Chang [4], p.19 and Papageorgiou–Kyritsi [14], p.237). Then for every $(\tau,u) \in [0,1]\times(X\setminus\overline{B}_R)$ we have

$$< h_\tau'(u), v_\tau(u) > \geq \|h_\tau'(u)\|_*^2 \tag{34}$$

The map $X \setminus \overline{B}_R \ni u \to -\frac{|\partial_\tau h_\tau(u)|}{\|h_\tau'(u)\|_*^2} v_\tau(u) = w_\tau(u) \in X$ is well defined and locally Lipschitz. Since $(\tau,u) \to h_\tau(u)$ is bounded, we can find $\eta \leq \beta$ s.t.

$$\eta < inf[h_\tau(u) : \tau \in [0,1], \|u\| \leq R].$$

We choose $\eta \leq \beta$ s.t. $h_0^\eta \neq 0$ or $h_1^\eta \neq \emptyset$ (if no such η can be found, then $C_k(h_0, \infty) = C_k(h_1, \infty) = \delta_{k,0}\mathbb{Z}$ for all $k \geq 0$ and so we are done). To fix things, we assume that $h_0^\eta \neq \emptyset$ and let $y \in h_0^\eta$. We consider the following Cauchy problem:

$$\frac{d\xi}{d\tau} = w_\tau(\xi)\ \tau \in [0,1], \xi(0) = y. \tag{35}$$

Since $w_\tau(u)$ is locally Lischitz, this Cauchy problem has a unique local flow (see, for example, Gasinski–Papageorgiou [9], p.618). We have

$$\begin{aligned}
\tfrac{d}{d\tau}h_\tau(\xi) \quad &=< h'_\tau(\xi), \frac{d\xi}{d\tau} > +\partial_\tau h_\tau(\xi) \\
&=< h'_\tau(\xi), w_\tau(\xi) > +\partial_\tau h_\tau(\xi) \text{ (see (35))} \\
&\leq -|\partial_\tau h_\tau(\xi)| + \partial_\tau h_\tau(\xi) \text{ (see (34))} \\
&\leq 0, \\
\Rightarrow \quad & \tau \to h_\tau(\xi(\tau, y)) \text{ is nonincreasing}, \\
\Rightarrow \quad & h_\tau(\xi(\tau, y)) \leq h_0(\xi(0, y)) = h_0(y) \leq \eta \leq \beta, \\
\Rightarrow \quad & (1 + \|\xi(\tau, y)\|)\|h'_\tau(\xi(\tau, y))\|_* \geq \delta \text{ (by hypothesis)}, \\
\Rightarrow \quad & h'_\tau(\xi(\tau, y)) \neq 0.
\end{aligned}$$

This shows that in fact the flow $\xi(\cdot, y)$ is global.

Then $\xi(1, \cdot)$ is a homeomorphism between h_0^η and a subset of h_1^η. Reversing the time $t \to 1 - t$, we establish that h_1^η is homeomorphic to a subset of h_0^η. Therefore h_0^η and h_1^η are homotopy equivalent and so

$$\begin{aligned}
& H_k(X, h_0^\eta) = H_k(X, h_1^\eta) \text{ for all } k \geq 0, \\
\Rightarrow \quad & C_k(h_0, \infty) = C_k(h_1, \infty) \text{ for all } k \geq 0.
\end{aligned}$$

□

Using this lemma we can have some useful information concerning the critical groups at infinity of φ.

PROPOSITION 2.6. If *hypotheses H hold,* **then** $C_{m+1}(\varphi, \infty) \neq 0$. *($m \geq 1$ as in hypothesis H(ii)).*

PROOF: Let $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and consider the C^1-functional $\chi : W^{1,p}_{\text{per}}(0, b) \to \mathbb{R}$ define by

$$\chi(u) = \frac{1}{p}\|u'\|_p^p - \frac{\mu}{p}\|u\|_p^p \text{ for all } u \in W^{1,p}_{\text{per}}(0, b).$$

We consider the homotopy

$$h(\tau, u) = (1 - \tau)\varphi(u) + \tau\chi(u) \text{ for all } (\tau, u) \in [0, 1] \times W^{1,p}_{\text{per}}(0, b).$$

Clearly we may assume that K_φ is finite (otherwise we already have infinitely many distinct nontrivial solutions of (1) and so we are done). Note that $h(0, \cdot) =$

φ satisfies the *C–condition* (see Proposition 2.2) and $h(1,\cdot) = \chi$ also satisfies the *C–condition* since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$.

Claim: There exist $\beta \in \mathbb{R}$ and $\delta > 0$ s.t.

$$h(\tau, u) \le \beta \Rightarrow (1 + \|u\|)\|h'_u(\tau, u)\|_* \ge \delta \text{ for all } \tau \in [0,1].$$

We argue by contradiction. So, suppose that the Claim is not true. Since h is bounded, we can find $\{\tau_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\tau_n \to \tau, \|u_n\| \to \infty, h(\tau_n, u_n) \to -\infty \text{ and } (1 + \|u_n\|)h'_u(\tau_n, u_n) \to 0. \tag{36}$$

By virtue of the last convergence in (36), we have

$$|< A(u_n), h > -(1-\tau_n)\int_0^b f(t, u_n)hdt - \tau_n\mu\int_0^b |u_n|^{p-2}u_n hdt| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{37}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.

Let $y_n = \frac{u_n}{\|u_n\|}$ $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T). \tag{38}$$

From (37) we have

$$\begin{aligned} &|< A(y_n), h > -(1-\tau_n)\textstyle\int_0^b \frac{f(t,u_n)}{\|u_n\|^{p-1}}hdt - \tau_n\mu\int_0^b |y_n|^{p-2}y_n hdt| \\ &\qquad \le \tfrac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \text{ for all } n \ge 1. \end{aligned} \tag{39}$$

Recall (see (21)) that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\theta} = \xi|y|^{p-2}y \text{ in } L^1(T) \text{ with } \widehat{\lambda}_m \le \xi(t) \le \widehat{\lambda}_{m+1} \text{ a.e. on } T. \tag{40}$$

In (39), we choose $h = y_n - y$ and pass to the limit as $n \to \infty$. Using (38), we obtain

$$\begin{aligned} &\lim_{n\to\infty} < A(y_n), y_n - y >= 0, \\ \Rightarrow\ & y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and so } \|y\| = 1. \end{aligned} \tag{41}$$

Hence, if in (39) we pass to the limit as $n \to \infty$ and use (40) and (41), then

$$\begin{aligned} & < A(y), h >= (1-\tau)\int_0^b \xi|y|^{p-2}yhdt + \tau\mu\int_0^b |y|^{p-2}yhdt \\ & \text{for all } h \in W^{1,p}_{\text{per}}(0,b), \\ \Rightarrow\ & A(y) = \xi_\tau y \text{ with } \xi_\tau = (1-\tau)\xi + \tau\mu, \\ \Rightarrow\ & -(|y'(t)|^{p-2}y'(t))' = \xi_\tau(t)|y(t)|^{p-2}y(t) \text{ a.e. on } T, \\ & y(0) = y(b), y'(0) = y'(b). \end{aligned} \tag{42}$$

Note that $\widehat{\lambda}_m \leq \xi_\tau(t) \leq \widehat{\lambda}_{m+1}$ a.e. on T.
If $\tau \in (0,1]$, then

$$\begin{aligned} & \xi_\tau \neq \widehat{\lambda}_m \text{ and } \xi_\tau \neq \widehat{\lambda}_{m+1}, \\ \Rightarrow \quad & y = 0 \text{ (see (42) , which contradicts (41).} \end{aligned}$$

So, suppose $\tau = 0$. Then $\xi_0 = \xi$ and we proceed as in the proof of Proposition 2.2 to reach a contradiction, using hypothesis H($\underline{ii}$)) and the third convergence in (36). This proves the Claim.

Then we can apply Lemma 2.5 and we have

$$C_k(\varphi, \infty) = C_k(\chi, \infty) \text{ for all } k \geq 0. \tag{43}$$

Since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, $u = 0$ is the only critical point of χ. Hence

$$C_k(\chi, \infty) = C_k(\chi, 0) \text{ for all } k \geq 0. \tag{44}$$

Let $r > 0$ and set $E_0 = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u'\|_p^p < \mu\|u\|_p^p, \|u\| = r\}$ and $D = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u'\|_p^p \geq \mu\|u\|_p^p\}$. Evidently $E_0 \cap D = \emptyset$. Also $\partial B_r = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u\| = r\}$ is a Banach C^1-manifold, hence locally contractible. Since E_0 is an open subset of ∂B_r E_0 is locally contractible. Similarly $W^{1,p}_{\text{per}}(0,b) \setminus D$ is locally contractible. Note that since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, we have $i(E_0) = m + 1$, where i denotes the index introduced by Fadell–Rabinowitz [8]. Similarly $i(W^{1,p}_{\text{per}}(0,b) \setminus D) = m + 1$. Invoking Theorem 3.6 of Cingolani–Degiovanni [6], we know that there exists $C \subseteq W^{1,p}_{\text{per}}(0,b)$ compact s.t. the pair $(E_0 \cup C, E_0)$ and D homologically link in dimension $m+1$ and so $C_{m+1}(\chi, 0) \neq 0$ (see Chang [4], p.89). From (43) and (44) we conclude that $C_{m+1}(\varphi, \infty) \neq 0$.

□

Next we compute the critical groups at infinity of $\psi_\pm$.

PROPOSITION 2.7. If *hypotheses H hold,*
then $C_k(\psi_+, \infty) = C_k(\psi_-, \infty) = 0$ *for all* $k \geq 0$.

PROOF: We do the proof for ψ_+, the proof for ψ_- being similar.

Let $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and consider the C^1-functional $\sigma_+ : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \frac{\mu + \varepsilon}{p}\|u^+\|_p^p$$

for all $u \in W^{1,p}_{\text{per}}(0,b)$, with $\varepsilon \in (0, \widehat{\lambda}_2)$.

We consider the homotopy $h_+ : [0,1] \times W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$h_+(\tau, u) = (1-\tau)\psi_+(u) + \tau\sigma_+(u) \text{ for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

As before, without any loss of generality, we assume that K_{ψ_+} is finite.

<u>**Claim:**</u> There exist $\beta \in \mathbb{R}$ and $\delta > 0$ s.t.

$$h_+(\tau, u) \leq \beta \Rightarrow (1 + \|u\|)\|(h_+)'_u(\tau, u)\|_* \geq \delta \text{ for all } \tau \in [0,1].$$

As before, we argue by contradiction. So, suppose we can find $\{\tau_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\tau_n \to \tau \in [0,1], \|u_n\| \to \infty, h_+(\tau_n, u_n) \to -\infty \text{ and } (1+\|u_n\|)(h_+)'_u(\tau_n, u_n) \to 0. \tag{45}$$

From the last convergence in (45), we have

$$\begin{aligned} | < A(u_n), h > +\varepsilon \textstyle\int_0^b |u_n|^{p-2} u_n h dt - (1-\tau_n) \int_0^b g_+(t, u_n) h dt \\ -\tau_n(\mu+\varepsilon) \textstyle\int_0^b (u_n^+)^p h dt| \leq \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \end{aligned} \tag{46}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.

In (46) we choose $h = -u_n^- \in W^{1,p}_{\text{per}}(0,b)$ and

$$\begin{aligned} &\|(u_n^-)'\|_p^p + \varepsilon \|u_n^-\|_p^p \leq \varepsilon_n \text{ for all } n \geq 1, \\ \Rightarrow \quad & u_n^- \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ as } n \to \infty. \end{aligned} \tag{47}$$

From (45) (second convergence) and (47) it follows that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T). \tag{48}$$

From (46) and (47), we have

$$\begin{aligned} | < A(y_n), h > +\varepsilon \int_0^b y_n^{p-1} h dt - (1-\tau_n) \int_0^b \frac{g_+(t, u_n^+)}{\|u_n^+\|^{p-1}} h dt \\ -\tau_n(\mu+\varepsilon) \int_0^b y_n^{p-1} h dt| \leq \varepsilon_n' \|h\| \end{aligned} \tag{49}$$

with $\varepsilon_n' \to 0$.

In (49) we choose $h = y_n - y$. Passing to the limit as $n \to \infty$ and using (48) we obtain

$$\begin{aligned} &\lim_{n\to\infty} < A(y_n), y_n - y >= 0, \\ \Rightarrow \quad & y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and so } \|y\| = 1, y \geq 0. \end{aligned} \tag{50}$$

Recall that

$$\frac{g_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\theta}_+ = (\xi+\varepsilon) y^{p-1} \text{ in } L^1(T) \text{ and } \widehat{\lambda}_m \leq \xi(t) \leq \widehat{\lambda}_{m+1} \text{ for a.a. } t \in T. \tag{51}$$

Therefore, if in (49) we pass to the limit as $n \to \infty$ and use (50) and (51), then

$$
\begin{aligned}
&< A(y), h >= \int_0^b \xi_\tau y^{p-1} h dt \text{ for all } h \in W^{1,p}_{\text{per}}(0,b) \\
&\qquad\qquad \text{with } \xi_\tau = (1-\tau)\xi + \tau\mu, \\
\Rightarrow\ & A(y) = \xi_\tau y^{p-1}, \\
\Rightarrow\ & -(|y'(t)|^{p-2}y'(t))' = \xi_\tau(t) y(t)^{p-1} \text{ a.e. on } T, \\
& y(0) = y(b), y'(0) = y'(b).
\end{aligned} \tag{52}
$$

We know that $\widehat{\lambda}_m \leq \xi_\tau(t) \leq \widehat{\lambda}_{m+1}$ a.e. on T. If $\tau \in (0,1]$, then $\xi_\tau \neq \widehat{\lambda}_m$, $\xi_\tau \neq \widehat{\lambda}_{m+1}$ and so by virtue of (52), we have $y = 0$, which contradicts (50). The same is true if $\tau = 0$ and $\xi_0 \neq \widehat{\lambda}_m$, $\xi_0 \neq \widehat{\lambda}_{m+1}$. Finally, if $\tau = 0$ and $\xi_0 = \widehat{\lambda}_m$, or $\xi_0 = \widehat{\lambda}_{m+1}$ a.e. on T, then from (52) and since $m \geq 1$, $y(\cdot)$ must be nodal again a contradiction (see (50)). This proves the claim.

The claim permits the use of Lemma 2.5 and we have

$$
C_k(\psi_+, \infty) = C_k(\sigma_+, \infty) \text{ for all } k \geq 0. \tag{53}
$$

Since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, $u = 0$ is the only critical point of σ_+ and so

$$
C_k(\sigma_+, \infty) = C_k(\sigma_+, 0) \text{ for all } k \geq 0. \tag{54}
$$

Let $\eta \in L^\infty(\Omega)$, $\eta \geq 0$, $\eta \neq 0$ and consider the homotopy $\widehat{h}_+ : [0,1] \times W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$
\widehat{h}_+(\tau, u) = \sigma_+(u) - \tau\eta u \text{ for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).
$$

We claim that

$$
(\widehat{h}_+)'(\tau, u) \neq 0 \text{ for all } \tau \in [0,1], u \neq 0. \tag{55}
$$

Suppose that (55) is not true. We can find $\tau \in [0,1]$ and $u \neq 0$ s.t.

$$
\begin{aligned}
& (\widehat{h}_+)'(\tau, u) = 0, \\
\Rightarrow\ & A(u) + \varepsilon|u|^{p-2}u = (\mu + \varepsilon)(u^+)^{p-1} + \tau\eta.
\end{aligned} \tag{56}
$$

On (56) we act with $-u^- \in W^{1,p}_{\text{per}}(0,b)$ and obtain $\|(u^-)'\|_p^p + \varepsilon\|u^-\|_p^p = 0$, i.e., $u \geq 0$. So, (56) becomes

$$
A(u) = \mu u^{p-1} + \tau\eta, u \geq 0, u \neq 0. \tag{57}
$$

First suppose that $\tau = 0$. Then

$$
\begin{aligned}
& A(u) = \mu u^{p-1} \text{ (see (57))}, \\
\Rightarrow\ & -(|u'(t)|^{p-2}u'(t))' = \mu u(t)^{p-1} \text{ a.e. on } T \\
& u(0) = u(b), u'(0) = u'(b), \\
\Rightarrow\ & u \text{ must be nodal (recall } m \geq 1\text{), which contradicts (57).}
\end{aligned}
$$

So, we assume that $\tau \in (0,1]$. Then

$$\begin{aligned} & A(u) = \mu u^{p-1} + \tau\eta, \\ \Rightarrow\quad & -(|u'(t)|^{p-2}u'(t))' = \mu u(t)^{p-1} + \tau\eta(t) \text{ a.e. on } T, \\ & u(0) = u(b), u'(0) = u'(b), \end{aligned} \tag{58}$$

We have $u \in C_+ \setminus \{0\}$ and $(|u'(t)|^{p-2}u'(t))' \leq 0$ a.e. on T. It follows that $u \in int\widehat{C}_+$ (see Vazquez [17]).

Let $y \in \widehat{C}_+$ and consider

$$R(y,u)(t) = |y'(t)|^p - |u'(t)|^{p-2}u'(t)(\frac{y^p}{u^{p-1}})'(t)$$

From the generalized Picone identity of Allegretto–Huang [2], we have

$$\begin{aligned} 0 \ & \leq \int_0^b R(y,u)(t)dt \\ & = \|y'\|_p^p - \int_0^b -(|u'|^{p-2}u')'\frac{y^p}{u^{p-1}}dt \text{ (by integration by parts)} \\ & = \|y'\|_p^p - \int_0^b (\mu y^p + \tau\eta)dt \text{ (see (58))} \\ & \leq \|y'\|_p^p - \mu\|y\|_p^p \text{ (recall } \eta \geq 0). \end{aligned}$$

We choose $y = \widehat{u}_0 \in int\widehat{C}_+$. Then

$$0 \leq -\mu\widehat{u}_0^p b < 0, \text{ a contradiction.}$$

This proves that (55) holds. Then the homotopy invariance property of critical groups (see Chang [5], p. 334) implies that

$$C_k(\sigma_+, 0) = C_k(\widehat{\sigma}_+, 0) \text{ for all } k \geq 0, \tag{59}$$

where $\widehat{\sigma}_+(u) = \sigma_+(u) - \eta u$ for all $u \in W^{1,p}_{\text{per}}(0,b)$. From the previous argument, we know that $\widehat{\sigma}_+$ has no critical points. Then

$$\begin{aligned} & C_k(\widehat{\sigma}_+, 0) = 0 \text{ for all } k \geq 0, \\ \Rightarrow\quad & C_k(\psi_+, \infty) \text{ for all } k \geq 0 \text{ (see (59), (54) and (53)).} \end{aligned}$$

Similarly, we show that $C_k(\psi_-, \infty) = 0$ for all $k \geq 0$. □

Having this proposition, we can have a precise computation of the critical groups of φ at $u_0 \in int\widehat{C}_+$ and $v_0 \in -int\widehat{C}_+$. Recall that u_0, v_0 are the two constant sign solutions of (1) obtained in Proposition 2.4.

PROPOSITION 2.8. If *hypotheses H hold and $u_0 \in int\widehat{C}_+$ and $v_0 \in -int\widehat{C_+}$ are the two constant sign solutions of (1) obtained in Proposition 2.4,* **then** $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \geq 0$.

PROOF: We do the proof for u_0, the proof for v_0 being similar.

First note that we may assume that $\{0, u_0\}$ are the only critical points of ψ_+ (otherwise, we already have one more solution $y_0 \in int\widehat{C}_+$ of (1) distinct from $\{0, u_0, v_0\}$; note that $K_{\psi_+} \subseteq \widehat{C}_+$).

Let $\eta < 0 < \xi < \widehat{\eta}_+$ (see (27)) and consider the following triple of sets

$$\psi_+^{\eta} \subseteq \psi_+^{\xi} \subseteq W^{1,p}_{\text{per}}(0, b).$$

For this triple, we consider the long exact sequence of homology groups

$$\cdots \to H_k(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}) \xrightarrow{i_*} H_k(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}) \xrightarrow{\partial_*} H_{k-1}(\psi_+^{\xi}, \psi_+^{\eta}) \to \cdots. \tag{60}$$

By i_* we denote the group homomorphism induced by the inclusion $(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}) \xrightarrow{i} (W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi})$ and ∂_* is the boundary homomorphism. From the rank theorem, we have

$$\begin{aligned} rankH_k(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}) &= rank(ker\partial_*) + rank(im\partial_*) \text{ (see (60))}, \\ &= rank(imi_*) + rank(im\partial_*) \qquad (61) \\ &\qquad \text{(from the exactness of (60))}. \end{aligned}$$

Recalling that $\{0, u_0\}$ are the only critical points of ψ_+ and since

$$\eta < 0 = \psi_+(0) < \widehat{\eta}_+ \leq \psi_+(u_0),$$

we have

$$\begin{aligned} H_k(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}) &= C_k(\psi_+, \infty) = 0 \text{ for all } k \geq 0 \text{ (see Proposition 2.7)}, \\ \Rightarrow imi_* &= \{0\}. \end{aligned} \tag{62}$$

Also $H_{k-1}(\psi_+^{\xi}, \psi_+^{\eta}) = C_{k-1}(\psi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see Proposition 2.3). Therefore

$$rank(im\partial_*) \leq 1. \tag{63}$$

Finally since $0 < \xi < \widehat{\eta}_+ \leq \psi_+(u_0)$, we have

$$H_k(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}) = C_k(\psi_+, u_0) \text{ for all } k \geq 0\ . \tag{64}$$

So, if in (61), we use (62), (63), (64), then

$$rankC_1(\psi_+, u_0) \leq 1. \tag{65}$$

From the proof of Proposition 2.4, we know that u_0 is a critical point of ψ_+ of mountain pass type. Hence $C_1(\psi_+, u_0) \neq 0$ (see Chang [4], p.89). Combining this with (65) we infer that

$$C_k(\psi_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{66}$$

Consider the homotopy $\overline{h}_+ : [0,1] \times W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\overline{h}_+(\tau, u) = (1-\tau)\varphi(u) + \tau\psi_+(u) \text{ for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

Claim: We may assume that we can find $\rho \in (0,1)$ small s.t. u_0 is the only critical point for all $\tau \in [0,1]$ of $\overline{h}_+(\tau, \cdot)$ in $\overline{B}_\rho(u_0) = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u - u_0\| = \rho\}$.

Suppose we can find $\{\tau_n\}_{n\geq 1} \subseteq [0,1]$ and $\{\overline{u}_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\tau_n \to \tau \in [0,1], \overline{u}_n \to u_0 \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } (\overline{h}_+)'(\tau_n, \overline{u}_n) = 0 \text{ for all } n \geq 1. \quad (67)$$

We have

$$\begin{aligned} & A(\overline{u}_n) + \tau_n|\overline{u}_n|^{p-2}\overline{u}_n && = (1-\tau_n)N_{g_+}(\overline{u}_n) + \tau_n N_f(\overline{u}_n) \text{ for all } n \geq 1, \\ \Rightarrow \quad & -(|\overline{u}_n'(t)|^{p-2}\overline{u}_n'(t))' && = f(t, \overline{u}_n^+(t)) + (1-\tau_n)f(t, -\overline{u}_n^-(t)) \\ & && \quad +\tau_n(u_n^-)^{p-1} \text{ a.e. on } T, \\ & && \overline{u}_n(0) = \overline{u}_n(b), \overline{u}_n'(0) = \overline{u}_n'(b). \end{aligned} \quad (68)$$

From (68), arguing as in the proof of Proposition 3.3 of Kyritsi–Papageorgiou [12], we establish that $\{\overline{u}_n\}_{n\geq 1} \subseteq C^1(T)$ is relatively compact. Therefore we have

$$\overline{u}_n \to u_0 \text{ in } C^1(T) \text{ (see (67)).} \quad (69)$$

Recall that $u_0 \in int\widehat{C}_+$. So, we can find $n_0 \geq 1$ s.t.

$$\begin{aligned} & = \overline{u}_n \in int\widehat{C}_+ \text{for all } n \geq n_0 \text{ (see (69))}, \\ \Rightarrow \quad & -(|\overline{u}_n'(t)|^{p-2}\overline{u}_n'(t))' = f(t, \overline{u}_n(t)) \text{ a.e. on } T, \\ & \overline{u}_n(0) = \overline{u}_n(b), \overline{u}_n'(0) = \overline{u}_n'(b), \\ \Rightarrow \quad & \{\overline{u}_n\}_{n\geq 1} \subseteq int\widehat{C}_+ \text{ are nontrivial solutions of (1) and so we are done.} \end{aligned}$$

This proves the Claim.

Then the Claim and the homotopy invariance property of the critical groups (see Chang [5], p.334), we have

$$\begin{aligned} & C_k(\varphi, u_0) = C_k(\psi_+, u_0) \text{ for all } k \geq 0, \\ \Rightarrow \quad & C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (66))} \end{aligned}$$

In the similar fashion, using this time ψ_-, we show that $C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.

□

Now we can state the multiplicity theorem for problem (1) under double resonance conditions.

THEOREM 2.9. **If** *hypotheses H hold,*
then *problem (1) has at least three nontrivial solutions*

$$u_0 \in int\widehat{C}_+, v_0 \in -int\widehat{C_+} \text{ and } y_0 \in C^1(T).$$

PROOF: From Proposition 2.4, we already have two nontrivial constant sing solutions of (1)

$$u_0 \in int\widehat{C}_+ \text{ and } v_0 \in -int\widehat{C}_+.$$

From Proposition 2.8, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{70}$$

Also, by virtue of Proposition 2.3, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{71}$$

Recall that $C_{m+1}(\varphi, \infty) \neq 0$ (see Proposition 2.6). This implies that there exists $y_0 \in K_\varphi$ s.t.

$$C_{m+1}(\varphi, y_0) \neq 0, m \geq 2. \tag{72}$$

Comparing (72) with (70) and (71), we infer that $y_0 \notin \{0, u_0, v_0\}$. Also $y_0 \in C^1(T)$ and solves problem (1).

□

So our work here shows that multiplicity (producing at least three nontrivial solutions) can happen when we have double resonance at any spectral interval beyond the "principal" one $[\widehat{\lambda}_0 = 0, \widehat{\lambda}_1]$.

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