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Variational convergence of multidimensional conservation laws

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Abstract

We are concerned with a control problem related to the vanishing viscosity approximation to multidimensional scalar conservation laws. We investigate the Γ-convergence of the control cost functional, as the viscosity coefficient tends to zero. We prove various generalizations to arbitrary dimensions of previous results known in one spatial dimension.

1. Introduction

We are concerned with the scalar multi-dimensional conservation law

$$u_t + \nabla \cdot f(u) = 0 \tag{1.1}$$

where, given $T > 0$, $u = u(t,x) \in \mathbb{R}$, $(t,x) \in [0,T] \times \mathbb{R}^n$, $\nabla\cdot$ denotes the divergence operator, and the *flux* $f = (f_1, \ldots, f_n)$ is a vector-valued smooth function. As is well known, (1.1) does not admit classical solutions, and it should be interpreted weakly. However (1.1) admits in general infinitely many weak solutions, and an admissibility condition is required to select the correct "physical" weak solution. Given a function η, called *entropy*, a conjugated *entropy flux* $q = (q_1, \ldots, q_n)$ is a vector-valued function such that $q_i' = \eta' f_i'$ for any $i = 1, \ldots, n$. A weak solution to (1.1) is called *entropic* iff for each entropy – entropy flux pair (η, q) with η convex, the inequality

$$\eta(u)_t + \nabla \cdot q(u) \le 0$$

holds in the sense of distributions. Note that the entropy condition is always satisfied for smooth solutions to (1.1). The classical theory, see e.g. [5, 8], shows existence and uniqueness in $C\left([0,T]; L_1(\mathbb{R}^n)\right)$ of the entropic solution to the Cauchy problem associated with (1.1). Such a solution is obtained as the limit, as $\varepsilon \to 0$, of the solutions u_ϵ to the parabolic equation

$$u_{\epsilon\, t} + \nabla \cdot f(u_\epsilon) = \frac{\varepsilon}{2} \nabla \cdot \big(D(u_\epsilon) \nabla u_\epsilon\big), \tag{1.2}$$

where D is a uniformly elliptic smooth matrix-valued function, $\varepsilon > 0$ and ∇ denotes the gradient operator.

This paper discusses an extension of this classical convergence result. Let us introduce in (1.2) a *control* $E \equiv E(t,x) \in \mathbb{R}^n$

$$u_t + \nabla \cdot f(u) = \frac{\varepsilon}{2} \nabla \cdot \big(D(u)\nabla u\big) - \nabla \cdot \big(\sigma(u)E\big) \qquad (t,x) \in (0,T) \times \mathbb{R}^n \tag{1.3}$$

where $u \equiv u^{\epsilon,E}$, σ is a matrix-valued positive smooth function. If we think of u as a density of charge, then E can be naturally interpreted as the "controlling" external electric field and $\sigma(u)$ as the conductivity. The flow (1.3) conserves the total charge $\int dx\, u(t,x)$ whenever it is well defined. The *cost functional* I_ε associated with (1.2) can be now informally defined as the work done by the optimal controlling field E in (1.3), namely

$$I_\varepsilon(u) = \inf_E \frac{1}{2} \int_{[0,T]\times\mathbb{R}^n} dt\, dx\, \sigma(u) E \cdot E, \tag{1.4}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$, and the infimum is taken over the controls E such that (1.3) holds. See Section 2 for a precise definition of I_ε. We notice that the approach that we follow here for the study of the hyperbolic equation (1.1) can be related to the least squares method [2].

The variational convergence of the sequence $\{I_\varepsilon\}$ as $\varepsilon \to 0$ has been analyzed in [1] in spatial dimension $n = 1$, for a priori equibounded densities u, and with free initial condition. In this setting, the Γ-limit I of $\{I_\varepsilon\}$ has been characterized in a suitable space of Young measures. Since I tuns out to vanish on a "large" set (the so-called measure valued solutions to (1.1)) the Γ-convergence of

$$H_\varepsilon := \varepsilon^{-1} I_\varepsilon$$

was then analyzed. Equicoercivity of $\{H_\varepsilon\}$ and a Γ-liminf inequality were then established in [1] and a candidate Γ-limit H was thus identified. The Γ-limsup inequality was proved on a wide set of functions u, but could not be established for all u.

The aim of this paper is to investigate the Γ-convergence of $\{H_\varepsilon\}$ in arbitrary spatial dimension n, thus generalizing the results obtained in [1]. Here, in addition, we drop the a priori restriction of uniform boundedness of the densities u, and we introduce a fixed initial condition u_0 for (1.3). We establish equicoercivity of $\{H_\varepsilon\}$ and a Γ-liminf inequality. The Γ-limsup inequality is not considered in the present paper. Indeed, the main difficulties to prove it come from the lackness of (the proof of) a chain rule formula for the so-called divergence-measure vector fields. While some progress has been made on this subject in the last decade [7, 4], especially in the case $n = 1$ [6], as far as we know the expected results are still missing. Establishing the Γ-limsup inequality in dimension $n > 1$ seems to be considerably more difficult than in dimension $n = 1$.

2. Notation and results

Let $n \geq 1$ be an integer. We assume

- $f \in C^2(\mathbb{R};\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R};\mathbb{R}^n)$,
- D, $\sigma \in C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^n)$ to be uniformly elliptic, symmetric matrix-valued functions, hence there exists $c > 0$ such that

$$D(v) \geq c \qquad \text{and} \qquad \sigma(v) \geq c \qquad \text{for any } v \in \mathbb{R}.$$

Here $D(v) \geq c$ means that the matrix $D(v) - c\ \mathrm{Id}$ is semipositive definite (and the same for $\sigma(v) - c\ \mathrm{Id}$) and $C_{\mathrm{b}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^n)$ denotes the space of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}^n \times \mathbb{R}^n$.

Moreover a so-called *Einstein relation* is supposed to hold, that is D and σ are proportional. More precisely, we suppose the following key hypothesis:

- there exists $\chi \in C(\mathbb{R}; (0, +\infty))$ (the susceptibility in the physical interpretation of (1.3)) such that

$$D(v) = \frac{1}{\chi(v)}\ \sigma(v) \qquad \text{for all } v \in \mathbb{R}. \tag{2.1}$$

Finally we suppose

- $u_0 \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$.

Hereafter $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(\mathbb{R}^n)$ and $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the inner product in $L_2\left([0,T] \times \mathbb{R}^n\right)$. We will address the problem on the function space

$$\mathfrak{U} := L_2\left([0,T] \times \mathbb{R}^n\right)$$

equipped with its strong topology.

2.1. The functionals I_ε and H_ε

For any $\varepsilon > 0$, any function $u \in \mathfrak{U}$ such that $\nabla u \in L_{2,\mathrm{loc}}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right)$, and $\varphi \in C^\infty_{\mathrm{c}}\left([0,T) \times \mathbb{R}^n\right)$, we set

$$\ell^u_\varepsilon(\varphi) := -\langle u_0, \varphi(0)\rangle - \langle\langle u, \varphi_t \rangle\rangle - \langle\langle f(u), \nabla\varphi \rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u)\nabla u, \nabla\varphi \rangle\rangle. \tag{2.2}$$

We then define

$$I_\varepsilon(u) := \begin{cases} \displaystyle\sup_{\varphi \in C^\infty_{\mathrm{c}}([0,T) \times \mathbb{R}^n)} \left[\ell^u_\varepsilon(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\nabla\varphi, \nabla\varphi \rangle\rangle\right] \\ \qquad\qquad \text{if } \nabla u \in L_{2,\mathrm{loc}}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right) \text{ and } u \in C\left([0,T]; L_2(\mathbb{R}^n)\right), \\ \\ +\infty \qquad \text{otherwise.} \end{cases}$$

We let, as in the introduction,

$$H_\varepsilon(u) := \varepsilon^{-1} I_\varepsilon(u), \qquad u \in \mathfrak{U}. \tag{2.3}$$

A more intrinsic characterization of I_ε (and H_ε) is given below in Lemma 3.1. Notice that

$$I_\varepsilon(u) < +\infty \quad \Longrightarrow \quad u(0) = u_0.$$

2.2. The functional H

An element $u \in \mathfrak{U}$ is a *weak solution* to (1.1) iff for each $\varphi \in C_c^\infty\big((0,T)\times\mathbb{R}^n\big)$ it satisfies

$$\langle\langle u, \varphi_t\rangle\rangle + \langle\langle f(u), \nabla\varphi\rangle\rangle = 0.$$

Notice that if u is a weak solution to (1.1), then $t \mapsto \langle u(t), \varphi(t)\rangle$ is a continuous map for all $\varphi \in C_c^\infty\big([0,T]\times\mathbb{R}^n\big)$. In particular, if $u(0) = u_0$ in such a weak (distributional) sense, then we say that u satisfies the initial condition $u(0) = u_0$.

We denote by $C_b^2(\mathbb{R})$ the set of twice differentiable functions with bounded second derivative. A function (resp. a convex function) $\eta \in C_b^2(\mathbb{R})$ is called an *entropy* (resp. a *convex entropy*), and its *conjugated entropy flux* $q \in C^2(\mathbb{R};\mathbb{R}^n)$ is defined, up to an additive constant, by

$$q_i(w) := \int^w dv\, \eta'(v) f_i'(v) \qquad \text{for any } i = 1,\dots,n.$$

For a weak solution u to (1.1), for an entropy – entropy flux pair (η, q), the *η-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C_c^\infty\big((0,T)\times\mathbb{R}^n\big)$ as

$$\wp_{\eta,u}(\varphi) := -\langle\langle \eta(u), \varphi_t\rangle\rangle - \langle\langle q(u), \nabla\varphi\rangle\rangle. \tag{2.4}$$

The next proposition introduces a suitable class of solutions to (1.1). We denote by $M\big((0,T)\times\mathbb{R}^n\big)$ the set of Radon measures on $(0,T)\times\mathbb{R}^n$ that we consider equipped with the weak* topology. In the following, for $\varrho \in M\big((0,T)\times\mathbb{R}^n\big)$ we denote by $\varrho^\pm$ the positive and negative part of ϱ.

Proposition 2.1 *Let $u \in \mathfrak{U}$ be a weak solution to* (1.1). *The following statements are equivalent:*

(i) *for each entropy η, the η-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $(0,T)\times\mathbb{R}^n$;*

(ii) *there exists a bounded measurable map*

$$\varrho_u : \mathbb{R} \ni v \to \varrho_u(v; dt, dx) \in M\big((0,T)\times\mathbb{R}^n\big)$$

such that for any entropy η

$$\wp_{\eta,u}(\varphi) = \int dv\, \varrho_u(v; dt, dx)\, \eta''(v)\varphi(t,x). \tag{2.5}$$

Proof. In the case $n = 1$ the proof is given in [1] following [7, Prop. 3.1]. The same proof works for $n > 1$. □

A weak solution $u \in \mathfrak{U}$ that satisfies the equivalent conditions in Proposition 2.1 is called an *entropy-measure solution* to (1.1). We denote by

$$\mathcal{E} \subset \mathfrak{U}$$

the set of entropy-measure solutions to (1.1), and by

$$\mathcal{E}_{u_0} \subset \mathcal{E}$$

the set of entropy-measure solutions to (1.1) satisfying the initial condition $u(0) = u_0$.
A weak solution $u \in \mathfrak{U}$ to (1.1) is called an *entropic solution* iff for each convex entropy η the inequality $\wp_{\eta,u} \le 0$ holds. In particular entropic solutions are entropy-measure solutions such that $\varrho_u(v; dt, dx)$ is a negative Radon measure for each $v \in \mathbb{R}$. It is well known, see e.g. [5, 8], that for each $u_0 \in L_1(\mathbb{R}^n)$ there exists a unique entropic solution $\bar{u} \in C([0,T]; L_1(\mathbb{R}^n))$ to (1.1) such that $\bar{u}(0) = u_0$. Such a solution $\bar{u}$ is called the *Kruzkov solution* with initial datum u_0.

Recalling (2.1) we define $H : \mathfrak{U} \to [0, +\infty]$ as

$$H(u) := \begin{cases} \displaystyle\int dv\, \varrho_u^+(v; dt, dx)\, \frac{1}{\chi(v)} & \text{if } u \in \mathcal{E}_{u_0}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

The following proposition is proved in Section 4.

Proposition 2.2 *H is a lower semicontinuous functional on $\mathfrak{U}$.*

2.3. Γ-convergence

We briefly recall the definition of Γ-convergence, see e.g. [3]
A sequence of functionals $H_\varepsilon : \mathfrak{U} \to [0, +\infty]$ is *equicoercive* on $\mathfrak{U}$ iff for each $M > 0$ there exists $\varepsilon_0 \equiv \varepsilon_0(M) > 0$ such that $\cup_{\varepsilon \in (0,\varepsilon_0)} \{u \in \mathfrak{U} : H_\varepsilon(u) \le M\}$ is precompact is $\mathcal{U}$.
Given $u \in \mathfrak{U}$ we define

$$\begin{aligned} \big(\Gamma\text{–}\varliminf_{\varepsilon \to 0} H_\varepsilon\big)(u) &:= \inf\big\{ \varliminf_{\varepsilon \to 0} H_\varepsilon(u^\varepsilon),\ \{u^\varepsilon\} \subset \mathfrak{U} \,:\, u^\varepsilon \to u \big\}, \\ \big(\Gamma\text{–}\varlimsup_{\varepsilon \to 0} H_\varepsilon\big)(u) &:= \inf\big\{ \varlimsup_{\varepsilon \to 0} H_\varepsilon(u^\varepsilon),\ \{u^\varepsilon\} \subset \mathfrak{U} \,:\, u^\varepsilon \to u \big\}. \end{aligned}$$

Whenever $\Gamma\text{–}\varliminf_{\varepsilon} H_\varepsilon = \Gamma\text{–}\varlimsup_{\varepsilon} H_\varepsilon = H$ we say that $\{H_\varepsilon\}$ Γ-converges to H in $\mathfrak{U}$. Equivalently, the sequence $\{H_\varepsilon\}$ Γ-converges to H iff for each $u \in \mathfrak{U}$:

- for any sequence $\{u^\varepsilon\}$ converging to u, $\varliminf_\varepsilon H_\varepsilon(u^\varepsilon) \ge H(u)$ (Γ-*liminf inequality*);
- there exists a sequence $\{u^\varepsilon\}$ converging to u such that $\varlimsup_\varepsilon H_\varepsilon(u^\varepsilon) \le H(u)$ (Γ-*limsup inequality*).

We remark that the equicoercivity and the Γ-liminf inequality for a sequence $\{H_\varepsilon\}$ imply a lower bound for infima over closed sets, see e.g. [3, Prop. 1.18]. This is a relevant information for the control problem introduced in (1.3)-(1.4).
The main results of this paper are the following two theorems.

Theorem 2.1 *Let H_ε and H be defined as in* (2.3) *and* (2.6) *respectively. Then the sequence of functionals $\{H_\varepsilon\}$ satisfies the* Γ*-liminf inequality*

$$\Gamma\text{-}\varliminf_{\varepsilon} H_\varepsilon \ge H \qquad \text{on } \mathfrak{U}.$$

Theorem 2.2 *Assume the nondegeneracy condition*

$$\operatorname{meas}\big(\{v \in \mathbb{R} \,:\, \alpha + f'(v)\cdot\zeta = 0\}\big) = 0 \qquad \textit{for all } \alpha \in \mathbb{R},\ \zeta \in \mathbb{S}^{n-1}. \tag{2.7}$$

Then the sequence of functionals $\{H_\varepsilon\}$ is equicoercive on $\mathfrak{U}$.

3. The functional I_ε and a priori bounds

In this section we establish some properties of the functional I_ε. The proofs are adapted from the one-dimensional case [1]. Given a bounded measurable matrix-valued function $a \geq 0$ on $[0,T]\times\mathbb{R}^n$ let $\mathcal{D}^1_a$ be the Hilbert space obtained by identifying and completing the functions $\varphi \in C^\infty_{\rm c}([0,T]\times\mathbb{R}^n)$ with respect to the seminorm $\langle\langle \nabla\varphi, a\,\nabla\varphi\rangle\rangle^{1/2}$. Let $\mathcal{D}^{-1}_a$ be its dual space. The corresponding norms are denoted respectively by $\|\cdot\|_{\mathcal{D}^1_a}$ and $\|\cdot\|_{\mathcal{D}^{-1}_a}$. Since σ is uniformly elliptic, $\mathcal{D}^1_{\sigma(u)}$ is independent of $\sigma(u)$, while the norm $\|\cdot\|_{\mathcal{D}^1_{\sigma(u)}}$ obviously depends on $\sigma(u)$.

We first establish the connection between the cost functional I_ε and the perturbed parabolic problem (1.3).

Lemma 3.1 *Fix $\varepsilon > 0$ and let $u \in \mathfrak{U}$ be such that*

$$I_\varepsilon(u) < +\infty.$$

Then there exists $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that u is a weak solution to (1.3) *with $E = \nabla\Psi^{\varepsilon,u}$, namely for each $\varphi \in C^\infty_{\rm c}([0,T]\times\mathbb{R}^n)$*

$$\begin{aligned}&\langle u(T),\varphi(T)\rangle - \langle u_0,\varphi(0)\rangle\\ &\quad - \Big[\langle\langle u,\varphi_t\rangle\rangle + \big\langle\big\langle f(u) - \frac{\varepsilon}{2}D(u)\nabla u + \sigma(u)\nabla\Psi^{\varepsilon,u},\nabla\varphi\big\rangle\big\rangle\Big] = 0.\end{aligned} \tag{3.1}$$

The function $\Psi^{\varepsilon,u}$ is unique and

$$I_\varepsilon(u) = \frac{1}{2}\Big\|u_t + \nabla\cdot f(u) - \frac{\varepsilon}{2}\nabla\cdot\big(D(u)\nabla u\big)\Big\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} = \frac{1}{2}\,\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}}. \tag{3.2}$$

Proof. The functional ℓ^ε_u defined in (2.2) can be extended to a linear functional on $C^\infty_{\rm c}([0,T]\times\mathbb{R}^n)$ by setting

$$\begin{aligned}\ell^u_\varepsilon(\varphi) =& \langle u(T),\varphi(T)\rangle - \langle u_0,\varphi(0)\rangle - \langle\langle u,\varphi_t\rangle\rangle - \langle\langle f(u),\nabla\varphi\rangle\rangle\\ &+ \frac{\varepsilon}{2}\langle\langle D(u)\nabla u,\nabla\varphi\rangle\rangle.\end{aligned} \tag{3.3}$$

Since for any $\varphi \in C^\infty_{\rm c}([0,T]\times\mathbb{R}^n)$ the map $[0,T] \ni t \mapsto \langle u(t),\varphi(t)\rangle \in \mathbb{R}$ is continuous, it is not difficult to see that

$$I_\varepsilon(u) = \sup_{\varphi\in C^\infty_{\rm c}([0,T]\times\mathbb{R}^n)} \Big\{\ell^u_\varepsilon(\varphi) - \frac{1}{2}\langle\langle\sigma(u)\nabla\varphi,\nabla\varphi\rangle\rangle\Big\}.$$

We claim that ℓ_ε^u defines a bounded linear functional on $\mathcal{D}^1_{\sigma(u)}$. Indeed, since $I_\varepsilon(u) < +\infty$, we have

$$\ell_\varepsilon^u(\varphi) \le I_\varepsilon(u) + \frac{1}{2}\langle\langle \sigma(u)\,\nabla\varphi, \nabla\varphi\rangle\rangle = I_\varepsilon(u) + \frac{1}{2}\|\varphi\|^2_{\mathcal{D}^1_{\sigma(u)}}$$

and thus the linearity of ℓ_ε^u implies a fortiori

$$|\ell_\varepsilon^u(\varphi)| \le \sqrt{2\,I_\varepsilon(u)}\|\varphi\|_{\mathcal{D}^1_{\sigma(u)}}.$$

Therefore $\ell_\varepsilon^u(\varphi)$ is compatible with the identification in the definition of $\mathcal{D}^1_{\sigma(u)}$ above, and bounded in such a space. It can thus be extended by compatibility and density to a continuous linear functional on $\mathcal{D}^1_{\sigma(u)}$. Still denoting by ℓ_ε^u such an extension, we get

$$I_\varepsilon(u) = \sup_{\varphi\in\mathcal{D}^1_{\sigma(u)}} \Big\{ \ell_\varepsilon^u(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\nabla\varphi, \nabla\varphi\rangle\rangle \Big\}, \tag{3.4}$$

which is equivalent to the first equality in (3.2). Riesz representation theorem on Hilbert spaces implies existence and uniqueness of $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that

$$\ell_\varepsilon^u(\varphi) = \big(\Psi^{\varepsilon,u}, \varphi\big)_{\mathcal{D}^1_{\sigma(u)}} \qquad \text{for any } \varphi \in \mathcal{D}^1_{\sigma(u)}, \tag{3.5}$$

in particular (3.1). Riesz representation also yields

$$I_\varepsilon(u) = \frac{1}{2}\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}}.$$

□

Lemma 3.2 *Let $\varepsilon > 0$ and $u \in \mathfrak{U}$ be such that*

$$I_\varepsilon(u) < +\infty,$$

and let $\Psi^{\varepsilon,u}$ be as in Lemma 3.1. Then for each entropy – entropy flux pair (η, q), each $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^n)$, and each $t \in [0,T]$ we have

$$\begin{aligned}
&\langle \eta(u(t)), \varphi(t)\rangle - \langle \eta(u_0), \varphi(0)\rangle - \int_{[0,t]} ds\,\big[\langle \eta(u), \varphi_s\rangle + \langle q(u), \nabla\varphi\rangle\big] \\
&\quad = -\frac{\varepsilon}{2}\int_{[0,t]} ds\,\Big[\langle \eta''(u)D(u)\nabla u, \varphi\,\nabla u\rangle + \langle \eta'(u)D(u)\nabla u, \nabla\varphi\rangle\Big] \\
&\qquad + \int_{[0,t]} ds\Big[\langle \sigma(u)\eta''(u)\,\nabla u, \varphi\nabla\Psi^{\varepsilon,u}\rangle + \langle \sigma(u)\eta'(u)\,\nabla\Psi^{\varepsilon,u}, \nabla\varphi\rangle\Big].
\end{aligned} \tag{3.6}$$

Furthermore, there exists a constant $C > 0$ independent of u and ε, such that

$$\sup_{t\le T}\int_{\mathbb{R}^n} dx\, u(t,x)^2 + \varepsilon\int_{[0,T]\times\mathbb{R}^n} ds\,dx\,|\nabla u(s,x)|^2 \le C\big[H_\varepsilon(u) + 1\big]. \tag{3.7}$$

Proof. Let ε, u, η and φ be as in the statement. Set

$$\theta := -f(u) + \frac{\varepsilon}{2} D(u)\,\nabla u - \sigma(u)\,\nabla \Psi^{\varepsilon,u} \in L_{2,\mathrm{loc}}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right).$$

Recall also that the linear functional ℓ_ε^u on $\mathcal{D}^1_{\sigma(u)}$ is defined as the extension of (3.3). By (3.1) it follows that

$$u_t = \nabla \cdot \theta \qquad \text{weakly.}$$

Since $u \in C\left([0,T]; L_2(\mathbb{R}^n)\right)$ and $\nabla u \in L_{2,\mathrm{loc}}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right)$, integrations by parts are allowed in the first line on the right hand side of (3.3), namely for each measurable compactly supported $\phi : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ with $\nabla \phi \in L_2\big([0,T] \times \mathbb{R}^n; \mathbb{R}^n\big)$

$$\ell_\varepsilon^u(\phi) = \langle\langle u_t, \phi \rangle\rangle + \langle\langle \nabla \cdot f(u), \phi \rangle\rangle + \frac{\varepsilon}{2} \langle\langle D(u) \nabla u, \nabla \phi \rangle\rangle, \tag{3.8}$$

where indeed we understand $\langle\langle u_t, \phi \rangle\rangle \equiv -\langle\langle \theta, \nabla \phi \rangle\rangle$. Since $\eta \in C^2_{\mathrm{b}}(\mathbb{R})$ and $\nabla u \in L_{2,\mathrm{loc}}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^n\right)$, $\eta'(u)\,\varphi$ has compact support and its weak gradient is square integrable. One can thus evaluate (3.8) with

$$\phi := \eta'(u)\varphi,$$

to obtain

$$\begin{aligned} &\langle\langle u_t, \eta'(u)\varphi \rangle\rangle + \langle\langle \nabla \cdot f(u), \eta'(u)\varphi \rangle\rangle + \frac{\varepsilon}{2} \langle\langle D(u) \nabla u, \nabla(\eta'(u)\varphi) \rangle\rangle \\ &\qquad = \ell_\varepsilon^u(\phi) = \big(\Psi^{\varepsilon,u}, \eta'(u)\varphi\big)_{\mathcal{D}^1_{\sigma(u)}}, \end{aligned}$$

where in the last equality we use (3.5). Integrations by parts are still allowed by the regularity of u, and thus (3.6) follows.
To prove (3.7), let (η, q) be an entropy – entropy flux pair, and $\varphi \in C_{\mathrm{c}}^\infty\left([0,T] \times \mathbb{R}^n\right)$. By (3.4), (3.5) and (3.8) it follows

$$\begin{aligned} I_\varepsilon(u) &\geq \ell_\varepsilon^u(\varepsilon\, \eta'(u)\, \varphi) - \frac{\varepsilon^2}{2} \big\langle\big\langle \nabla\big(\eta'(u)\varphi\big), \sigma(u) \nabla\big(\eta'(u)\varphi\big) \big\rangle\big\rangle \\ &= \varepsilon \langle \eta(u(T)), \varphi(T) \rangle - \varepsilon \langle \eta(u_0), \varphi(0) \rangle - \varepsilon \big[\langle\langle \eta(u), \varphi_t \rangle\rangle + \langle\langle q(u), \nabla \varphi \rangle\rangle \big] \\ &\quad + \frac{\varepsilon^2}{2} \Big[\langle\langle \eta''(u) D(u) \nabla u, \varphi \nabla u \rangle\rangle + \langle\langle \eta'(u) D(u) \nabla u, \nabla \varphi \rangle\rangle \\ &\qquad - \langle\langle \sigma(u) \eta''(u)^2 \nabla u, \varphi^2 \nabla u \rangle\rangle - \langle\langle \sigma(u) \eta'(u)^2 \nabla \varphi, \nabla \varphi \rangle\rangle \\ &\qquad - 2 \langle\langle \sigma(u) \eta''(u) \eta'(u) \nabla u, \varphi\, \nabla \varphi \rangle\rangle \Big]. \end{aligned} \tag{3.9}$$

On the other hand

$$\begin{aligned} &2 \big| \langle\langle \sigma(u) \eta''(u) \eta'(u) \nabla u, \varphi\, \nabla \varphi \rangle\rangle \big| \\ &\qquad \leq \langle\langle \sigma(u) \eta''(u)^2 \nabla u, \varphi^2 \nabla u \rangle\rangle + \langle\langle \sigma(u) \eta'(u)^2 \nabla \varphi, \nabla \varphi \rangle\rangle. \end{aligned}$$

Hence if η is such that the two matrices σ and D satisfy

$$0 \leq \sigma \eta'' \leq D, \tag{3.10}$$

formula (3.9) yields, recalling (2.3),

$$
\begin{aligned}
&\langle \eta(u(T)), \varphi(T)\rangle - \langle\langle \eta(u), \varphi_t\rangle\rangle + \frac{\varepsilon}{2}\langle\langle \eta''(u)D(u)\nabla u, (\varphi - 2\varphi^2)\nabla u\rangle\rangle \\
&\qquad \le H_\varepsilon(u) + \langle \eta(u_0), \varphi(0)\rangle + \langle\langle q(u), \nabla\varphi\rangle\rangle \\
&\qquad\quad - \frac{\varepsilon}{2}\langle\langle \eta'(u)D(u)\nabla u, \nabla\varphi\rangle\rangle + \varepsilon\langle\langle \sigma(u)\eta'(u)^2\nabla\varphi, \nabla\varphi\rangle\rangle.
\end{aligned}
\tag{3.11}
$$

Choose now

$$\eta(v) = cv^2$$

for a constant $c > 0$ such that (3.10) holds (such a constant exists since σ is bounded and D is uniformly elliptic). Choose also

$$\varphi(s, x) = \alpha(s)\phi(x)$$

for some $\alpha \in C^\infty([0, T])$ and $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ such that $0 \le \phi \le 1/4$ (in particular $2\phi^2 \le \phi/2$). Fix $t \in [0, T]$. As one lets α converge to the characteristic function $1\!\!1_{[0,t]}$ of $[0, t]$, (3.11) yields

$$
\begin{aligned}
&\langle u(t)^2, \phi\rangle + \varepsilon\int_0^t ds\, \langle D(u(s))\nabla u(s), \phi\nabla u(s)\rangle \\
&\qquad \le C\Big[H_\varepsilon(u) + \langle u_0^2, \phi\rangle + \int_0^t ds\, \Big(\langle q(u(s)), \nabla\phi\rangle \\
&\qquad\quad - \frac{\varepsilon}{2}\langle u(s)D(u(s))\nabla u(s), \nabla\phi\rangle + \varepsilon\langle \sigma(u(s))u(s)^2\nabla\phi, \nabla\phi\rangle\Big)\Big],
\end{aligned}
\tag{3.12}
$$

for a constant C independent of u and ε. Since f is Lipschitz, one can choose $C_1 > 0$ so that $|q(v)| \le C_1 v^2$ for any $v \in \mathbb{R}$. Moreover

$$-\frac{1}{2}\langle u\, D(u)\nabla u, \nabla\phi\rangle = \sum_{i,j=1}^n \langle \zeta_{ij}(u), \nabla_i\nabla_j\phi\rangle,$$

where ζ_{ij} is defined by $\zeta_{ij}(v) := \frac{1}{2}\int_0^v dw\, wD_{ij}(w)$, and thus there exists $C_2 > 0$ such that $|\zeta(v)| \le C_2 v^2$ for all $v \in \mathbb{R}$. Recalling again the assumptions on σ and D we then have

$$
\begin{aligned}
&\langle u(t)^2, \phi\rangle + \varepsilon\int_0^t ds\, \langle |\nabla u(s)|^2, \phi\rangle \\
&\qquad \le C\Big[H_\varepsilon(u) + \langle u_0^2, \phi\rangle + \int_0^t ds\, \langle u(s)^2, |\nabla\phi| + \varepsilon\sum_{i,j}|\nabla_i\nabla_j\phi| + \varepsilon|\nabla\phi|^2\rangle\Big],
\end{aligned}
\tag{3.13}
$$

for a possibly different constant $C > 0$. Now we observe that given $L > 0$ one can find $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n; [0, 1/4])$ such that $\phi(x) = 1/4$ for $|x| \le L$, and

$$|\nabla\phi| + \varepsilon|\nabla\phi|^2 + \varepsilon\sum_{i,j}|\nabla_i\nabla_j\phi| \le 1.$$

Therefore as one lets $L \to +\infty$ (3.13) yields (with the same C as in (3.13))

$$\begin{aligned}&\|u(t)\|^2_{L_2(\mathbb{R}^n)} + \varepsilon \int_0^t ds\, \|\nabla u(s)\|^2_{L_2(\mathbb{R}^n)} \\ &\quad \le 4\,C\Big[H_\varepsilon(u) + \|u_0\|^2_{L_2(\mathbb{R}^n)} + \int_0^t ds\, \|u(s)\|^2_{L_2(\mathbb{R}^n)}\Big],\end{aligned}$$

and (3.7) follows by Gronwall's inequality. □

4. Γ-convergence of H_ε

We denote by $C^{2,\infty}_{\mathrm{b,c}}\big(\mathbb{R}\times[0,T)\times\mathbb{R}^n\big)$ the space of bounded functions $\vartheta(v,t,x)$, twice boundedly differentiable in the v variable, and infinitely differentiable (up to the boundary) and compactly supported in the $(t,x) \in [0,T)\times\mathbb{R}^n$ variables. We denote by ϑ' the derivative of ϑ with respect to the v variable. We say that a function $\vartheta \in C^{2,\infty}_{\mathrm{b,c}}\big(\mathbb{R}\times[0,T)\times\mathbb{R}^n\big)$ is an *entropy sample*, and its *conjugated entropy flux sampler* $Q : \mathbb{R}\times[0,T)\times\mathbb{R}^n \to \mathbb{R}^n$ is defined up to an additive function of (t,x) by

$$Q(v,t,x) := \int^v dw\, \vartheta'(w,t,x) f'(w).$$

If $u \in \mathfrak{U}$, we understand $\partial_t\vartheta(u(t,x),t,x)$ as the partial derivative with respect to the time variable of ϑ, and $D_x\vartheta(u(t,x),t,x)$ as the partial gradient with respect to the space variable. We also introduce the short hand notation $\vartheta(u)(t,x) \equiv \vartheta(u(t,x),t,x)$. So for instance $(\nabla\vartheta(u))(t,x) = \vartheta'(u(t,x),t,x)\nabla u(t,x) + D_x\vartheta(u(t,x),t,x)$. A similar notation holds for Q: in this case $D_x \cdot Q(u(t,x),t,x)$ stands for the partial divergence of Q with respect to x.
For (ϑ, Q) an entropy sampler – conjugated entropy flux sampler pair and $u \in \mathfrak{U}$ we let

$$P_{\vartheta,u} := -\int dx\, \theta(u_0(x),0,x) - \int dt\, dx\, \Big[\big(\partial_t\vartheta\big)\big(u(t,x),t,x\big) + \big(D_x\cdot Q\big)\big(u(t,x),t,x\big)\Big]. \tag{4.1}$$

Notice that if u is a weak solution to (1.1) and $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some entropy η and some $\varphi \in C^\infty_{\mathrm{c}}\big((0,T)\times\mathbb{R}^n\big)$, then

$$P_{\vartheta,u} = \wp_{\eta,u}(\varphi).$$

Lemma 4.1 *Assume $u \in \mathfrak{U}$ is such that $t \mapsto \langle u(t), \varphi\rangle$ is continuous for all $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, and $u(0) = u_0$ in this distributional sense. Then the functional H in (2.6) has the representation*

$$H(u) = \sup_\vartheta P_{\vartheta,u} \tag{4.2}$$

where the supremum is taken over the entropy samplers ϑ such that

$$0 \le \vartheta''(v,t,x)\chi(v) \le 1 \qquad \text{for all } (v,t,x) \in \mathbb{R}\times[0,T]\times\mathbb{R}^n. \tag{4.3}$$

Proof. If u is not a weak solution to (1.1), then take $\vartheta(v,t,x) = v\varphi(t,x)$ for an arbitrary $\varphi \in C_c^\infty\big([0,T)\times\mathbb{R}^n\big)$. It is immediate to check that

$$\sup_{\varphi\in C_c^\infty([0,T)\times\mathbb{R}^n)} P_{v\varphi,u} = +\infty,$$

and a fortiori $\sup_\vartheta P_{\vartheta,u} = +\infty$. Hence (4.2) is valid.
If u is a weak solution to (1.1), but not an entropy measure solution, namely $u \notin \mathcal{E}$, then there exists an entropy $\eta \in C_b^2(\mathbb{R})$ such that $\wp_{\eta,u}$ is not a Radon measure. Therefore, taking ϑ of the form $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some $\varphi \in C_c^\infty\big((0,T)\times\mathbb{R}^n\big)$, we have

$$\sup_{\varphi\in C_c^\infty((0,T)\times\mathbb{R}^n)} P_{\eta\varphi,u} = \sup_{\varphi\in C_c^\infty((0,T)\times\mathbb{R}^n)} \wp_{\eta,u}(\varphi) = +\infty,$$

since $\wp_{\eta,u}$ is not a Radon measure. Hence (4.2) is still valid.
Eventually, assume that $u \in \mathcal{E}_{u_0}$. Fix an entropy sampler ϑ, and take sequences $\{\eta^i\}$ and $\{\varphi^i\}$ of entropies η^i and functions $\varphi^i \in C_c^\infty\big([0,T)\times\mathbb{R}^n\big)$ such that

$$\sum_{i=1}^m \eta^i(v)\varphi^i(t,x) \to \vartheta(v,t,x) \qquad \text{as } m \to +\infty$$

locally in $C^2(\mathbb{R}\times[0,T)\times\mathbb{R}^n)$. Then

$$\lim_{m\to+\infty}\sum_{i=1}^m \Big(-\langle \eta^i(u_0), \varphi^i(0)\rangle - \langle\langle \eta^i(u), \varphi^i_t\rangle\rangle - \langle\langle q^i(u), \nabla\varphi^i\rangle\rangle \Big) = P_{\vartheta,u}.$$

Thus (2.4) and (2.5) yield

$$P_{\vartheta,u} = \int dv\, \varrho_u(v;dt,dx)\, \vartheta''(v,t,x). \tag{4.4}$$

Using (4.3) and recalling the definition (2.6) of H, (4.2) follows. □

Proof of Proposition 2.2. The proof follows [1, Proposition 2.6] up to minor changes. Let $u \in \mathfrak{U}$ and let $\{u_m\} \subset \mathcal{E}_{u_0}$ be a sequence converging to u in $\mathfrak{U}$. For each $\varphi \in C_c^\infty(\mathbb{R}^n)$ and L such that B_L contains the support of φ, $f(u_m)$ is uniformly bounded in $L_2\left([0,T]\times B_L\right)$ and thus

$$|\langle u_m(t) - u_m(s), \varphi\rangle| \le C_{f,L}\sqrt{|t-s|}\|\nabla\varphi\|_{L_2(\mathbb{R}^n)},$$

for any $s,t \in [0,T]$, where $C_{f,L} \ge 0$ depends on f and L and is independent of m. Since $u_m(0) = u_0$, by the Ascoli-Arzelà theorem it follows that the sequence $\{u_m\}$ is compact in $C\left([0,T], H_{-1,\mathrm{loc}}(\mathbb{R}^n)\right)$, and thus $u \in C\left([0,T], H_{-1,\mathrm{loc}}(\mathbb{R}^n)\right)$ satisfies $u(0) = u_0$. Since the set of weak solutions is closed with respect to convergence in $\mathfrak{U}$, and $P_{\vartheta,u_m} \to P_{\vartheta,u}$ for all entropy samplers ϑ, the lower semicontinuity of H follows from Lemma 4.1. □

Lemma 4.2 *Let $\{u^\varepsilon\} \subset \mathfrak{U}$ be such that $H_\varepsilon(u^\varepsilon)$ is bounded uniformly with respect to ε. Then $\{u^\varepsilon\}$ is compact in $C\left([0,T]; H_{-1,\mathrm{loc}}(\mathbb{R}^n)\right)$.*

Proof. Let $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^n)$ and let L be large enough such that the support of φ is contained in B_L. By (3.1) we have, for $t\in[0,T]$,

$$
\begin{aligned}
\big|\langle u^\varepsilon(t)-u^\varepsilon(s),\varphi\rangle\big| \le & \int_s^t ds\ \big|\langle f(u^\varepsilon),\nabla\varphi\rangle\big| \\
& + \frac{\varepsilon}{2}\int_s^t ds\ \big|\langle D(u^\varepsilon)\nabla u^\varepsilon,\nabla\varphi\rangle\big| + \int_s^t ds\ \big|\langle \sigma(u^\varepsilon)\nabla\Psi^{\varepsilon,u^\varepsilon},\nabla\varphi\rangle\big|.
\end{aligned}
$$

Since f is Lipschitz, $f(u^\varepsilon)$ is bounded in $L_2\left([0,T]\times B_L\right)$ uniformly in ε. Recalling (3.7), $\varepsilon\, D(u^\varepsilon)\nabla u^\varepsilon$ is bounded in $L_2\left([0,T]\times B_L\right)$ uniformly in ε. Finally, since σ is bounded,

$$
\|\sigma(u^\varepsilon)\nabla\Psi^{\varepsilon,u^\varepsilon}\|^2_{L_2([0,T]\times\mathbb{R}^n)} \le C_\sigma\,\varepsilon\, H_\varepsilon(u^\varepsilon)
$$

is also bounded uniformly in ε. Therefore, for a suitable $C\ge 0$ independent of ε

$$
\big|\langle u^\varepsilon(t)-u^\varepsilon(s),\varphi\rangle\big| \le C\sqrt{|t-s|}\,\|\nabla\varphi\|_{L_2([0,T]\times\mathbb{R}^n)}.
$$

Since $u^\varepsilon(0)=u_0$, the stated compactness is then a consequence of the Ascoli-Arzelà theorem. $\square$

Proof of Theorem 2.1. Let $\{u^\varepsilon\}$ be a sequence converging to u in $\mathfrak{U}$. We can suppose without loss of generality that $H_\varepsilon(u^\varepsilon)$ is uniformly bounded with respect to ε. Lemma 4.2 implies that, if φ is a smooth test function, then the map $t\mapsto\langle u(t),\varphi\rangle$ is continuous and $u(0)=u_0$.
By Lemma 4.1, it is then enough to prove that for each entropy sampler ϑ such that

$$
0\le\sigma(v)\vartheta''(v,t,x)\le D(v), \qquad (v,t,x)\in\mathbb{R}\times(0,T)\times\mathbb{R}^n \tag{4.5}
$$

one has

$$
\varliminf_{\varepsilon} H_\varepsilon(u^\varepsilon)\ge P_{\vartheta,u}. \tag{4.6}
$$

We introduce the short hand notation $\big(\vartheta'(u^\varepsilon)\big)(t,x)\equiv\vartheta'(u^\varepsilon(t,x),t,x)$, $\big(\vartheta''(u^\varepsilon)\big)(t,x)\equiv\vartheta''(u^\varepsilon(t,x),t,x)$, $\big((D_x\vartheta')(u^\varepsilon)\big)(t,x)\equiv\big(D_x\vartheta'\big)(u^\varepsilon(t,x),t,x)$.
Let

$$
\varphi^\varepsilon(t,x)=\varepsilon\vartheta'(u^\varepsilon)(t,x).
$$

As we assumed $H_\varepsilon(u^\varepsilon)<+\infty$, ∇u^ε is locally square integrable, and since $\vartheta(u^\varepsilon)$ is compactly supported we have

$$
\nabla\varphi^\varepsilon=\varepsilon\vartheta''(u^\varepsilon)\,\nabla u^\varepsilon+\varepsilon(D_x\vartheta')(u^\varepsilon)\in L_2\left([0,T]\times\mathbb{R}^n\right).
$$

The representation (3.8) of $\ell_\varepsilon^{u^\varepsilon}(\varphi^\varepsilon)$ thus holds, and recalling (4.1) we get, taking Q

an entropy flux sampler conjugated to ϑ,

$$
\begin{aligned}
H_\varepsilon(u^\varepsilon) &\geq \varepsilon^{-1}\ell_\varepsilon^{u^\varepsilon}(\varphi^\varepsilon) - \frac{\varepsilon^{-1}}{2}\|\varphi^\varepsilon\|^2_{\mathcal{D}^1_{\sigma(u^\varepsilon)}} \\
&= \langle\langle u_t^\varepsilon, \vartheta'(u^\varepsilon)\rangle\rangle + \langle\langle \nabla\cdot f(u^\varepsilon), \vartheta'(u^\varepsilon)\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)\nabla u^\varepsilon, \vartheta''(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\
&\quad + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)\nabla u^\varepsilon, \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)\nabla u^\varepsilon, \vartheta''(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \\
&\quad - \varepsilon\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)\nabla u^\varepsilon, \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\big(D_x\vartheta'\big)(u^\varepsilon), \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle \\
&= -\int dx\, \theta(u_0(x),0,x) - \int dt\, dx\, \Big[\big(\partial_t\vartheta\big)\big(u^\varepsilon(t,x),t,x\big) + \big(D_x\cdot Q\big)\big(u^\varepsilon(t,x),t,x\big)\Big] \\
&\quad + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon), \vartheta''(u^\varepsilon)|\nabla u^\varepsilon|^2\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)\nabla u^\varepsilon, \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle \\
&\quad - \varepsilon\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)\nabla u^\varepsilon, \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\big(D_x\vartheta'\big)(u^\varepsilon), \big(D_x\vartheta'\big)(u^\varepsilon)\rangle\rangle.
\end{aligned}
$$

By the bound (3.7), the last three terms in the above formula vanish as $\varepsilon \to 0$, while (4.5) implies

$$\langle\langle [D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)]\nabla u^\varepsilon, \vartheta''(u^\varepsilon)\nabla u^\varepsilon\rangle\rangle \geq 0$$

Therefore, taking the limit as $\varepsilon \to 0$, we obtain (4.6). □

Proof of Theorem 2.2. Take a sequence $\{u^\varepsilon\} \subset \mathfrak{U}$ such that $H_\varepsilon(u^\varepsilon)$ is bounded uniformly in ε. We need to prove that $\{u^\varepsilon\}$ is compact in $\mathfrak{U}$.
For each $\varepsilon > 0$, u^ε satisfies (3.1) for a suitable $\Psi^\varepsilon \equiv \Psi^{\varepsilon,u^\varepsilon} \in \mathcal{D}^1_{\sigma(u^\varepsilon)}$. Following [8], let

$$\bar{\chi}(v,w) := \begin{cases} +1 & \text{if } 0 < v < w, \\ -1 & \text{if } w < v < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and consider the equation in the variables (v,t,x)

$$b_t + f'(v)\cdot\nabla b - \frac{\varepsilon}{2}\nabla\cdot\big(D(v)\nabla b\big) = m_0^\varepsilon + \frac{d}{dv}m_1^\varepsilon \tag{4.7}$$

where

$$m_0^\varepsilon(dv,t,x) := \sigma(v)\nabla\Psi^\varepsilon\delta_{u(t,x)=v}$$

and

$$m_1^\varepsilon(v,t,x) := \Big[\frac{\varepsilon}{2}(D(v)\nabla u^\varepsilon\cdot\nabla u^\varepsilon) + \sigma(v)\nabla\Psi^\varepsilon\cdot\nabla u^\varepsilon\Big]\delta_{u(t,x)=v}$$

are Radon measures on $\mathbb{R}\times[0,T]\times\mathbb{R}^n$, where δ denotes the Dirac delta.
Define $b^\varepsilon : \mathbb{R}\times[0,T]\times\mathbb{R}^n \to \mathbb{R}$ as

$$b^\varepsilon(v,t,x) := \bar{\chi}(v,u^\varepsilon(t,x)).$$

Note that (3.6) states that b^ε is a weak solution to the (linear) equation (4.7). Indeed, let η be an entropy and $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^n)$ a smooth function. Evaluate the left hand side of (4.7) for $b = b^\varepsilon$ and multiply it by $\eta'(v)\varphi(t,x)$. Equation (3.6) is then obtained integrating by parts. On the other hand, checking (3.6) againt test functions of the form $\eta'(v)\varphi(t,x)$ is equivalent to checking it against general test functions $\vartheta(v,t,x)$. The theory developed in [8, Chap. 5.2-5.3-5.4] proves that, if

- $D(v) \equiv 0$ in (4.7),
- $b^\varepsilon = \bar{\chi}(v, u^\varepsilon(t,x))$ is a solution to (4.7),
- $u^\varepsilon \in C([0,T]; L_2(\mathbb{R}^n))$ is bounded in $\mathfrak{U}$ uniformly in ε,
- the nondegeneracy condition (2.7) holds,
- the total variations $|m_0^\varepsilon|$ and $|m_1^\varepsilon|$ of m_0^ε and m_1^ε are bounded on $\mathbb{R} \times [0,T] \times \mathbb{R}^n$ uniformly in ε,

then $\{u^\varepsilon\}$ is compact in $\mathfrak{U}$. It is immediate to extend these results to the case $D(v) \geq 0$. Indeed [8, Lemma 5.3.1], and thus the whole technique, goes through by replacing the function $\eta \mapsto \beta|\eta|^2$ in [8, Lemma 5.3.1] with $\eta \mapsto \beta|\eta|^2 + \frac{\varepsilon}{2} D(v)\eta \cdot \eta$.
Recalling that we are assuming (2.7), the only condition that we need to check is the last one, namely the boundedness of the total variations of the measures m_0^ε and m_1^ε. First notice that $H_\varepsilon(u^\varepsilon) < +\infty$ implies $u^\varepsilon \in C([0,T]; L_2(\mathbb{R}^n))$. Since $H_\varepsilon(u^\varepsilon)$ is bounded uniformly in ε, estimate (3.7) implies that $\{u^\varepsilon\}$ is bounded in $\mathfrak{U}$. Notice that by Cauchy-Schwartz inequality

$$|m_0^\varepsilon|(\mathbb{R} \times [0,T] \times \mathbb{R}^n) = \int dt\, dx\, |\sigma(u(t,x)) \nabla \Psi^\varepsilon(t,x)| \leq C_\sigma \varepsilon H_\varepsilon,$$

where $C_\sigma := \sup_{v \in \mathbb{R}} \sigma(v) < +\infty$. Moreover by Lemma 3.2 and Cauchy-Schwartz inequality

$$\begin{aligned} |m_1^\varepsilon|(\mathbb{R} \times [0,T] \times \mathbb{R}^n) = \int dt \int_{\mathbb{R}^n} dx\, \Big| \frac{\varepsilon}{2} (D(u^\varepsilon(t,x)) \nabla u^\varepsilon(t,x) \cdot \nabla u^\varepsilon(t,x) \\ + \sigma(u^\varepsilon(t,x)) \nabla \Psi^\varepsilon(t,x) \cdot \nabla u^\varepsilon(t,x) \Big| \\ \leq C(1 + H_\varepsilon(u^\varepsilon)). \end{aligned}$$

Since $H_\varepsilon(u^\varepsilon)$ is bounded uniformly in ε, $|m_0^\varepsilon|$ and $|m_1^\varepsilon|$ are therefore bounded uniformly in ε. Compactness of $\{u^\varepsilon\}$ in $\mathfrak{U}$ then follows as in [8, Theorem 5.4.1]. □

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