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A note on cyclic separability of groups*

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Abstract

We give a proof that in certain classes of groups residual finiteness and cyclic subgroup separability are equivalent properties.

1. Introduction

Definition 1.1 1. A group G is called cyclic subgroup separable (or π_c for short) if for each cyclic subgroup H and $x \in G - H$, there exists a normal subgroup N of finite index in G such that $x \in G - HN$. Clearly a cyclic subgroup separable group is residually finite.

2. A group G is called subgroup separable (or LERF for short) if for each finitely generated subgroup H and $x \in G - H$, there exists a normal subgroup N of finite index in G such that $x \in G - HN$. Clearly a subgroup separable group is cyclic subgroup separable.

There is another equivalent definition of subgroup separability:

Definition 1.2 A group G is called subgroup separable or locally extended residually finite (LERF) if every finitely generated subgroup of G is closed in the profinite topology, the topology whose open basis consists of the cosets of finite index subgroups of G.

The concept of cyclic subgroup separability was introduced by Stebe [2] in 1968 and he used it to prove the residual finiteness of a class of knot groups. Many classes of groups, including the free groups and the polycyclic-by-finite groups are cyclic subgroup separable [3]. Moreover, a finite extension of a cyclic subgroup separable group is again cyclic subgroup separable (Stebe [2]). On the other hand, not much is known about the cyclic subgroup separability of HNN-extensions. For example, the HNN-extension $< a, t \mid tat^{-1} = a^2 >$ is residually finite but not cyclic subgroup separable (see [4]) while another HNN-extension, the Baumslag-Solitar group, $< t, a \mid ta^2t^{-1} = a^3 >$ is not even residually finite (see [5]). Kim [6] and Kim and Tang [7] give characterisations for HNN-extensions of cyclic subgroup separable groups with cyclic associated subgroups to be again cyclic subgroup separable. They then apply their results to give characterisations for the HNN-extensions of a finitely

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generated abelian group with cyclic associated subgroups to be cyclic subgroup separable and show that certain HNN-extensions of finitely generated torsion-free nilpotent groups with cyclic associated subgroups are again cyclic subgroup separable. An example in the literature is the family BN of all groups with an HNN presentation of the form $< t, K|tKt^{-1} = A >$ where A is a proper subgroup of K . Such an HNN- extension is called *ascending*. The above groups are not subgroup separable by the results of Blass and Neumann in [8]. Notice that the BN groups include the residually finite, non-subgroup separable Baumslag-Solitar groups, namely the groups $BS_{n,m} =< x, a|xa^n x^{-1} = a^m >$ with either $\mid n \mid = 1$ or $\mid m \mid = 1$ and $\mid n \mid \neq \mid m \mid$. Shalen showed [9] that non-subgroup separable Baumslag-Solitar groups cannot be subgroups of 3-manifold groups.

In this note, we give a proof that in certain classes of groups, residual finiteness and cyclic subgroup separability are equivalent properties.

2. The theorem

Theorem 2.1 *Let G be the fundamental group of a graph of cyclic groups without ascending HNN-extension subgroups. The following are equivalent:*

1 G is a cyclic separable group

2 G is a residually finite group

Proof. If G is a cyclic separable group then it is a residually finite group, because the trivial subgroup is closed in the profinite topology see definitions 1.

Let now G be a residually finite group and K be a cyclic subgroup of G. If K is a finite group the proof is obvious from the definition of the profinite topology. In general, every finite subgroup in a residually finite group G is closed in the profinite topology. Let K be an infinite cyclic subgroup. We consider M a maximal abelian subgroup of G that contains K. From the hypothesis and the structure theorem of groups acting on trees, M is either a subgroup of some conjugate of a vertex group or else it is infinite cyclic. It follows in all cases that M is an infinite cyclic group.

Let $\overline{M}$ be the closure of M in the profinite topology and $x, y \in \overline{M}$. Then $x = m_1 n_1, y = m_2 n_2, m_1, m_2 \in M, n_1, n_2 \in N, N \lhd_f G$ because $\overline{M} = \bigcap_{N \lhd_f G} MN$. Let $h = [x, y]$ be the commutator of x, y. It is easy to see that $hN = [m_1 n_1, m_2 n_2]N = N$, so the commutator $[x, y] \in N$, for every normal subgroup of finite index in G. But G is a residually finite group. It follows that $[x, y] = 1$, and the group $\overline{M}$ is an abelian group. We conclude that $\overline{M} = M$, because M is a maximal abelian group. See also the work of D.Long [1].

Since M is an infinite cyclic group, K is a subgroup of finite index in M. It follows that K is closed in the profinite topology of G.

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