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Construction of Smooth Fractal Surfaces Using Hermite Fractal Interpolation Functions

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Abstract

This paper approaches the Hermite interpolation problem using fractal interpolation procedures. We generalise some theorems provided by Barnsley and others regarding the differentiability of fractal interpolation functions, when recurrent iterated function systems are used. In addition, we generalise the construction given by Navascués and Sebastián in [11] and we provide a construction of smooth (C^1) fractal surfaces using C^1 Hermite Fractal Interpolation Functions.

Keywords: Fractals, recurrent iterated function systems, fractal interpolation, fractal dimension

1. Introduction

The most commonly used technique for the construction of fractal sets, defined in a complete metric space X, is the use of Iterated Function Systems (IFS). IFS are sets of contractive mappings that take effect on $\mathcal{H}(X)$, the space containing all the non-empty compact subsets of X equipped with the Hausdorff metric h. This method was introduced by Barnsley and Demko in [3] and generalised by Barnsley, Elton and Hardin in [4] to include Recurrent Iterated Function Systems (RIFS), using stochastic matrices with probabilities. IFS and RIFS are able to produce very complicated sets using only a handful of mappings.

A special construction of IFS, presented in [5] and [4], produces continuous functions whose fractal dimension is greater than 1. These are called Fractal Interpolation Functions (FIF) and were used especially in signal processing. Recently it was shown that FIF generalise Hermite interpolation polynomials (see [11]). In spite of the fact that this construction was extensively studied (see for example [1], [2], [9]) it has remained restricted in the case of 1-dimensional data points. Several attempts were made to generalise the notion on $\mathbb{R}^2$, but only in very specific cases (see [10]). The most general construction involving Fractal Interpolation Surfaces was published recently (see [7]), but still it involves interpolation points that are confined in some manner.

In [8] FIF were used to construct Fractal Interpolation Surfaces on arbitrary points defined on grids. In this paper, our intention is to provide a similar construction that gives rise to smooth (i.e. C^1) surfaces.

2. Iterated Function Systems

An *Iterated Function System* $\{X; w_{1-N}\}$ is defined as a pair of a complete metric space (X, ρ) together with a finite set of continuous contractive mappings $w_i : X \to X$, with respective contraction factors s_i for $i = 1, 2, \ldots, N$ $(N \geq 2)$. The attractor of an IFS is the unique set E for which $E = \lim_{k\to\infty} W^k(A_0)$ for every starting compact set A_0, where

$$W(A) = \bigcup_{i=1}^{N} w_i(A) \text{ for all } A \in \mathcal{H}(X),$$

and $\mathcal{H}(X)$ is the complete metric space of all nonempty compact subsets of X with respect to the Hausdorff metric h (for the definition of the Hausdorff metric and properties of $\langle \mathcal{H}(X), h\rangle$ see [6]). A simple example of an IFS defined on $\mathbb{R}^2$ is the one that produces the well-known Sierpinski's Triangle (see figure 1(b)), which consists of the three mappings:

$$w_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad w_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, + \begin{pmatrix}0\\50\end{pmatrix}$$

$$w_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}50\\0\end{pmatrix}.$$

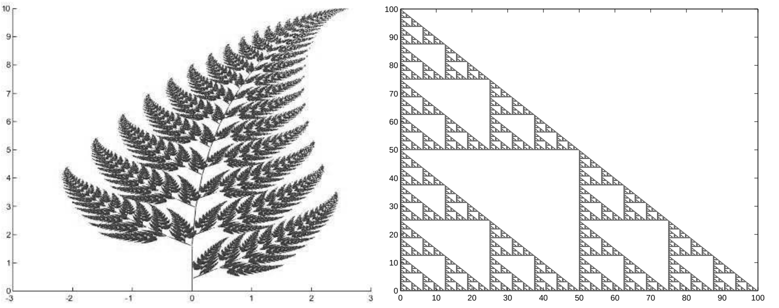
The attractor of the following IFS looks like a natural fern (see 1(a)).

$$w_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0.16\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}, \qquad w_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.85 & 0.04\\-0.04 & 0.85\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, + \begin{pmatrix}0\\1.6\end{pmatrix}$$

$$w_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.2 & -0.26\\0.23 & 0.22\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\1.6\end{pmatrix}, \qquad w_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-0.15 & 0.28\\0.26 & 0.24\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0.44\end{pmatrix}.$$

As we mentioned earlier, a more general concept, that allows the construction of even more complicated sets, is that of the *Recurrent Iterated Function System*, or *RIFS* for short, which consists of the IFS $\{X; w_{1-N}\}$ together with an irreducible row-stochastic matrix $P = (p_{\nu,\mu})^N$ $(p_{\nu,\mu} \in [0,1] : \ \nu, \mu = 1, \ldots, N)$, such that

$$\sum_{\mu=1}^{N} p_{\nu,\mu} = 1, \quad \nu = 1, \ldots, N. \tag{1}$$



(a) (b)

Figure 1: Two known attractors that arise from IFS. (a) a fern, (b) Sierpinski's Triangle.

The recurrent structure is given by the (irreducible) *connection matrix* $C = (c_{\nu,\mu})^N$ which is defined by

$$c_{\nu,\mu} = \begin{cases} 1, & \text{if } p_{\mu,\nu} > 0 \\ 0, & \text{if } p_{\mu,\nu} = 0 \end{cases},$$

where $\nu, \mu = 1, 2, \ldots, N$. The transition probability for a certain discrete time Markov process is $p_{\nu,\mu}$, which gives the probability of transfer into state μ given that the process is in state ν. Condition (1) says that whichever state the system is in (say ν), a set of probabilities is available that sum to one and describe the possible states to which the system transits at the next step.

In this case the construction of the contractive map $\boldsymbol{W}$ needs a little more effort. First, we define the mappings

$$W_{i,j} : \mathcal{H}(X) \to \mathcal{H}(X), \text{ with } W_{i,j}(A) = \begin{cases} w_i(A), & p_{j,i} > 0 \\ \emptyset, & p_{j,i} = 0 \end{cases}, \tag{2}$$

for all $A \in \mathcal{H}(X)$ and the metric space

$$\tilde{\mathcal{H}}(X) = \mathcal{H}(X)^N = \mathcal{H}(X) \times \mathcal{H}(X) \times \cdots \times \mathcal{H}(X)$$

equipped with the metric

$$\tilde{h}\left(\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}, \begin{pmatrix} B_1 \\ B2 \\ \vdots \\ B_N \end{pmatrix}\right) = \max\{h(A_i, B_i); i = 1, 2, \ldots, N\}.$$

Then $\langle \tilde{\mathcal{H}}(X), \tilde{h} \rangle$ is a complete metric space. The map $\boldsymbol{W}$ is now defined by

$$\boldsymbol{W} : \tilde{\mathcal{H}}(X) \to \tilde{\mathcal{H}}(X) : \boldsymbol{W} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1N} \\ W_{21} & W_{22} & \dots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{N1} & W_{N2} & \dots & W_{NN} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} \bigcup_{j \in I(1)} w_1(A_j) \\ \bigcup_{j \in I(2)} w_2(A_j) \\ \vdots \\ \bigcup_{j \in I(N)} w_N(A_j) \end{pmatrix},$$

where $I(i) = \{j : p_{j,i} > 0\}$, for $i = 1, 2, \dots, N$. If w_i are contractions, then $\boldsymbol{W}$ is a contraction and there is $\boldsymbol{E} = (E_1, E_2, \dots, E_N)^t \in \tilde{\mathcal{H}}(X)$ such that $\boldsymbol{W}(\boldsymbol{E}) = \boldsymbol{E}$ and $E_i = \bigcup_{j \in I(i)} w_i(E_j)$, for $i = 1, 2, \dots, N$.

Let $A \in \mathcal{H}(X)$. We define the sequences $\{\boldsymbol{A}_n\}_{n \in \mathbb{N}}$ in $\tilde{\mathcal{H}}(X)$ and $\{A_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}(X)$ as follows: $\boldsymbol{A}_0 = (A, A, \dots, A)^t$, $\boldsymbol{A}_n = \boldsymbol{W}(\boldsymbol{A}_{n-1})$ and $A_n = \bigcup_{i=1}^N (\boldsymbol{A}_n)_i$, for $n \in \mathbb{N}$, where $\boldsymbol{A}_n = ((\boldsymbol{A}_n)_1, (\boldsymbol{A}_n)_2, \dots, (\boldsymbol{A}_n)_N)$. Then, the set $G = \bigcup_{i=1}^N E_i$ is called the attractor of the RIFS $\{X;\ w_{1-N},\ P\}$. Evidently

$$G = \lim_{n \to \infty} A_n.$$

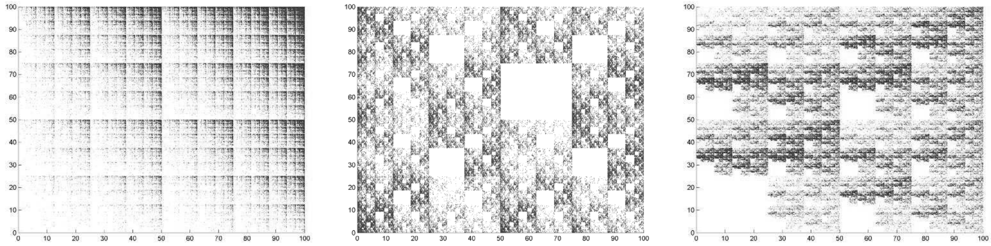
We emphasize that the attractor of a RIFS depends not only from the corresponding IFS, but also from the stochastic matrix. For example, the following IFS equipped with different stochastic matrices produces different attractors (see figure 2).

$$w_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad w_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 50 \end{pmatrix}, \tag{3}$$

$$w_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 50 \\ 0 \end{pmatrix}, \qquad w_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 50 \\ 50 \end{pmatrix}. \tag{4}$$

3. Fractal Interpolation Functions

In this section we briefly describe the construction of fractal interpolation functions based on RIFSs as we will use it in our method. Let $X = [0,1] \times \mathbb{R}$ and $\Delta = \{(x_i, y_i) : i = 0, 1, \dots, N\}$ be an interpolation set with N+1 interpolation points such that $0 = x_0 < x_1 < \dots < x_N = 1$. The interpolation points divide $[0,1]$ into N intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \dots, N$, which we call *domains*. In addition, let $\hat{\Delta} = \{(\hat{x}_j, \hat{y}_j) : j = 0, 1, \dots, M\}$ be a subset of Δ, such that $0 = \hat{x}_0 < \hat{x}_1 < \dots < \hat{x}_M = 1$.



(a) (b) (c)

$$P = \frac{1}{21}\begin{pmatrix} 1 & 4 & 4 & 12 \\ 1 & 4 & 4 & 12 \\ 1 & 4 & 4 & 12 \\ 1 & 4 & 4 & 12 \end{pmatrix} \qquad P = \frac{1}{10}\begin{pmatrix} 5 & 3 & 2 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad P = \frac{1}{15}\begin{pmatrix} 0 & 2 & 4 & 9 \\ 1 & 2 & 4 & 8 \\ 4 & 2 & 1 & 8 \\ 4 & 2 & 1 & 8 \end{pmatrix}$$

Figure 2: The attractors of the IFS defined by equations 3-4, equipped with various stochastic matrices.

We also assume that for every $j = 0, 1, \ldots, M-1$ there is at least one i such that $\hat{x}_j < x_i < \hat{x}_{j+1}$. Thus, the points of $\hat{\Delta}$ divide $[0,1]$ into M intervals $J_j = [\hat{x}_{j-1}, \hat{x}_j]$, $j = 1, \ldots, M$, which we call *regions*. Finally, let $\mathbb{J}$ be a labelling map such that $\mathbb{J}\colon \{1, 2, \ldots, N\} \to \{1, 2, \ldots, M\}$ with $\mathbb{J}(i) = j$. Let $x_i - x_{i-1} = \delta_i$, $i = 1, 2, \ldots, N$, and $\hat{x}_j - \hat{x}_{j-1} = \psi_j$, $j = 1, 2, \ldots, M$. It is evident that each region contains an integer number of domains. In the special case where the interpolation points are equidistant (that is $x_i - x_{i-1} = \delta$, $i = 1, 2, \ldots, N$, and $\hat{x}_j - \hat{x}_{j-1} = \psi$, $j = 1, 2, \ldots, M$), each region contains exactly $\alpha = \psi/\delta \in \mathbb{N}$ domains.

We define N mappings of the form:

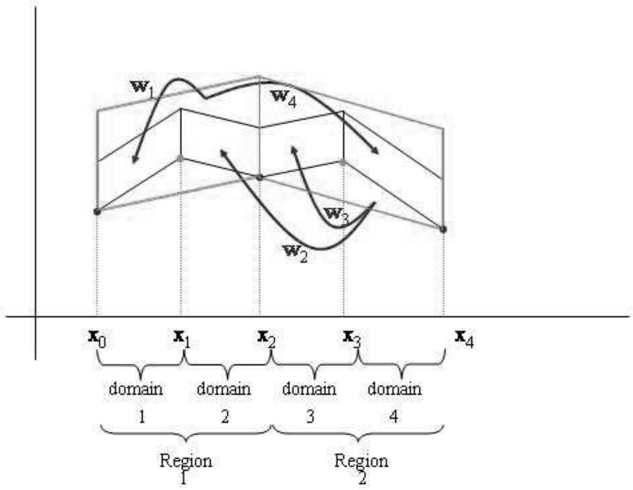
$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x,y) \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N, \tag{5}$$

where $L_i(x) = a_i x + b_i$, $F_i(x,y) = s_i y + q_i(x)$ and $q_i(x)$ is a polynomial. Each map w_i is constrained to map the endpoints of the region $J_{\mathbb{J}(i)}$ to the endpoints of the domain I_i (see figure 3). That is,

$$w_i \begin{pmatrix} \hat{x}_{j-1} \\ \hat{y}_{j-1} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \quad w_i \begin{pmatrix} \hat{x}_j \\ \hat{y}_j \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N. \tag{6}$$

Vertical segments are mapped to vertical segments scaled by the factor s_i. The parameter s_i is called the *vertical scaling factor* or the *contraction factor* of the map w_i.

It is easy to show that if $|s_i| < 1$, then there is a metric d equivalent to the Euclidean metric, such that w_i is a contraction (i.e., there is $\hat{s}_i : 0 \leq \hat{s}_i < 1$ such that



$$(p_{nm}) = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, \quad (c_{nm}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \text{V=(1, 2, 2, 1)}$$

Figure 3: In the above figure, the set Δ consists of five interpolation points, while the set $\hat{\Delta}$ consists of three points. The stochastic matrix, the connection matrix and the connection vector are shown below the figure.

$d(w_i(\vec{x}), w_i(\vec{y})) \leq \hat{s}_i d(\vec{x}, \vec{y})$, $\overrightarrow{x}$, $\overrightarrow{y} \in \mathbb{R}^2$, see [6]).

The $N \times N$ stochastic matrix $(p_{nm})^N$ is defined by the labelling function $\mathbb{J}$ as follows:

$$p_{nm} = \begin{cases} \frac{1}{\gamma_n}, & \text{if } I_n \subseteq J_{\mathbb{J}(m)} \\ 0, & \text{otherwise.} \end{cases},$$

where γ_n is the number of positive entries of the line n, $n = 1, 2, \ldots, N$.
This means that p_{nm} is positive, if the transformation L_m, maps the region containing the nth domain (i.e. I_n) to the mth domain (i.e. I_m). If we take a point in $I_n \times \mathbb{R}$, $n = 1, \ldots, N$, we say that we are in state n. The number p_{nm} shows the probability of applying the map w_m to that point, so that the system transits to state m. Sometimes, it is more efficient to describe the matrix P through the connection matrix C or the connection vector V, which are defined as follows:

$$c_{nm} = \begin{cases} 1, & p_{mn} > 0 \\ 0, & \text{otherwise} \end{cases},$$

$$V = (\mathbb{J}(1), \mathbb{J}(2), \ldots, \mathbb{J}(N)).$$

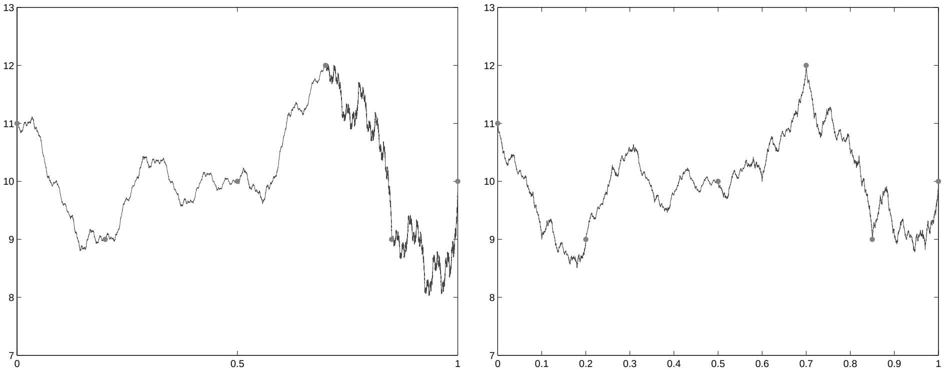


Figure 4: The two FIFs shown above interpolate the points of the same set Δ (consisting of six points). The difference is due to the selection of two distinct stochastic matrices.

Next, we consider $\langle C([x_0, x_N]), \|\cdot\|_\infty \rangle$, where $\|\phi\|_\infty = \max\{|\phi(x)|,\ x \in [x_0, x_N]\}$ and the complete metric subspace $\mathcal{F}_\Delta = \{g \in C([x_0, x_N]) : g(x_i) = y_i,\ \text{for } i = 0, 1, \ldots, N\}$. The Read-Bajraktarevic operator $T_{\Delta,\hat{\Delta}} : \mathcal{F}_\Delta \to \mathcal{F}_\Delta$ is defined as follows

$$(T_{\Delta,\hat{\Delta}} g)(x) = F_i\left(L_i^{-1}(x), g\left(L_i^{-1}(x)\right)\right), \qquad \text{for } x \in I_i, i = 1, 2, \ldots, N.$$

It is easy to verify that $T_{\Delta,\hat{\Delta}} g$ is well defined and that $T_{\Delta,\hat{\Delta}}$ is a contraction with respect to the $\|\cdot\|_\infty$ metric. According to the Banach fixed-point theorem, there exists a unique $f \in \mathcal{F}_\Delta$ such that $T_{\Delta,\hat{\Delta}} f = f$. If f_0 is any interpolation function and $f_n = T^n_{\Delta,\hat{\Delta}} f_0$, where $T^n_{\Delta,\hat{\Delta}} = T_{\Delta,\hat{\Delta}} \circ T_{\Delta,\hat{\Delta}} \circ \cdots \circ T_{\Delta,\hat{\Delta}}$, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. The graph of the function f is the attractor of the RIFS $\{X, w_{1-N}, (p_{ij})^N\}$ associated with the interpolation points (see [6]). Note that f interpolates the points of Δ for any selection of the parameters of the polynomials p_i that satisfies (6). We will refer to a function of this nature as *Recurrent Fractal Interpolation Function (RFIF)*. In the case where all the elements of the stochastic matrix are equal to 1 (i.e. we have an IFS instead of a RIFS), the function will be simply referred to as Fractal Interpolation Function (FIF). We emphasize that a RFIF satisfies

$$f(L_i(x)) = s_i f(x) + q_i(x) \tag{7}$$

for all $x \in I_i$, $i = 1, 2, \ldots, N$.

The most extensively studied case is that, where $q_i(x) = c_i x + f_i,\ x \in I_i$, which means that w_i are affine:

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x,y) \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N. \tag{8}$$

The FIF that corresponds to the above RIFS is called *affine* FIF.

From eq. (6) four linear equations arise, which can always be solved for $a_i,\ c_i,\ e_i,\ f_i$ in terms of the coordinates of the interpolation points and the vertical scaling factor s_i. Thus, once the vertical scaling factor s_i for each map has been chosen, the remaining parameters may be easily computed (see [6]).

3.1. **The integral of a RFIF**

In [2] it is shown that the integral of a FIF defined from an IFS is also a FIF defined from a different IFS. Here we will show that the same is true in the case of the RFIF. The following theorems are extensions of the ones presented in [2].

Proposition 3.1 *Let f be a RFIF constructed from the RIFS $\{\mathbb{R}^2;\ w_{1-N},\ P\}$, where*

$$w_i\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}L_i(x)\\ s_i y + q_i(x)\end{pmatrix},$$

with $|s_i| < a_i$, $i = 1,2,\ldots,N$, and associated with the interpolation points $\Delta = \{(x_i, y_i) : i = 0,1,\ldots,N\}$ the points $\hat{\Delta} = \{(\hat{x}_j, \hat{y}_j) : j = 0,1,\ldots,M\}$ and the labelling map $\mathbb{J}$ as given above. If

$$\tilde{f}(x) = \tilde{y}_0 + \int_{x_0}^{x} f(t)dt,$$

then the function $\tilde{f}$ is the RFIF constructed from the RIFS $\{\mathbb{R}^2;\ \tilde{w}_{1-N},\ P\}$, with

$$\tilde{w}_i\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}L_i(x)\\ \tilde{s}_i y + \tilde{q}_i(x)\end{pmatrix},$$

where $\tilde{s}_i = a_i \cdot s_i$,

$$\tilde{q}_i(x) = \tilde{y}_{i-1} - a_i s_i \hat{\tilde{y}}_{j-1} + a_i \int_{\hat{x}_{j-1}}^{x} q_i(t)dt$$

and P is defined by the function $\mathbb{J}$ as mentioned above. for $i = 1,2,\ldots,N$, $j = \mathbb{J}(i)$ and $\hat{\tilde{y}}_j = \tilde{y}_{\mathbb{I}(j)}$, where $\mathbb{I}(j) = i$ iff $x_i = \hat{x}_j$. The function $\tilde{f}$ is associated with the interpolation points $\tilde{\Delta} = \{(x_i, \tilde{y}_i) : i = 0,1,\ldots,N\}$ the points $\hat{\tilde{\Delta}} = \{(\hat{x}_j, \hat{\tilde{y}}_j) : j = 0,1,\ldots,M\}$ and the labelling map $\mathbb{J}$. The value $\tilde{y}_0$ is arbitrary, while the values $\tilde{y}_1,\ldots,\tilde{y}_N$ are computed as solutions of the linear system

$$\tilde{y}_i = \tilde{y}_{i-1} + a_i s_i(\hat{\tilde{y}}_j - \hat{\tilde{y}}_{j-1}) + a_i \int_{\hat{x}_{j-1}}^{\hat{x}_j} q_i(t),$$

$i = 1,2,\ldots,N$, $j = \mathbb{J}(i)$.

Proof.

$$\tilde{f}(L_i(x)) = \tilde{y}_0 + \int_{x_0}^{L_i(x)} f(t)dt = \tilde{y}_0 + \int_{x_0}^{x_{i-1}} f(t)dt + \int_{x_i-1}^{L_i(x)} f(t)dt$$
$$= \tilde{y}_{i-1} + \int_{x_{i-1}}^{L_i(x)} f(t)dt.$$

Substituting with $t = L_i(u)$, $dt = a_i du$ we have that

$$\tilde{f}(L_i(x)) = \tilde{y}_{i-1} + a_i \int_{\hat{x}_{j-1}}^{x} f(L_i(u))du, \ \textit{with}\ j = \mathbb{J}(i).$$

In addition, if we use the relation (7), we have that

$$\begin{aligned}\tilde{f}(L_i(x)) &= \tilde{y}_{i-1} + a_i s_i \int_{\hat{x}_{j-1}}^{x} f(t)dt + a_i \int_{\hat{x}_{j-1}}^{x} q_i(t)dt \\ &= \tilde{y}_{i-1} + a_i s_i \int_{x_0}^{x} f(t)dt - a_i s_i \int_{x_0}^{\hat{x}_{j-1}} f(t)dt + a_i \int_{\hat{x}_{j-1}}^{x} q_i(t)dt \\ &= \tilde{y}_{i-1} + a_i s_i(\tilde{f}(x) - \tilde{y}_0) - a_i s_i(\hat{\tilde{y}}_{j-1} - \tilde{y}_0) + a_i \int_{\hat{x}_{j-1}}^{x} q_i(t)dt\end{aligned}$$

$$\tilde{f}(L_i(x)) = a_i s_i \tilde{f}(x) + \tilde{y}_{i-1} - a_i s_i \hat{\tilde{y}}_{j-1} + a_i \int_{\hat{x}_{j-1}}^{x} q_i(t)dt, \tag{9}$$

for all $x \in I_i$. Thus $\tilde{f}$ is a RFIF as stated by the theorem. Substituting $x = \hat{x}_j$ in (9) we obtain the linear system

$$\tilde{y}_i = \tilde{y}_{i-1} + a_i s_i(\hat{\tilde{y}}_j - \hat{\tilde{y}}_{j-1}) + a_i \int_{\hat{x}_{j-1}}^{\hat{x}_j} q_i(t),$$

for $i = 1, 2, \ldots, N$, $j = \mathbb{J}(i)$. □

Corollary 3.1 *Let f, $\tilde{f}$ be two RFIF produced by the RIFS $\{\mathbb{R}^2;\ w_{1-N},\ P\}$ and $\{\mathbb{R}^2;\ \tilde{w}_{1-N},\ P\}$ respectively, where*

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ s_i y + q_i(x) \end{pmatrix}, \quad \tilde{w}_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ \tilde{s}_i y + \tilde{q}_i(x) \end{pmatrix},$$

with $|s_i| < 1$, $|\tilde{s}_i| < a_i$ for $i = 1, 2, \ldots, N$, and associated with the points of the sets Δ, $\hat{\Delta}$ and $\tilde{\Delta}$, $\hat{\tilde{\Delta}}$ and the labelling map $\mathbb{J}$ as described above. Then $\tilde{f}' = f$, if and only if

$$\tilde{s}_i = a_i s_i \quad \textit{and} \quad \tilde{q}_i'(x) = a_i q_i(x), \quad \textit{for all} \quad x \in I_i,\ i = 1, 2, \ldots, N. \tag{10}$$

Proof. The if part is immediate from the proposition 3.1. For the converse we have:

$$\begin{aligned}\tilde{f}(x_i) - \tilde{f}(x_{i-1}) &= \tilde{y}_i - \tilde{y}_{i-1} \\ &= \tilde{F}_i(\hat{x}_j, \hat{\tilde{y}}_j) - F_i(\hat{x}_{j-1}, \hat{\tilde{y}}_{j-1}) \\ &= a_i s_i \hat{\tilde{y}}_j + \tilde{q}_i(\hat{x}_j) - a_i s_i \hat{\tilde{y}}_{j-1} - \tilde{q}_i(\hat{x}_{j-1}) \\ &= a_i s_i (\hat{\tilde{y}}_j - \hat{\tilde{y}}_{j-1}) + a_i \int_{\hat{x}_{j-1}}^{\hat{x}_j} q_i(t)dt.\end{aligned}$$

Thus $\tilde{y}_i$, $\tilde{y}_{i-1}$, $i = 1, 2, \ldots, N$, satisfy the relations of proposition 3.1. Therefore $\tilde{f}'$ is the RFIF constructed from the RFIS $\{\mathbb{R}^2, \tilde{w}_{1-N}, P\}$. Due to (10), $\tilde{f}' = f$. □

For a function g we symbolize $g^{(0)} = g$ and $g^{(k)}$ as its k-th order derivative. In addition, we consider $C^n([0,1])$ as the space of the functions that have continuous n-th order derivative, equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(n)}\|_\infty$. The following theorem may be easily deduced from proposition 3.1.

Theorem 3.1 *Consider the RIFS $\{\mathbb{R}^2, w_{1-N}, P\}$, whose attractor is the graph of a RFIF associated with the data points Δ, $\hat{\Delta}$ and the labelling map $\mathbb{J}$, where*

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ s_i y + q_i(x) \end{pmatrix},$$

$|s_i| < a_i^n$, $q_i \in C^n([0,1])$, for $i = 1, 2, \ldots, N$. We define

$$F_{k,i}(x, y) = \frac{s_i y + q_i^{(k)}(x)}{a_i^k}.$$

If for any $k = 0, 1, \ldots, n$, each one of the $2N \times 2N$ linear systems

$$\begin{aligned} y_{k,i-1} &= F_{k,i}(\hat{x}_{j-1}, \hat{y}_{k,j-1}) = \frac{a_i \hat{y}_{k,j-1} + q_i^{(k)}(\hat{x}_{j-1})}{a_i^k}, \\ y_{k,i} &= F_{k,i}(\hat{x}_j, \hat{y}_{k,j}) = \frac{a_i \hat{y}_{k,j} + q_i^{(k)}(\hat{x}_j)}{a_i^k}, \end{aligned}$$

with $\hat{y}_{k,j} = y_{k,\mathbb{I}(j)}$, $\mathbb{I}(j) = i$ iff $\hat{x}_j = x_i$, $j = \mathbb{J}(i)$ $i = 1, 2, \ldots, N$, has a unique solution for $y_{k,i}$, then the RFIF $f \in C^n([0,1])$ and $f^{(k)}$ is the RFIF defined by the RIFS $\{\mathbb{R}^2;\ w_{k,1-N}, P\}$, where

$$w_{k,i} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_{k,i}(x, y) \end{pmatrix},$$

for $k = 1, \ldots, n$.

3.2. **Hermite Fractal Interpolation Functions**

Consider an interval $[a, b]$. The Hermite interpolation problem is the search for a polynomial q of degree $2p+1$ which satisfy

$$q^{(k)}(a) = a_k, \quad q^{(k)}(b) = b_k,$$

where a_k, b_k are given real numbers for $k = 0, 1, \ldots, p$. It has been proven that this problem has always a unique solution. The corresponding polynomial q is called Hermite interpolation polynomial. A more general definition is that of the Hermite interpolation function. Let $D = \{x_0 < x_1 < \cdots < x_N\}$ be a given partition of an interval $[x_0, x_N]$, where as usual $I_i = [x_{i-1}, x_i]$. Then the space of the Hermite interpolation functions of order p, denoted as $\mathcal{H}_D^{p+1}$ $(p \in \mathbb{N})$, is defined as follows

$$\mathcal{H}_D^{p+1} = \{\phi : [x_0, x_N] \to \mathbb{R};\ \phi \in C^p([x_0, x_N]),\ \phi|_{I_i} \in P_{2p+1},\ i = 1, 2, \ldots, N\},$$

where the space P_{2p+1} is composed from the polynomials of degree at most $2p+1$. In order to approximate a given function $y \in C^p([x_0, x_N])$ with a function $\phi \in \mathcal{H}_D^{p+1}$, it is sufficient to choose the polynomials $q_i = \phi|_{I_i}$ such that

$$q_i^{(k)}(x_{i-1}) = y^{(k)}(x_{i-1}), \quad p_i^{(k)}(x_i) = y^{(k)}(x_i).$$

The following theorem generalizes the concept of Hermite interpolation using RFIF. We note that in [11] Navascués and Sebastián gave a similar theorem using FIFs. Their work may be considered as a special case of theorem 3.2.

Theorem 3.2 *Consider $x_0 < x_1 < \cdots < x_N$ a partition of the interval $[x_0, x_N]$, $(N \in \mathbb{N},\ N \geq 2)$, and $\hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_M$, $(M \in \mathbb{N})$ a subset of it, such that $\hat{x}_0 = x_0$, $\hat{x}_M = \hat{x}_N$. Consider, also, a mapping $\mathbb{J} : \{0, 1, \ldots, N\} \to \{0, 1, \ldots, M\}$, a set of numbers $D_y = \{y_{k,i} :\ k = 0, 1, \ldots, p,\ i = 0, 1, \ldots, N\}$ and $a_i = (x_i - x_{i-1})/(\hat{x}_j - \hat{x}_{j-1})$, $|a_i| < 1$. If the real numbers $s_1, s_2, \ldots, s_N$ satisfy $|s_i| < a_i^p$, $i = 1, 2, \ldots, N$, then there is a unique RFIF $f \in C^p$ such that*

$$f^{(k)}(x_i) = y_{k,i}$$

for all $k = 0, 1, \ldots, p$, $i = 0, 1, \ldots, N$. The function f is constructed as the attractor of a RIFS $\{\mathbb{R}^2;\ w_{1-N},\ P\}$ with

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ s_i y + q_i(x) \end{pmatrix},$$

where q_i are polynomials of degree at most $2p+1$.

Proof. Again, we define the mapping $\mathbb{I} : \{0, 1, \ldots, M\} \to \{0, 1, \ldots, N\}$ as follows: $\mathbb{I}(j) = i$, iff $\hat{x}_j = x_i$. Using this mapping we construct the set $\hat{D}_y = \{\hat{y}_{k,j};\ k =$

$0, 1, \ldots, p, j = 0, 1, \ldots, M\}$, such that $\hat{y}_{k,j} = y_{k,\mathbb{I}(j)}$. The coefficients of the polynomial q_i are computed as solutions of the linear system

$$F_{k,i}(\hat{x}_{j-1}, \hat{y}_{k,j-1}) = \frac{s_i \hat{y}_{k,j-1} + q_i^{(k)}(\hat{x}_{j-1})}{a_i^k} = y_{k,i-1},$$

$$F_{k,i}(\hat{x}_j, \hat{y}_{k,j}) = \frac{s_i \hat{y}_{k,j} + q_i^{(k)}(\hat{x}_j)}{a_i^k} = y_{k,i},$$

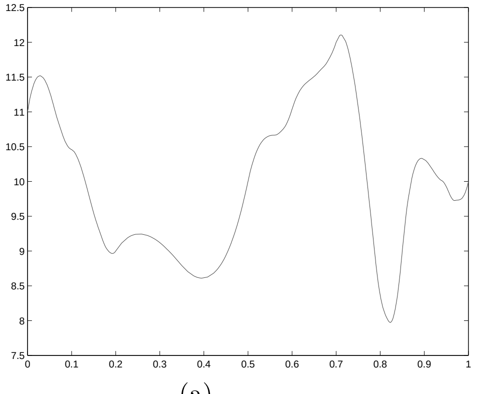
for $k = 0, 1, \ldots, p$, $i = 1, 2, \ldots, N$. We may rewrite this linear system as

$$(q_i \circ L_i^{-1})^{(k)}(x_{i-1}) = y_{k,i-1} - \frac{s_i \hat{y}_{k,j-1}}{a_i^k},$$

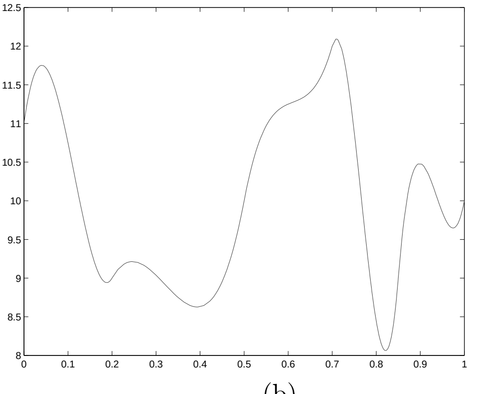
$$(q_i \circ L_i^{-1})^{(k)}(x_i) = y_{k,i} - \frac{s_i \hat{y}_{k,j}}{a_i^k},$$

$k = 0, 1, \ldots, p$, $i = 1, 2, \ldots, N$. Thus $(q_i \circ L_i^{-1})$ is an Hermite polynomial of degree p, for all $i = 1, 2, \ldots, N$. This fact ensures that the linear system has a unique solution. In addition, the conditions of theorem 3.1 are satisfied, therefore $f \in C^p([x_0, x_N]^n)$. □

We note that in the special case where $s_i = 0$, $i = 1, 2, \ldots, N$, the RFIF f, is identical to the classical Hermite interpolation function (the proof is straightforward).



(a)



(b)

Figure 5: Two Hermite RFIF that interpolate the same set of data using (a) non zero contraction factors, (b) zero contractions factors.

4. Fractal Interpolation Surfaces derived from RFIF

The construction of fractal surfaces has drawn the attention of many researchers. Massopust was the first who gave a valid construction in [10], using data points placed on a triangular domain. A lot of other attempts followed, however most of them need strongly restricted data (for example the interpolation points that are placed on the boundary of the domain need to be colinear). The most general case was given in [7]. Here, we describe a method that uses Hermite Fractal Interpolation Functions of order 3 to construct C^1 Fractal Interpolation Surfaces on a rectangular grid of arbitrary interpolation points.

Consider the interpolation points $\Delta = \{(x_i, y_j, z_{ij}) : i = 0, 1, \ldots N; j = 0, 1, \ldots, M\} \subseteq [0,1] \times [0,1] \times \mathbb{R}$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_M = 1$ and $x_i - x_{i-1} = \delta_i$, $i = 0, 1, \ldots, N-1$, $y_j - y_{j-1} = \tilde{\delta}_j$, $j = 0, 1, \ldots, M-1$. Let $S = \{s_1, s_2, \ldots, s_N\}$, $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_M\}$ be two sets of contraction factors and let $P = (p_{nm})^N$, $\tilde{P} = (\tilde{p}_{nm})^M$ be two stochastic matrices with dimensions $N \times N$ and $M \times M$, respectively. Also, let $\hat{\Delta} = \{(\hat{x}_k, \hat{y}_l, \hat{z}_{kl}) : k = 0, 1, \ldots K; l = 0, 1, \ldots, L\}$ be a subset of Δ such that $\hat{x}_0 = 0$, $\hat{x}_K = 1$, $\hat{y}_0 = 0$, $\hat{y}_L = 1$ and $\hat{x}_k - \hat{x}_{k-1} = \psi_k$, $\hat{y}_l - \hat{y}_{l-1} = \tilde{\psi}_l$, $k = 0, 1, \ldots, K$, $l = 0, 1, \ldots, L$. Let $\mathbb{J}$ and $\tilde{\mathbb{J}}$ be defined as in section 3 associated with the matrices P and $\tilde{P}$, respectively, with $\mathbb{J}(i) = k$, $\tilde{\mathbb{J}}(j) = l$. The points $\{x_0, x_1, \ldots, x_N\}$ divide $[0,1]$ into N domains $I_1, I_2, \ldots, I_N$, while the points $\{y_0, y_1, \ldots, y_M\}$ divide $[0,1]$ into M domains $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_M$. Consequently, the points $\{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K\}$ divide $[0,1]$ into K regions $J_1, J_2, \ldots, J_K$, while the points $\{\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L\}$ divide $[0,1]$ into L regions $\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_L$. In addition, we define the mappings

$$\mathbb{I} : \{0, 1, \ldots, K\} \to \{0, 1, \ldots, N\}$$
$$\tilde{\mathbb{I}} : \{0, 1, \ldots, L\} \to \{0, 1, \ldots, M\}$$

such that $\hat{x}_k = x_{\mathbb{I}(k)}$ and $\hat{y}_l = y_{\tilde{\mathbb{I}}(l)}$.

We consider arbitrary continuous functions u_i, that interpolate the sets $\tilde{\Delta}_{x_i} = \{(x_i, y_j, z_{ij}) : j = 0, 1, \ldots, M\}$, for $i = 0, 1, \ldots, N$. Then, for $y \in [0,1]$, we construct an affine RIFS associated with the interpolation points $\Delta_y = \{(x_i, y, u_i(y)) : i = 0, 1, \ldots, N\}$, $\hat{\Delta}_y = \{\hat{x}_k, y, u_{\mathbb{I}(k)}(y)), k = 0, 1, \ldots, K\}$, the set of contraction factors S together with the matrix P, which produce an affine RFIF $f_y : [0,1] \to \mathbb{R}$ (see figure 6). We define the function

$$F : [0,1] \times [0,1] \to \mathbb{R} \text{ such that } F(x,y) = f_y(x).$$

Similarly, we consider arbitrary continuous functions v_j, that interpolate the sets $\Delta_{y_j} = \{(x_i, y_j, z_{ij}) : i = 0, 1, \ldots, N\}$ for $j = 0, 1, \ldots, M$. As before, for $x \in [0,1]$ we construct an affine RIFS associated with the interpolation points $\tilde{\Delta}_x = \{(x, y_j, v_j(x)) : j = 0, 1, \ldots, M\}$, $\tilde{\hat{\Delta}}_x = \{(x, \hat{y}_l, v_{\tilde{\mathbb{I}}(l)}(x)), l = 0, 1, \ldots, L\}$, the set of contraction factors $\tilde{S}$ together with the matrix $\tilde{P}$, which produce an affine FIF $\tilde{f}_x : [0,1] \to \mathbb{R}$. Thus, we define the function

$$\tilde{F} : [0,1] \times [0,p] \to \mathbb{R} \text{ such that } \tilde{F}(x,y) = \tilde{f}_x(y).$$

The functions F, $\tilde{F}$ are continuous functions that interpolate the data set Δ (see [8]).

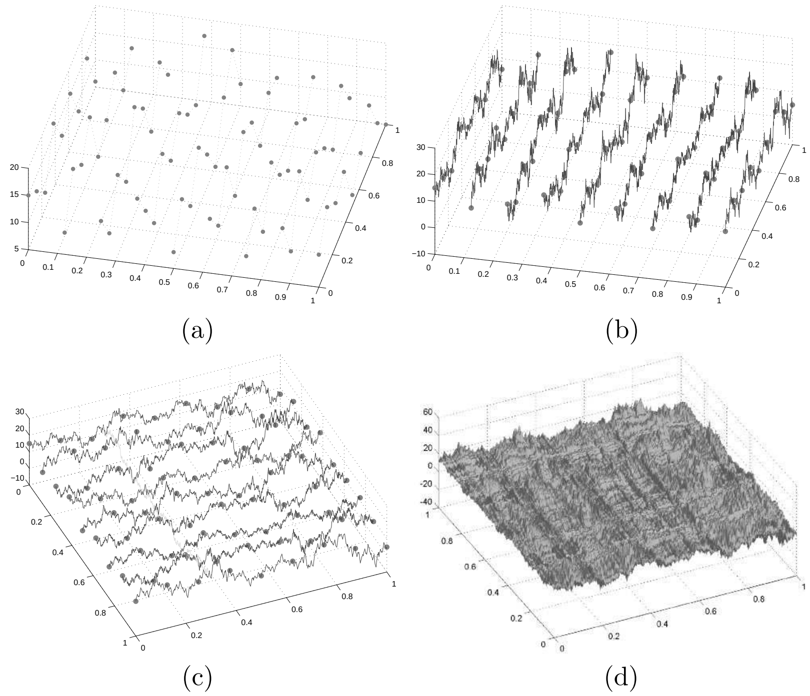


Figure 6: An example of the construction of the function F is shown. (a) The points of Δ, where $N = M = 8$, $p = 1$. (b) The nine interpolation functions $u_0, u_1, \dots, u_8$, (c) One of the FIFs f_y (green line), (d) The graph of the function F.

Using a similar method we may proceed in a construction of fractal interpolation surfaces of class C^1. In this case, we consider arbitrary C^1 functions u_i, that interpolate the sets $\tilde{\Delta}_{x_i} = \{(x_i, y_j, z_{ij}) : j = 0, 1, \dots, M\}$, for $i = 0, 1, \dots, N$. We, also, consider some other arbitrary continuous functions u_i^* defined on $[0, 1]$ that satisfy the Lipschitz condition. These functions will be used as the x-partial derivative of the constructed surface. Then, for $y \in [0, 1]$, we construct a Hermite RFIF of order 1 associated with the interpolation points $\Delta_y = \{(x_i, y, u_i(y), u_i^*(y)) : i = 0, 1, \dots, N\}$, $\hat{\Delta}_y = \{(\hat{x}_k, y, u_{\mathbb{I}(k)}(y), u^*_{\mathbb{I}(k)}(y)) : \ k = 0, 1, \dots, K\}$, the set of contraction factors S together with the matrix P. The corresponding RIFS is $\{\mathbb{R}^2;\ w_{y,1-N}, P\}$, where $w_{y,j}$ are of the form

$$w_{y,j}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ s_i z + q_{y,i}(x) \end{pmatrix},$$

for $i = 1, 2, \dots, N$, as defined in section 3.2. We define the function

$$F : [0, 1] \times [0, 1] \to \mathbb{R} \text{ such that } F(x, y) = f_y(x).$$

We can easily show that F is a C^1 function. To this end we construct a RIFS whose attractor is identical to F.

We begin with the set $\Delta = \{(x_i, y_j, z_{ij}) : i = 0, 1, \ldots N; j = 0, 1, \ldots, M\} \subseteq [0, 1] \times [0, 1] \times \mathbb{R}$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_M = 1$. We define a subset $\hat{\Delta}'$ of Δ as follows, $\hat{\Delta}' = \{(\hat{x}_k, y_j, z_{\mathbb{I}(k),j}) : k = 0, 1, \ldots K; j = 0, 1, \ldots M\}$. The corresponding domains are $D_i = [x_{i-1}, x_i] \times [y_0, y_M]$, $i = 1, 2, \ldots, N$ and the regions are $R_k = [\hat{x}_{k-1}, \hat{x}_k] \times [y_0, y_M]$. The mappings w'_i are constructed as follows

$$w'_i \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ y \\ s_i z + Q_i(x, y) \end{pmatrix} = \begin{pmatrix} L_i(x) \\ y \\ s_i z + Q_i(x, y) \end{pmatrix},$$

for $i = 1, 2, \ldots, N$, $(x, y) \in D_i$ such that $Q_i(x, y) = q_{y,i}(x)$, where $q_{y,i}$ are the polynomials that are used in the RIFS whose attractor is the Hermite RFIF f_y. The fact that $q_{y,i}$ are Hermite type polynomials of order 3 such that

$$\begin{aligned} q_{y,i}(x_{i-1}) &= u_{i-1}(y) - s_i u_{k-1}(y), \\ q_{y,i}(x_i) &= u_i(y) - s_i u_k(y), \\ q'_{y,i}(x_{i-1}) &= u^*_{i-1}(y) - \frac{s_i u^*_{k-1}(y)}{a_i}, \\ q'_{y,i}(x_i) &= u^*_i(y) - \frac{s_i u^*_k(y)}{a_i}, \end{aligned}$$

where $k = \mathbb{J}(i)$, for all $y \in [0, 1]$, ensures that Q_i is a C^1 function for all $i = 1, 2, \ldots, N$. We define the set $\mathcal{F} = \{f \in C^1([0, 1]^2) : f|_{x_i \times [0,1]} = u_i, \frac{\partial f}{\partial x}|_{x_i \times [0,1]} = u^*_i, i = 0, 1, \ldots, N\}$ and the operator $\boldsymbol{T} : \mathcal{F} \to \mathcal{F}$ such that $\boldsymbol{T}f(x, y) = s_i f(L_i^{-1}(x), y) + Q_i(L_i^{-1}(x), y)$, for all $(x, y) \in D_i$, $i = 1, 2, \ldots, N$. We may easily show that $\boldsymbol{T}$ is well defined (based on the construction of f_y and the fact that Q_i are C^1) and that it is a contraction. Therefore, it has a unique fixed point which is identical to F. Hence F is a C^1 function.

In figure 7 we used sets of the form $\Delta = \{(x_i, y_j, z_{i,j}, z^*_{i,j}, z^{**}_{i,j}) : i = 0, 1, \ldots N; j = 0, 1, \ldots, M\}$, where the values $z^*_{i,j}, z^{**}_{i,j}$ are the values of the x and y partial derivatives. We constructed the functions u_i as Hermite RFIF that are associated with the data sets $\{(x_i, y_j, z_{i,j}, z^{**}_{i,j}) : j = 0, 1, \ldots, M\}$, for $i = 0, 1, \ldots N$. The functions u^*_i were constructed as the linear interpolation functions passing through the points $\{(x_i, y_j, z^*_{i,j}) : j = 0, 1, \ldots, M\}$, for $i = 0, 1, \ldots N$.

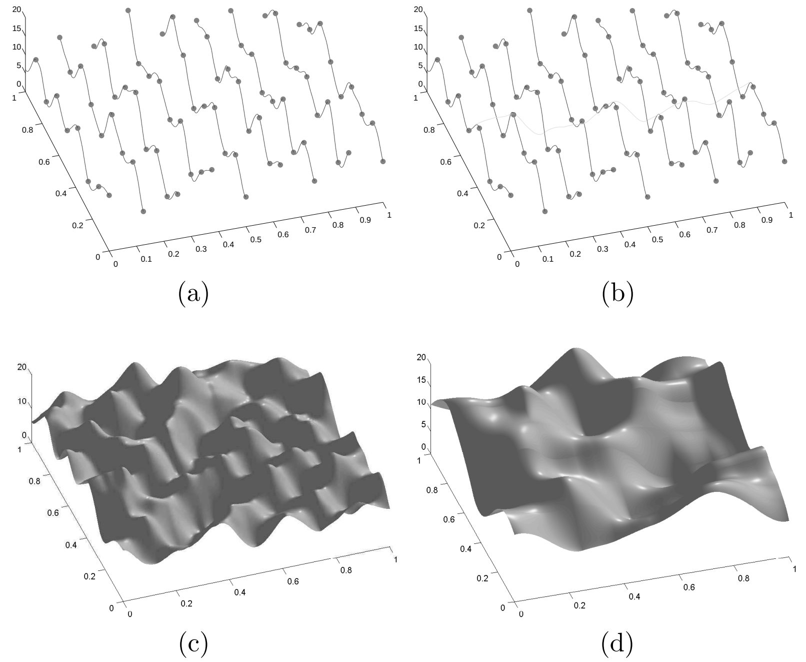


Figure 7: An example of the construction of the smooth function F is shown. (a) The nine interpolation functions $u_0, u_1, \ldots, u_8$, (b) One of the FIFs f_y (green line), (c) The graph of the C^1 surface, (d) Another example of a C^1, surface using a different set of data.

References

1. B. F. Barnsley, J. Elton, D. Hardin, P. Massopust, Hidden Variable Fractal Interpolation Functions, SIAM J. Math. Anal., (20),1218–1242,1989.
2. B. F. Barnsley, A. N. Harrington, The calculus of Fractal Interpolation Functions, J. Approx. Theory, (57),14–43,1989.
3. M. F. Barnsley, S. Demko, Iterated function systems and the global construction of fractals, Proc. Roy. Soc. London Ser A, (399),243–275,1985.
4. M. F. Barnsley, J. H. Elton, D. P. Hardin, Recurrent Iterated Function Systems, Constr. Approx., (5),3–31,1989.
5. M. F. Barnsley, Fractal functions and interpolation, Constr. Approx., (2),303–329,1986.
6. M. F. Barnsley, Fractals everywhere, 2nd ed., Academic Press Professional,1993.
7. P. Bouboulis, L. Dalla, V. Drakopoulos, Construction of Recurrent Bivariate Fractal Interpolation Surfaces and Computation of their Box-Counting Dimension, J. Approx. Theory, (141),99–117,2006.
8. P. Bouboulis, L. Dalla, Fractal interpolation surfaces derived from fractal interpolation functions, J. Math. Anal. Appl. (to appear).

9. G. C. Donovan, J.S. Geronimo, D. P. Hardin, P. R. Massopust, Construction of Orthogonal Wavelets Using Fractal Interpolation Functions, SIAM J. Math. Anal., (27),1158–1192,1996.
10. Peter R. Massopust, Fractal Functions, Fractal Surfaces and Wavelets, Academic Press,1994.
11. M. A. Navascues and M. V. Sebastian, Generalization of Hermite functions by fractal interpolation, J. Approx. Theory, (131), 19–29, 2004.

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