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On the center of galbed algebras

Mart Abel

Abstract

It is shown that every unital Mackey advertibly σ-complete strongly galbed Hausdorff algebra with bounded elements is central, if there is a closed maximal left or right ideal M of A such that $M \cap Z(A) = \{\theta_A\}$.

Keywords: Galbed algebra, central algebra

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{N} = \{0, 1, 2, ...\}$, $k > 0$, l^k the set of all $\mathbb{C}$-valued sequences (α_n), for which the series

$$\sum_{v=0}^{\infty} |\alpha_v|^k$$

converges, l^0 the set of all $\mathbb{C}$-valued sequences (α_n) such that the set $\{k \in \mathbb{N} : \alpha_k \neq 0\}$ is finite $l = l^1 \setminus l^0$ and

$$l^{(0,1]} = \bigcap_{k \in (0,1]} l^k.$$

Let A be an associative topological algebra over $\mathbb{C}$ with separately continuous multiplication (in short, a *topological algebra*).

Remember, that an element $a \in A$ is *bounded*, if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ and for every neighbourhood O of zero in A there exists $\mu_O > 0$ such that

$$\left\{ \left(\frac{a}{\lambda} \right)^n : n \in \mathbb{N} \right\} \subset \mu_O O.$$

Throughout this paper, let $a \circ b = a + b + ab$.

Definition 1. A sequence $(a_n)_{n \in N}$ of elements of a topological algebra A is *advertibly convergent*[1] in A, if there exists an element $a \in A$ such that both $(a \circ a_n)_{n \in N}$ and $(a_n \circ a)_{n \in N}$ converge to θ_A.

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[1]The term "advertibly null" is also in use by A. Mallios.

Definition 2. A topological algebra A is *advertibly σ-complete* if every advertibly convergent Cauchy sequence of A converges in A.

Definition 3. A sequence $(a_n)_{n\in N}$ of elements of a topological algebra A is *Mackey convergent* in A to an element $a_0 \in A$, if there exists a bounded set $B \subset A$ and for every $\epsilon > 0$ an element $n_\epsilon \in \mathbb{N}$ such that $a_n - a_0 \in \epsilon B$ for any $n > n_\epsilon$.

Definition 4. A sequence $(a_n)_{n\in N}$ of elements of a topological algebra A is *Mackey advertibly convergent* in A, if there exists an element $a \in A$ such that both $(a \circ a_n)_{n\in N}$ and $(a_n \circ a)_{n\in N}$ Mackey converge to θ_A.

Definition 5. A topological algebra A is *Mackey advertibly σ-complete* if every Mackey advertibly convergent Cauchy sequence of A converges in A.

Definition 6. A topological algebra A is a *galbed algebra*, if there exists a sequence $(\alpha_n) \in l$ such that for each neighbourhood O of zero in A there is another neighbourhood U of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$.

Furthermore, if there exists a sequence $(\alpha_n) \in l$ with $\alpha_0 \neq 0$ and

$$\alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0$$

such that the previous condition is fulfilled, then we say that A is a *strongly galbed algebra*. We call α the "module of galbness" of A.[2]

In case we have already specified the sequence $(\alpha_n) \in l$, then we talk about (α_n)-*galbed algebra* and *strongly (α_n)-galbed algebra*.

The center of a primitive ring is always an integral domain[3] (see [7], Lemma 2.1.3, p. 45) and any commutative integral domain can be the center of a primitive ring[4] (see [8], Chapter II.6, Example 3, p. 36). Herewith, every field is a commutative integral domain, but any commutative integral domain does not have to be a field. It is known (see [3]), that if R is a unital primitive locally A-pseudoconvex Hausdorff algebra or a unital primitive locally pseudoconvex Fréchet Q-algebra, then R is central (for Banach algebras a similar result has been given in [10], Corollary 2.4.5, see also [5], p. 127; [9], Theorem 4.2.11, and [6], Theorem 2.6.26 (ii); for k-Banach algebras in [4], Corollary 9.3.7; for locally m-convex Q-algebras in [12], Corollary 2, and for locally A-convex algebras in which all maximal ideals are closed in [13], Theorem 3).

[2] The previous terminology is due to A. Mallios.

[3] A ring R is an *integral domain*, if from $a, b \in R$ and $ab = \theta_R$ follows that $a = \theta_R$ or $b = \theta_R$.

[4] The author would like to express his gratitute to Professor Laszlo Marki for informing him about this result.

Some results about the center of topologically primitive galbed algebras can be found in [1] and (for the special case of *exponentially* or (2^{-n})-*galbed* algebras) in [2]. The motivation of the present paper is to generalize the results from [1] for a larger class of topological algebras.

2. Preliminary results

In this chapter we prove some preliminary results about the convergence of some series.

Lemma 1. *A subset A of a topological vector space L is bounded in L if and only if for every sequnence (λ_n) of $\mathbb{C}$ and every sequence (x_n) of A from*

$$(\lambda_n) \xrightarrow{n} 0$$

follows that

$$(\lambda_n x_n) \xrightarrow{n} \theta_L.$$

Proof. See the proof of the statement 5.3 in [11], p. 26-27.

Proposition 1. *Let A be a strongly galbed algebra with bounded elements and let $\lambda_0 \in \mathbb{C}, a_0 \in A$. Then there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that the sequence $(S_n(\lambda))$, where*

$$S_n(\lambda) = \sum_{k=0}^{n} (\lambda - \lambda_0)^k {a_0}^k$$

is a Mackey advertibly convergent Cauchy sequence for each $\lambda \in O(\lambda_0)$.

Proof. The proof of the fact that $(S_n(\lambda))$ is a Cauchy sequence, is done exactly the same way as in [1], first part of the proof of Proposition 2.2.

It remains to show that $(S_n(\lambda))$ is Mackey advertibly convergent. As in [1], we can find $\mu_0 \in \mathbb{C}\backslash\{0\}$ such that

$$\left\{ \left(\frac{a_0}{\mu_0}\right)^n : n \in \mathbb{N} \right\}$$

is bounded in A. Remember, that (see [1])

$$O(\lambda_0) = \lambda_0 + \left\{ \lambda \in \mathbb{C} : |\lambda| < \frac{\alpha}{|\mu_0|} \right\}$$

where α is the module of galbness of A. Therefore,

$$[\mu_0(\lambda - \lambda_0)]^{n+1} \xrightarrow{n} 0$$

for every $\lambda \in O(\lambda_0)$. Every bounded set is contained in a balanced bounded set. Thus, there exists a balanced bounded set B in A such that

$$\left\{\left(\frac{a_0}{\mu_0}\right)^n \;:\; n \in \mathbb{N}\right\} \subset B.$$

Let us fix an arbitrary $\epsilon > 0$. We can suppose that $1 > \epsilon$. Then there exists such $n_\epsilon \in \mathbb{N}$ that $[\mu_0(\lambda - \lambda_0)]^{n+1} < \epsilon$ for every $n > n_\epsilon$. Now

$$S_n(\lambda) \circ [(\lambda_0 - \lambda)a_0] = [(\lambda_0 - \lambda)a_0] \circ S_n(\lambda) = [(\lambda_0 - \lambda)a_0]^{n+1} =$$

$$= [\mu_0(\lambda_0 - \lambda)]^{n+1}\left(\frac{a_0}{\mu_0}\right)^{n+1} \subset [\mu_0(\lambda_0 - \lambda)]^{n+1}B = \epsilon\frac{[\mu_0(\lambda_0 - \lambda)]^{n+1}}{\epsilon}B \subset \epsilon B,$$

if $n > n_\epsilon$. Thus, $(S_n(\lambda))$ is Mackey advertibly convergent for each $\lambda \in O(\lambda_0)$.

Corollary 1. *Let A be a strongly galbed algebra with bounded elements and let $\lambda_0 \in \mathbb{C}, a_0 \in A$. Then there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that the sequence $(S_n(\lambda))$, where*

$$S_n(\lambda) = \sum_{k=0}^{n}(\lambda - \lambda_0)^k {a_0}^k$$

is an advertibly convergent Cauchy sequence for each $\lambda \in O(\lambda_0)$.

Proof. Since every Mackey advertibly convergent sequence is advertibly convergent, then we obtain the desired result directly from Proposition 1.

One can easily conclude the following:

Corollary 2. *Let A be a strongly galbed algebra with bounded elements and let $\lambda_0 \in \mathbb{C}, a_0 \in A$. Then there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that*

$$S_n(\lambda) = \sum_{k=0}^{n}(\lambda - \lambda_0)^k {a_0}^k$$

is a Cauchy sequence.

Proposition 2. *Let A be a unital strongly galbed algebra with bounded elements, being also Mackey advertibly σ-complete or a nil algebra. Moreover, let $\lambda_0 \in \mathbb{C}$ and $a_0 \in A$. Then there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that*

$$(e_A + (\lambda_0 - \lambda)a_0)^{-1} = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a_0^k$$

for each $\lambda \in O(\lambda_0)$.

Proof. Take $(S_n(\lambda))$ and $O(\lambda_0)$ as in Proposition 1. Then $S_n(\lambda)$ is Mackey advertibly convergent Cauchy sequence for every $\lambda \in O(\lambda_0)$. If A is Mackey advertibly σ-complete, then $(S_n(\lambda))$ converges for every $\lambda \in O(\lambda_0)$. The case when A is a nil algebra is studied in [1], Proposition 2.2. In [1] it is also shown that

$$(e_A + (\lambda_0 - \lambda)a_0)^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k a_0^k.$$

Proposition 3. *Let A be a unital strongly galbed algebra with bounded elements. If A is an Mackey advertibly σ-complete or a nil algebra, then for each $a_0 \in A$ there exists a number $R > 0$ such that*

$$\sum_{k=0}^{\infty} \frac{a_0^k}{\mu^{k+1}}$$

converges in A, whenever $|\mu| > R$.

Proof. Apply Corollary 2.3 in [1] with Proposition 2 above, in place of Proposition 2.2 in [1].

3. Main results

Now we are ready to prove our main results. First of all, remember that in every central algebra A we have $M \cap Z(A) = \{\theta_A\}$ for every closed maximal left or right ideal M of A. In next result we prove the converse for some algebras.

Theorem 1. *Let A be a unital Mackey advertibly σ-complete strongly galbed Hausdorff algebra with bounded elements. If there exists a closed maximal left or right ideal M of A such that $M \cap Z(A) = \{\theta_A\}$, then A is a central algebra.*

Proof. Notice, that in the proof of Theorem 3.1 in [1] we need A to be topologically primitive only for finding a closed maximal left or right ideal M of A such that $M \cap Z(A) = \{\theta_A\}$. Therefore, we can follow exactly the proof of Theorem 3.1 in [1] using Proposition 2 from the present paper instead of Proposition 2.2 from [1] and Proposition 3 from the present paper instead of the Corollary 2.3 from [1].

Corollary 3. *Let A be a unital Mackey advertibly σ-complete topologically primitive strongly galbed Hausdorff algebra with bounded elements. Then A is a central algebra.*

Proof. The result follows from Theorem 1, since for algebra A being topologically primitive guarantees the existence of a closed maximal left or right ideal M of A such that $M \cap Z(A) = \{\theta_A\}$.

Concerning the above, we further remark that Proposition 2, Proposition 3, Theorem 1 and Corollary 3 are also valid in the following particular cases:

a) A is advertibly σ-complete;

b) A is σ-complete.

An open problem. *Does there exist a topological algebra A which is not topologically primitive but has a closed maximal left or right ideal M such that $M \cap Z(A) = \{\theta_A\}$?*

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References

[1] Mart Abel, *The center of topologically primitive galbed algebras*, Topological Algebras, their Applications and Related Topics, Banach Center Publications **67**, Warszawa, 2005, 45–54.

[2] Mart Abel, Mati Abel, *The center of topologically primitive exponentially galbed algebras*, (submitted).

[3] Mati Abel, *The center of primitive locally psedoconvex algebras*, Bull. Belg. Math. Soc. Simon Stevin **11** (2004), no. 2, 191–199.

[4] V. K. Balachandran, *Topological Algebras*, North-Holland Math. Studies **185**, Elsevier, Amsterdam, 2000.

[5] F. F. Bonsall, J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin 1973.

[6] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs. New Series **24**, Oxford Univ. Press, New York, 2000.

[7] I. N. Herstein, *Noncommutative rings*, John Wiley & Sons, Inc., New York, 1968.

[8] N. Jacobson, *Structure of rings*, Amer. Math. Soc., Colloquium Publ., **37**, Providence, Rhode Island, 1968 (revised edition).

[9] Th. W. Palmer, *Banach algebras and the general theory of *-algebras*, Vol. I. Algebras and Banach algebras. Encyclopedia of Math. and Appl. **49**, Cambridge Univ. Press, Cambridge, 1994.

[10] C. E. Rickart, *General Topology of Banach Algebras*, D. van Nostrad, Princeton, 1960.

[11] H. H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Math. **3**, Springer-Verlag, New York-Heidelberg-Berlin, 1980.

[12] Y. Tsertos, *On primitive topological algebras*, Bull. Greek Math. Soc., **28** (1987), 81–92.

[13] Y. Tsertos, *On primitive A-convex algebras*, Comment. Math. Prace Mat., **41** (2001), 203–219.

◇ Mart Abel
2 Liivi Str. Room 615, Institute of Pure Mathematics,
University of Tartu, 50409 Tartu, Estonia
mart.abel@ut.ee