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Compactness Theorems for Saddle Surfaces in Metric Spaces of Bounded Curvature

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Abstract

The notion of a non-regular saddle surface is well known in Euclidean spaces. In this work we extend the idea of a saddle surface to non-regular spaces of curvature bounded from above and we establish the compactness property for the family of saddle surfaces with uniformly bounded areas. In spaces of constant curvature we obtain a stronger result based on an isoperimetric inequality for saddle surfaces.

Keywords: Saddle surface, CAT(κ) space, curvature in the sense of A.D.Aleksandrov, isoperimetric inequality, compactness.
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1. Introduction

The class of saddle surfaces is dual to the class of general convex surfaces. In the three-dimensional Euclidean space, a non-regular saddle surface is described by the property that one cannot cut off a crust from it by any plane. Such definition is readily available in higher dimensional Euclidean spaces but it makes no sense in non-Euclidean spaces. Major contribution to the theory of non-regular saddle surfaces in Euclidean spaces has been made by S.Z. Shefel in 1960s ([13], [14], [15]). Historically, saddle surfaces are connected with the Plateau problem in Euclidean spaces ([10]).

In this work we extend the idea of a saddle surface to geodesically connected metric spaces and we study the class of saddle surfaces in spaces of curvature bounded from above in the sense of A.D. Aleksandrov (also known as CAT(κ) spaces). Spaces of bounded curvature inherit basic properties of Riemannian manifolds while they can have strong topological and metric singularities.

In section 2 we recall what it means for a metric space to be of curvature bounded from above and we give the definition and some examples of saddle surfaces in such spaces. Saddle surfaces in Euclidean spaces are usually defined by means of the operation of cutting off crusts by hyperplanes. Therefore, a generalization of the classical definition to any metric space is not quite obvious. The definition we propose here for a saddle surface in a general geodesically connected metric space (Definition 1) makes use of the concept of convex hull and goes back to a property that a regular solution of a Plateau problem satisfies: the convex hull of a closed Jordan curve on a minimal surface (not intersecting the boundary of the surface) always contains the interior part of the surface enclosed by the curve ([12], Lemma 7.1).

In section 3 we present a condition for compactness of a family of saddle surfaces in a compact space of curvature bounded from above, which assumes restrictions on areas and lengths of bounding curves (Theorem 1). In spaces of constant curvature we show that an isoperimetric inequality for the class of saddle surfaces (Theorem 2) reduces compactness of a family of saddle surfaces to the compactness of the set of bounding curves (Theorem 3). These results generalize difficult theorems of S.Z. Shefel on compactness of saddle surfaces in a Euclidean space ([15]).

2. Saddle surfaces in metric spaces of curvature bounded from above by κ

Metric spaces of curvature bounded from above by κ

A notion of curvature of metric space can be defined by comparing triangles in a metric space with the corresponding model triangles in the κ-plane with sides of the same length. The definition is due to A.D. Aleksandrov and the curvature is usually referred to as the curvature in the sense of A.D. Aleksandrov. Aleksandrov's spaces are a natural generalization of Riemannian manifolds but they are of more general nature. For details, see [2], [3].

The n-dimensional κ-space S^n_κ (κ-plane, for $n = 2$) is the hyperbolic space H^n_κ for $\kappa < 0$, the Euclidean space E^n_κ for $\kappa = 0$, and the upper open hemisphere $S^n_+(\kappa^{-1/2})$ of E^n_κ of radius $\kappa^{-1/2}$ with the induced metric, when $\kappa > 0$. Every S^n_κ is a Riemannian simply connected manifold of constant sectional curvature κ such that any pair of points can be joined by a unique geodesic segment.

An R_κ domain, abbreviated by R_κ, is a metric space satisfying the following axioms:

Axiom 1: Any two points in R_κ can be joined by a geodesic segment.

Axiom 2: If $\kappa > 0$, then the perimeter of each triangle in is R_κ less than $\dfrac{2\pi}{\sqrt{\kappa}}$.

Axiom 3: Each triangle in R_κ has nonpositive κ-excess; that is, for the angles α, β, γ of a triangle ABC

$$\alpha + \beta + \gamma - (\alpha_\kappa + \beta_\kappa + \gamma_\kappa) \leq 0,$$

where α_κ, β_κ, γ_κ are the corresponding angles of a triangle $A^\kappa B^\kappa C^\kappa$ on the κ-plane with sides of the same length as ABC.

Another term for an R_κ domain is a CAT(κ) space. However, we shall use Aleksandrov's original notation. ([1]). It is evident that any κ-space is an R_κ domain.

A space of curvature bounded from above by κ in the sense of A.D. Aleksandrov is a metric space, each point of which is contained in some neighborhood of the original space, which is an R_κ domain.

Examples of such spaces are any Riemannian manifold with sectional curvature not greater than κ, any polyhedron, trees, Euclidean buildings, Hilbert spaces and other infinite dimensional symmetric spaces (see [3]). Also, if X is a space of curvature bounded from above by κ and M is a finite-volume Riemannian manifold, then the space $L^2(M, X)$ is also a space of curvature bounded from above by κ.

Saddle surfaces

Let M be a geodesically connected metric space with intrinsic metric. A (parametrized) surface f in M is any continuous mapping $f : D \to M$, where D denotes the closed unit disk on the plane.

In the Euclidean space E^n we say that a hyperplane P cuts off a crust from the surface f if among the connected components of $f^{-1}(f(D)\backslash P)$ there is one with positive distance from the boundary of D. If U is such a component, then the set $f(U)$ is called a crust. A surface f in E^n is said to be a saddle surface if it is impossible to cut off a crust from it by any hyperplane.

We extend the idea of a saddle surface to any geodesically connected metric space making use of the concept of convex hull. The convex hull $conv(A)$ of a subset A of M is defined to be the smallest closed convex subset containing A.

Definition 1. A surface f on a metric space M is said to be a saddle surface if

$$f(\text{int}\gamma) \subset conv(f(\gamma))$$

for every Jordan curve $\gamma \subset D$ having positive distance from the unit circle.

The following theorem shows the equivalence of Definition 1 with the classical one in the case of a Euclidean space.

Theorem 1. ([6]) *If f is a surface in E^n then the following are equivalent:*

a) It is impossible to cut off a crust from f by any hyperplane.

b) $f(\text{int}\gamma) \subset conv(f(\gamma))$, for every Jordan curve $\gamma \subset D$ having positive distance from the unit circle.

Examples

1. Any regular surface in S^3_κ with Gaussian curvature not greater than κ is a saddle surface ([7]).

2. Saddle surfaces are preserved under the action of any continuous geodesic mapping. Since the inclusion mapping $id : H^n_\kappa \to E^n$, where H^n_κ is interpreted by the Beltrami-Klein model of the hyperbolic space, and the central projection $\varphi : S^n_\kappa(\kappa^{-1/2}) \to E^n$ are continuous geodesics mappings, then the following characterization for saddle surfaces in S^n_κ holds: (i) a surface in H^n_κ is a saddle surface if and only if it is a saddle

surface in E^n, and (ii) a surface in $S^n_+(\kappa^{-1/2})$ is a saddle surface if and only if its image under the central projection is a saddle surface in E^n.
3. Any ruled surface in an R_κ domain is a saddle surface. The definition of a ruled surface in an R_κ domain is the following. Let L be a closed curve in an R_κ domain (whose length, when $\kappa > 0$, is less than $\frac{2\pi}{\sqrt{\kappa}}$), C be the unit circle, and $f : C \to R_\kappa$ be a parametrization of L. Let O be an arbitrary point on C. Consider the surface whose parametrization f is specified as follows: for any $X \in D$, lying on the line segment OY, with $Y \in C$, $f(X)$ lies on the geodesic segment $O'Y'$, where $O' = f(O)$, $Y' = f(Y)$ and $O'f(X) : O'Y' = OX : OY$. Because of the condition on the length of the curve L, $O'Y'$ depends continuously on $Y' \in L$ and therefore f is a parametrized surface. In this case we say that we have a ruled surface with vertex O' spanned on L. The inclusion of Definition 1 is trivially satisfied.

4. Any energy minimizing surface in a complete nonpositively curved space is a saddle surface ([8]). An energy minimizing surface is a Sobolev mapping with given trace which is stationary (among the mappings having the same trace) for the Sobolev energy. The energy of a mapping in a metric space was originally introduced by N. Korevaar and R. Shoen in [9] and independently by J. Jost in [5] in connection with the Dirichlet problem in nonpositively curved spaces.

3. Compactness Theorem

We recall that if $f_1 : D \to M$ and $f_2 : D \to M$ are two surfaces in a metric space (M, d), where D the unit closed disk, we say that they are Fréchet equivalent if given any positive number ε, there exists a homeomorphism $h_\varepsilon : D \to D$ such that $d(f_1(x), f_2(h_\varepsilon(x))) < \varepsilon$, for all $x \in D$. It is known that this relation is an equivalence relation in the collection of surfaces. If f is a surface then the set of all surfaces which are Fréchet equivalent with f is an equivalence class, denoted by $[f]$, and is called a Fréchet surface.

The Fréchet distance ρ is defined to be

$$\rho(f_1, f_2) = \inf \sup\{d(f_1(x), f_2(h(x))) : x \in D\}$$

where the infimum is taken over all possible homeomerphisms $h : D \to D$. It is also known that $\rho([f_1], [f_2]) = \rho(f_1, f_2)$ is a well defined relation. The space of Fréchet surfaces is a complete metric space relative to the Fréchet distance.

In this section we consider conditions for compactness of a family of saddle surfaces in a compact space of curvature bounded from above, which assumes restrictions on areas and lengths of bounding curves. In spaces of constant curvature we show that an isoperimetric inequality for the class of saddle surfaces reduces compactness of a family of saddle surfaces to the compactness of the set of bounding curves. These results generalize difficult theorems of S.Z. Shefel on compactness of saddle surfaces in a Euclidean space ([15]).

The area of a surface in a general metric space was originally introduced by I.G. Nikolaev in connection with the Plateau problem ([11]). The following theorem is our main result.

Theorem 1. *Let $\{[f_\alpha]\}_{\alpha\in A}$ be a family of saddle surfaces in a compact R_κ domain with Jordan bounding curves. If the areas and the lengths of the bounding curves of the surfaces of the given family are uniformly bounded, then $\{[f_\alpha]\}_{\alpha\in A}$ is a relatively compact subset of the space of surfaces with respect to the Fréchet metric.*

To prove Theorem 1 we need the following lemmas.

Lemma 1. ([11]) *Let there be given a sequence $\{L_n\}_{n\in N}$ of Jordan curves in a metric space which converges, in the Fréchet sense, to a Jordan curve L. Then, for any $\varepsilon > 0$ there exists a $\tau(\varepsilon) > 0$ such that if the distance between two points on any of the curves L_n and L is less than $\tau(\varepsilon)$, the diameter of one of the two arcs into which these points divide the chosen curve is less than ε.*

Fix a point x in the closed unit disk D of the Euclidean plane. Denote by S_r the arc of the circle centered at x of radius r which is contained in D. Let $f : D \to R_\kappa$ be a surface. Denote by ℓ_r the length of the curve $f(S_r)$ in R_κ. We say that the surface f satisfies Courant's condition relative to a constant $C > 0$ if for each $x \in D$ and $\delta \in (0,1)$ there exists a $\rho \in \lfloor \delta, \sqrt{\delta} \rfloor$ such that:

$$\ell_\rho \leq \frac{C}{\sqrt{\ln \dfrac{1}{\delta}}}.$$

Lemma 2. ([11]) *If the areas of a sequence $\{f_n\}_{n\in N}$ of surfaces in a complete, locally compact R_κ domain are uniformly bounded, then each f_n, $n = 1, 2, \dots$ satisfies the Courant's condition relative to a constant $C > 0$ independent of n.*

Lemma 3. (D. Hilbert) *Let F be a family of rectifiable curves in a metric space (M, d) with traces in a compact subset of M. If lengths of curves in F are uniformly bounded, then the family F is sequentially compact relative to the Fréchet metric.*

Proof of Theorem 1. It suffices to prove that each sequence $\{[f_n]\}_{n\in N}$ of the given family of saddle surfaces is a relatively compact set, or equivalently, by the Arzella-Ascoli theorem (since the space R_κ is a compact space) that: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f_n(x), f_n(y)) < \varepsilon$, for all $n \in N$ provided $x, y \in D$ and $|x - y| < \delta$, where d is the metric of R_κ.

The lengths of the bounding curves of $\{f_n\}_{n\in N}$ are uniformly bounded therefore, by the Hilbert's theorem (Lemma 3), the sequence of these bounding curves is a relatively compact set. So it has a convergent subsequence in the Fréchet sense. For the sake of simplicity let us assume that the sequence of bounding curves converges.

Let $\tau(\varepsilon)$ be the uniform constant ensured by Lemma 1. Also, since the areas of the sequence $\{f_n\}_{n\in N}$ are uniformly bounded, then by Lemma 2, each f_n satisfies Courant's condition relative to a uniform constant $C > 0$. Choose $\delta > 0$ such that

$$\frac{C}{\sqrt{\ln \frac{1}{\delta}}} < \min \left\{ \frac{\varepsilon}{3}, \tau\left(\frac{\varepsilon}{3}\right) \right\}. \tag{1}$$

Let $x, y \in D$ with $|x - y| < \delta$. Take $z = \dfrac{x+y}{2}$. Then by Courant's condition, there exists a $\rho \in \lfloor \delta, \sqrt{\delta} \rfloor$ such that

$$\ell_{\rho,n} \leq \frac{C}{\sqrt{\ln \frac{1}{\delta}}}, \quad \text{for all} \quad n \in N \tag{2}$$

with $\ell_{\rho,n}$ to be the length of $f_n(S_\rho)$, where S_ρ is the arc of the circle centered at z of radius ρ which is contained in D.

Let U be the intersection of D with the closed disk centered at z of radius ρ, and γ be its boundary. There are two cases: either $U \subset D$ or not. In both cases we shall show that each set $f_n(U)$ is a subset of a ball or radius $\dfrac{\varepsilon}{3}$ which terminates the proof, since if $|x - y| < \delta$, then $x, y \in U$ and hence, $d(f_n(x), f_n(y)) < \varepsilon$ for all $n \in N$.

Suppose $U \subset D$. Then $S_\rho = \gamma$ and therefore $\ell(f_n(\gamma)) < \dfrac{\varepsilon}{3}$. So $f_n(\gamma)$ is contained in a closed ball in R_κ of radius $\dfrac{\varepsilon}{3}$. But f_n is a saddle surface and in any R_κ domain closed balls of sufficiently small radius are convex sets. Hence, $f_n(U) \subset conv(f_n(\gamma)) \subset$ ball of radius $\dfrac{\varepsilon}{3}$.

In the case when U is not a subset of D, then the boundary of U consists of two curves: the arc S_ρ and an arc of the unit circle. These two arcs intersect at two points. Let A_n, B_n be the images of these two points under f_n. Then, by (1) and (2), $d(A_n, B_n) < \tau\left(\dfrac{\varepsilon}{3}\right)$ for all $n \in N$. The points A_n, B_n divide the boundary curve of f_n into two arcs. By Lemma 2, the diameter of one of them, say $f_n(\gamma_1)$, is less than $\dfrac{\varepsilon}{3}$ so, since f_n is a saddle surface, $f_n(U) \subset conv(f_n(\gamma_1) \cup f_n(S_\rho)) \subset$ ball of radius $\dfrac{\varepsilon}{3}$. The proof is complete. □

In spaces of constant curvature saddle surfaces satisfy the following isoperimetric inequality.

Theorem 2. *If a saddle surface in a compact subset K of S^n_κ is bounded by a curve of length ℓ, then its area S satisfies the inequality*

$$k_n S - \ell^2 \leq 0,$$

where k_n is a positive constant depending on n and the compact set K.

Proof. The case $\kappa = 0$ is due to S.Z. Shefel ([14]). Let $\kappa \neq 0$ and f be a saddle surface in S^n_κ. It is not difficult to see that the geodesic mappings φ of Example 2 are both bi-Lipschitz mappings on K. This observation has two useful consequences; there are positive constants k_1 and k_2 such that for any curve γ and for any surface f in the compact set K we have $k_2\ell(\gamma) \leq \ell(\varphi \circ \gamma) \leq k_1\ell(\gamma)$ and $k_2^2 S(f) \leq S(\varphi \circ f) \leq k_1^2 S(f)$ where ℓ denotes length and S denotes the Lebesgue area. Based on the conclusion (i) and (ii) of Example 2, the proof is complete. $\square$

Theorem 2 enables us to strengthen Theorem 1 when R_κ is a compact subset of S^n_κ.

Theorem 3. *If the lengths of the Jordan bounding curves of a family of saddle surfaces in a compact subset of S^n_κ are uniformly bounded, then the family being considered is a relatively compact subset of the space of surfaces with respect to the Fréchet metric.*

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