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"POLYNOMIALS ON CERTAIN BANACH SPACES"

BY
I. Sarantopoulos

ABSTRACT. *We investigate Bernstein's theorem for polynomials on normed spaces. We also determine the supremum of the values of* $\|L\| / \|\hat{L}\|$, $L \not\equiv 0$, *where L varies over all continuous summetric m-linear forms on vector valued* L^p*-spaces and* c_p *spaces (*$\hat{L}$ *is the homogeneous polynomial associated to L).*

1. INTRODUCTION

Throughout this paper K denotes either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. If the field is not specifield the results are valid in both cases. For a fixed normed space E over K, we denote by B_E, $\bar{B}_E$ the open unit ball, closed unit ball respectively. Let $\mathscr{L}^s_m(E, K)$ denotes the space of all continuous symmetric m-linear mapping $L : E^m \to K$ and define

$$\hat{L}(x) = L(x, ..., x).$$

A mapping $P : E \to K$ is said to be a homogeneous polynomial of degree m if $P = \hat{L}$ for some $L \in \mathscr{L}^s_m(E, K)$ and it is said to be a polynomial of degree m if

$$P = \sum_{k=0}^{m} P_k, \quad P_m \not\equiv 0$$

where $P_k : E \to K$ is a homogeneous polynomial of degree k, for $k = 1, ..., m$, and $P_0 : E \to K$ is a constant mapping.

If $L \in \mathscr{L}^s_m(E, K)$ we define the norms of $\hat{L}$ and L by

$$\|\hat{L}\| = \sup \{|\hat{L}(x)| : x \in \bar{B}_E\}$$

$$\|L\| = \sup \{|L(x_1, ..., x_m)| : x_i \in \bar{B}_E \ i = 1, ..., m\}.$$

If $n_1, ..., n_k$ are nonnegative integers whose sum is m, then we write $L(x_1^{n_1} \dots x_k^{n_k})$ for $L(x_1, ..., x_1, ..., x_k, ..., x_k)$, where x_1 appears n_1 times, x_2 appears n_2 times, and so on. We define $K(n_1, ..., n_k; E)$ to be the smallest number M with the property that for every $L \in \mathscr{L}^s_m(E, K)$

$$\sup \{|L(x_1^{n_1} \dots x_k^{n_k})| : x_i \in \bar{B}_E, \quad i = 1, ..., k\} \leq M \|\hat{L}\|.$$

We shall write $K(m; E)$ instead of $K(n_1, \dots, n_k; E)$ if $n_1 = \dots = n_k = 1$. It is shown in [7] that if E is a complex normed space then

$$\mathbb{C}(n_1, \dots, n_k; E) \leq \frac{n_1! \dots n_k! \, m^m}{n_1^{n_1} \dots n_k^{n_k} \, m!} \tag{1}$$

and in particular $\mathbb{C}(m; E) \leq \frac{m^m}{m!}$. Mazur and Orlich in the Scottish Book [11] conjectured that

$$K(m; E) \leq \frac{m^m}{m!}$$

and this conjecture was subsequently proved by Matrin [9]. If $E = \ell^1$, the space of all K-valued sequences $x = (x_i)$ such that $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ is finite, then the constant $\frac{m^m}{m!}$ is the best possible, that is

$$K(m; \ell^1) = \frac{m^m}{m!}.$$

In fact it was shown by Nachbin (see [5], p. 44) that there is an $L \in \mathscr{L}_m^s(\ell^1, K)$ such that $\|L\| = \frac{m^m}{m!} \|\hat{L}\|$.

If E is a real Banach space, Harris has given estimates for $\mathbb{R}(n_1, \dots, n_k; E)$ in the Scottish Book ([11], Problem 74). We now give an example to show that (1) does not hold when real Banach spaces are considered. First of all it is easy to see that for every $L \in \mathscr{L}_4^s(E, K)$ we have

$$\hat{L}(x) + \hat{L}(y) + 6L(x^2y^2) = \int_0^1 \hat{L}(r_1(t)x + r_2(t)y)dt$$

for any x, y in E, where r_n is the nth Rademacher function defined on $[0, 1]$ by $r_n(t) = \text{sign} \sin 2^n \pi t$. Thus

$$|L(x^2y^2)| \leq 3 \, \|\hat{L}\|$$

for all x, y in $\bar{B}_E$, and the following example show that

$$\mathbb{R}(2, 2; \ell_4^\infty) = 3 > \mathbb{C}(2, 2; E)$$

where ℓ_4^∞ is the space $\mathbb{R}^4$ equipped with the norm

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_4|\}.$$

Example 1. We choose $L \in \mathscr{L}^s_4(\ell^\infty_4, \mathbb{R})$ whose associated homogeneous polynomial is defined by

$$\hat{L}(x) = (x_1^2 - x_2^2)^2 - (x_3^2 - x_4^2)^2$$

where $x = (x_1, x_2, x_3, x_4) \in \ell^\infty_4$. It is easy to verify that $\|\hat{L}\| = 1$. We have

$$L(x^2y^2) = x_1^2y_1^2 + x_2^2y_2^2 - x_3^2y_3^2 - x_4^2y_4^2 - \frac{1}{3}\{x_1^2y_2^2 + 4x_1x_2y_1y_2 + x_2^2y_1^2\}$$

$$+ \frac{1}{3}\{x_3^2y_4^2 + 4x_3x_4y_3y_4 + x_4^2y_3^2\},$$

for every $x = (x_1, \ldots, x_4)$, $y = (y_1, \ldots, y_4)$ in ℓ^∞_4. By taking $x = (1, 1, 0, 1)$, $y = (1, -1, 1, 0)$ we get $|L(x^2y^2)| = 3$. Hence

$$\sup\{|L(x^2y^2)| : \|x\|_\infty = \|y\|_\infty = 1\} = 3\,\|\hat{L}\|.$$

This example is due to A.M. Tonge.

2. Bernstein's theorem for polynomials on normed spaces

If H is a Hilbert space, it is an old result that for every $L \in \mathscr{L}^s_m(H, K)$ we have $\|L\| = \|\hat{L}\|$. Kellogg [8] in 1928 and van der Corput and Schaake [4] in 1936 proved this result in the case where H is a finite dimensional real Hilbert space. Banach [1] also proved this result under the assumption that H is real and separable. For modern expositions, see [3] or [7].

If E is a normed space, $L \in \mathscr{L}^s_m(E, K)$ and $D\hat{L}(x)$ is the Fréchet derivative of $\hat{L}$ at x, we can verify easily that

$$D\hat{L}(x)y = mL(x^{m-1}y)$$

for every x, y in E. Thus $K(m; E) = 1$ iff $\|D\hat{L}(x)\| \leq m\,\|\hat{L}\|$ for every $L \in \mathscr{L}^s_m(E, K)$ and every x in B_E. In other words $K(m; E) = 1$ iff Bernstein's theorem holds for every homogeneous polynomial of degree m on E. We recall that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree m, the classical Bernstein's theorem asserts that its derivative P' satisfies the inequality

$$\|P'\|_\infty \leq m\,\|P\|_\infty$$

where the symbol $\|\cdot\|_\infty$ denotes the supremum norm taken over the unit disc. Harris in his commentary to Problem 73 in the Scottish Book ([11], pp. 144-145) asked whether there is a similar result for polynomials on C(K) spaces. The answer to this question is negative. In fact A. Tonge [14] gave an example of a homogeneous polynomial of degree 2 on the complex space l_3^∞ for which the inequality proposed by Harris fails.

If $T_m(\vartheta) = \sum_{k=-m}^{m} c_k e^{ik\vartheta}$ is a complex trigonometric polynomial of degree m, which satisfies $|T_m(\vartheta)| \le 1$ for all real ϑ, then

$$|T_m'(\vartheta)| \le m \quad \text{(Bernstein's inequality)} \tag{2}$$

for all real ϑ. It is an interesting fact that this inequality implies $\mathbb{C}(m; H) = 1$, where H is a complex Hilber space.

Proposition 1. *Let H be a complex Hilbert space and suppose $P : H \to \mathbb{C}$ is a polynomial of degree m satisfying $|P(x)| \le 1$ for every $x \in \bar{B}_H$. Then (2) implies*

$$\|DP(x)\| \le m$$

for all x in $\bar{B}_H$.

Proof: We put

$$T_m(\vartheta) = (x\cos\vartheta + i\sigma y\sin\vartheta)$$

for every x, y in $\bar{B}_H$, where $\sigma = \begin{cases} 1, & \text{if } (x, y) = 0 \\ \dfrac{(x, y)}{|(x, y|)}, & \text{if } (x, y) \neq 0. \end{cases}$

We see that $T_m(\vartheta)$ is a complex trigonometric polynomial of degree less or equal to m. Since $\|x\cos\vartheta + i\sigma y\sin\vartheta\|_2 \le 1$, we have $|T_m(\vartheta)| \le 1$ for all real ϑ. Thus (2) implies $|T_m'(\vartheta)| \le m$ for all real ϑ, and in particular $|T_m'(0)| = |DP(x)i\sigma y| \le m$. It follows that

$$|DP(x)y| \le m$$

and this proves the proposition.

It is well known (see [13]) that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree m, satisfying $|P(z)| \le 1$ for every $|z| = 1$, then $|(z^{-m/2}P(z))'| \le \frac{m}{2}$ or

$$|mP(z) - 2zP'(z)| \le m. \tag{3}$$

Next, we obtain a result which is a generalization of inequality (3).

Proposition 2: *The following are equivalent:*
(a) Bernstein's inequality for complex trigonometric polynomials.
(b) If E is a complex normed space and $P : E \to \mathbb{C}$ is a polynomial of degree m satisfying $|P(x)| \leq 1$ for every $x \in \bar{B}_E$, then

$$|mP(x) - 2DP(x)x| \leq m \tag{4}$$

for all $x \in \bar{B}_E$.

Proof: If (b) holds, then for $E = \mathbb{C}$ we get inequality (3) and this implies Bernstein's inequality (see [13], p. 377).

We prove now that (a) implies (b). We define

$$Q(<x, \lambda>) = \sum_{i=0}^{m} P_i(x)\lambda^{m-i}$$

where $P(x) = \sum_{i=0}^{m} P_i(x)$, and $P_i : E \to \mathbb{C}$ is a homogeneous polynomial of degree i, $i = 0, \ldots, m$. Clearly Q is a homogeneous polynomial of degree m on $E \times \mathbb{C}$ and $Q(<x, \lambda>) = \lambda^m P\left(\frac{x}{\lambda}\right)$. We put

$$T_m(\vartheta) = Q(<x, 1> \cos\vartheta + i <-x, 1> \sin\vartheta)$$

for every $x \in \bar{B}_E$. Then $T_m(\vartheta)$ is a complex trigonometric polynomial of degree less or equal to m.

Since $|T_m(\vartheta)| = \left| P\left(\frac{x(\cos\vartheta - i\sin\vartheta)}{\cos\vartheta + i\sin\vartheta}\right)\right|$ and $\left\| \frac{x(\cos\vartheta - i\sin\vartheta)}{\cos\vartheta + i\sin\vartheta} \right\| \leq 1$, we have $|T_m(\vartheta)| \leq 1$ for all real ϑ. Thus

$$|T_m'(0)| = |DQ(<x, 1>)\, i <-x, 1>| \leq m. \tag{*}$$

Since it is easily verified that the identity

$$DQ(<x, 1>) <y, 1> = DP(x)y + mP(x) - DP(x)x$$

holds, (*) proves (4).

Note that (4) is obvious when P is a homogeneous polynomial of degree m.

If H is a complex Hilbert space and $P : H \to \mathbb{C}$ is a polynomial of degree m, satisfying $|P(x)| \leq 1$ for every $x \in \bar{B}_H$, Harris proved (see [7], p. 150) that

$$|mP(x) - DP(x)x| + \|DP(x)\| \leq m \tag{5}$$

for all x in $\bar{B}_H$. This is a refinement of inequality (4) in the case where E is a Hilbert space. Harris observed also that inequality (5) in the case $H = \mathbb{C}$, is eqivalent to $\mathbb{C}(m; \ell_2^\infty) = 1$.

Using (5), we can show that for the non-Hilbert space $H \times \mathbb{C}$, with the supremum norm, we have $\|L\| = \|\hat{L}\|$ for eyery $L \in \mathscr{L}_m^s(H \times \mathbb{C}, \mathbb{C})$.

Proposotion 3. *If H is a complex Hilbert space then for the space $H \times \mathbb{C}$, with the supremum norm, we have*

$$\mathbb{C}(m; H \times \mathbb{C}) = 1.$$

Proof: Suppose that $\dim(H) < \infty$. If Q is a homogeneous polynomial of degree m on $H \times \mathbb{C}$, then

$$Q(<x, z>) = z^m P\left(\frac{x}{z}\right)$$

where P is a polynomial of degree m on H and $x \in H$, $z \in \mathbb{C}$. We have, using the maximum modulus principle, that

$$\|Q\| = \sup_{\|<x,z>\|=1} \left| z^m P\left(\frac{x}{z}\right)\right| = \sup_{\|x\|_2 \leq 1} |P(x)| = \|P\|.$$

To prove the proposition, it is enough to prove that

$$|DQ(<x, z>) <y, w>| \leq m \|Q\|,$$

for every $<x, z>$, $<y, w>$ in $\bar{B}_{H\times\mathbb{C}}$.

From the maximum modulus principle, we have to show that

$$|DQ(<x, 1>) <y, 1>| \leq m \|Q\|$$

for every x, y in $\bar{B}_H$. But

$$DQ(<x, 1>) <y, 1> = DP(x) + mP(x) - DP(x)x$$

and (5) implies

$$|DQ(<x, 1>) <y, 1>| \leq m\|P\| = m \|Q\|$$

for every x, y in $\bar{B}_H$. The proof, in the case where H is an arbitrary Hilbert space, follows easily.

3. Polynomials on vector valued L^p-spaces and c_p spaces.

If E is a Banach space, we denote by $L^p(\mu, E) = L^p(\Omega, \Sigma, \mu, E)$, $1 \leq p < \infty$, the Banach space of equivalence classes of μ-Bochner integrable functions $f : \Omega \to E$ such that

$$\|f\|_p = \left(\int_\Omega \|f\|^p \, d\mu\right)^{1/p} < \infty.$$

By $L^\infty(\mu, E)$ we denote all (equivalence classes of) essentially bounded μ-Bochner integrable functions $f : \Omega \to E$ such thtat

$$\|f\|_\infty = \text{ess sup}(\|f(\omega)\| : \omega \in \Omega) < \infty.$$

The Banach space $\ell^p(E)$, $1 \leq p \leq \infty$, is the collection of all sequences $x = (x_i)$ of elements of E, such that

$$\|x\|_p = \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{1/p} \quad (\|x\|_\infty = \sup\{\|x_i\| : i \in \mathbb{N}\} \quad \text{if} \quad p = \infty)$$

is finite. When E is the Banach space of scalars, we use the symbols $L^p(\mu)$, ℓ^p for $L^p(\mu, E)$ and $\ell^p(E)$ respectively. We have mentioned in the introduction that $K(m; \ell^1) = \dfrac{m^m}{m!}$. The next example shows that ℓ^1 may be replaced by $\ell^1(E)$.

Example 2. Consider the Banach space $\ell^p(E)$, $1 \leq p < \infty$, and let $a \in E$ with $\|a\| = 1$. By the Hahn-Banach theorem there is a $\phi \in E^*$ with norm 1, such that $\phi(a) = \|a\| = 1$. Let $L \in \mathscr{L}_m^s(\ell^p(E), K)$ be defined by

$$L(x^1, \ldots, x^m) = \frac{1}{m!} \sum_{\sigma \in S_m} \phi(x_1^{\sigma(1)}) \ldots \phi(x_m^{\sigma(m)})$$

where $x^i = (x_n^i)_{n=1}^\infty$ for $i = 1, \ldots, m$ and S_m is the set of permutations of the first m natural numbers. Take

$$e^j = (0, \ldots, 0, a, 0, \ldots) \quad (a \text{ at the jth coordinate})$$

and define

$$y^1 = n_1^{-1/p}\,(e^1 + \ldots + e^{n_1})$$

$$\vdots$$

$$y^k = n_k^{-1/p}\,(e^{n_1 + \ldots + n_{k-1} + 1} + \ldots + e^{n_1 + \ldots + n_k})$$

where $n_1, \ldots, n_k$ are nonnegative integers whose sum is m. Then an easy calculation, as in [12] example 1, shows that $y^1, \ldots, y^k$ are unit vectors in $\ell^p(E)$ and

$$|L((y^1)^{n_1} \ldots (y^k)^{n_k})| \geq \frac{n_1! \ldots n_k! \, m^{m/p}}{n^{n_1/p} \ldots n^{n_k/p} \, m!} \|\hat{L}\|.$$

In particular, by taking $n_1 = \dots = n_k = 1$ and $p = 1$, we have

$$\|L\| = \frac{m^m}{m!} \|\hat{L}\|.$$

The previous example implies that if $m \leq k$, then $K(m; \ell_k^1(E)) = \frac{m^m}{m!}$.

The following result gives the best constant $\mathbb{C}(m; \ell_k^1)$ for any positive integers $m, k = 2, 3, \dots$

Proposition 4. *For any positive integers* $m, k = 2, 3, \dots$ *we have*

$$\mathbb{C}(m; \ell_k^1) = \begin{cases} \dfrac{m^m}{m!}, & \text{if } m \leq k \\ \max \dfrac{n_1! \dots n_k!\, m^m}{n_1^{n_1} \dots n_k^{n_k}\, m!}, & \text{if } m > k \end{cases} \tag{6}$$

where the maximum is taken over all k-tuples $(n_1, \dots, n_k)$ *of nonnegative integers whose sum is m.*

Proof. We have to consider only the case where $m > k$. First observe that as $L \in \mathscr{L}_m^s(\ell_k^1, \mathbb{C})$ is symmetric, and as the norm of a multilinear form is the greatest absolute value at the basis vectors e_j,

$$\|L\| = \max |L(e_1^{n_1} \dots e_i^{n_i})|$$

where the maximum is taken over all i-tuples $(n_1, \dots, n_i)$ of nonnegative integers whose sum is m, $1 \leq i < m$. The proof now follows from inequality (1) and from the fact that (1) becomes equality when $E = \ell_k^1$, $k < m$ (see [7], theorem 1).

Note that if $k = m - 1$, then we have from (6)

$$\mathbb{C}(m; \ell_{m-1}^1) = \frac{1}{2} \frac{m^m}{m!}.$$

Next, by taking E to be an $L^q(\nu)$ space, we give estimates for $K(m; L^p(\mu, E))$. We shall show that most of the results for $K(m; L^p(\mu))$ in [12] hold for appropriate values of p and q. First we need a lemma. If we use the notation $\|f\|_{q,p}$ for the norm of $f \in L^p(\mu, L^q(\nu))$, we have:

Lemma 1. *Let* Ω *be a measure space with a finite measure* λ, *and let* $1 \leq p \leq 2$, $p \leq q \leq p'$, *where* p' *is the conjugate exponent of* p. *If* *we have an orthonormal set of K-valued measurable functions* s_i *satisfying* $|s_i(t)| \leq 1$ *a.e.,* $1 \leq i \leq m$, *then*

$$\left[\int_\Omega \left\| \sum_{i=1}^m s_i(t) f_i \right\|_{q,p}^{p'} d\lambda(t)\right]^{1/p'} \leq \left(\sum_{i=1}^m \|f_i\|_{q,p}^p\right)^{1/p} \tag{7}$$

for each $f_1, \ldots, f_m$ *in* $L^p(\mu, L^q(\nu))$.

Proof. Working as in the proof of lemma 1 in [12] (see also [15]) we get the following inequalities:

$$\left[\int_\Omega \left\| \sum_{i=1}^m s_i(t) f_i \right\|_{p,p}^{p'} d\lambda(t)\right]^{1/p'} \leq \left(\sum_{i=1}^m \|f_i\|_{p,p}^p\right)^{1/p} \tag{a}$$

$$\left[\int_\Omega \left\| \sum_{i=1}^m s_i(t) f_i \right\|_{p',p}^{p'} d\lambda(t)\right]^{1/p'} \leq \left(\sum_{i=1}^m \|f_i\|_{p',p}^p\right)^{1/p} \tag{b}$$

where $1 \leq p \leq 2$. The result of applying an extended version of the Riesz-Thorin interpolation theorem [2] to (a) and (b) is the inequality (7).

If $\Omega = [0, 1]$, λ is Lebesgue measure on $[0, 1]$ and $s_i = r_i$ is the ith Rademacher function, them we have the following generalized Clarkson inequality

$$\left[\int_0^1 \left\| \sum_{i=1}^m f_i r_i(t) \right\|_{q,p}^{p'} d\lambda(t)\right]^{1/p'} \leq \left(\sum_{i=1}^m \|f_i\|_{q,p}^p\right)^{1/p} \tag{8}$$

for each $f_1, \ldots, f_m$ in $L^p(\mu, L^q(\nu))$, where $1 \leq p \leq 2$, $p \leq q \leq p'$.

If E is a normed space over K and $L : E \to K$ is a symmetric m-linear mapping, then we have (see [12], lemma 2)

$$L(f_1, \ldots, f_m) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t)\, \hat{L}\left(\sum_{i=1}^m f_i r_i(t)\right) dt$$

for every $f_1, \ldots f_m$ in E.

The above polarization formula, together with inequality (8) and example 2, imply the following result (c.f. [12], theorem 2).

Proposition 5. ***For*** $1 \leq p \leq m'$, $p \leq q \leq p'$, *we have*

$$K(m; L^p(\mu, L^q(\nu))) = \frac{m^{m/p}}{m!}.$$

If $2 \leq p \leq \infty$, $p' \leq q \leq p$, we have another generalized Clarkson inequality:

$$\left[\int_0^1 \Big\| \sum_{i=1}^{m} f_i r_i(t) \Big\|_{q,p}^{p} dt\right]^{1/p} \leq \left(\sum_{i=1}^{m} \|f_i\|_{q,p}^{p'}\right)^{1/p'} \tag{9}$$

for each $f_1, ..., f_m$ in $L^p(\mu, L^q(\nu))$. We omit the proof of this inequality which is similar to the proof of lemma 1.

Proposition 6. *For* $p \geq m$ *and* $p' \leq q \leq p$ *we have*

$$K(m; L^p(\mu, L^q(\nu)) \leq \frac{m^{m/p'}}{m!}.$$

The proof of this proposition follows easily from inequality (9), using the polarization formula. The constant $\frac{m^{m/p'}}{m!}$ *is by no means the best possible, as the example in [12], p. 268 shows.*

If

$$K(m; p, q) = \sup\{K(m; L^p(\mu, L^q(\nu))) : \mu, \nu \text{ are measures}\}$$

the propositions 5 and 6 imply that $K(2; p, q) = 2^{2/p-1}$ *for* $1 \leq p \leq 2$, $p \leq q \leq p'$ *and* $K(2; p, q) \leq 2^{1-2/p}$ *for* $2 \leq p \leq \infty, p' \leq q \leq p$. *The next example shows that* $\mathbb{R}(2; p, q) = 2^{1-2/p}$, *for* $2 \leq p \leq \infty$, $p' \leq q \leq p$.

Example 3. Consider the real space ℓ_2^p, $1 \leq p \leq \infty$. Let L be a symmetric bilinear form on ℓ_2^p defined by

$$L(x, y) = x_1 y_1 - x_2 y_2$$

for every $x = (x_1, x_2), y = (y_1, y_2)$ in ℓ_2^p. Then

$$\|\hat{L}\| = \sup\{|x_1^2 - x_2^2| : |x_1|^p + |x_2|^p \leq 1\} = 1$$

and for $x = (2^{-1/p}, 2^{-1/p})$, $y = (2^{-1/p}, -2^{-1/p})$, $L(x, y) = 2^{1-2/p}$ Hence

$$\|L\| = 2^{1-2/p} \|\hat{L}\|.$$

Corollary. *For the real valued continuous symmetric bilinear forms on every* $L^p(\mu, L^q(\nu))$ *space,* $1 \leq p \leq \infty, p \leq q \leq p'$ $(p' \leq q \leq p)$, *the constant*

$$2^{|p-2|/p}$$

is the best possible

To find the estimates for $\mathbb{C}(m; p, q)$ *in the case where* $2 \le p \le m$, $m = 2^n$, *for appropriate values of* q, *we need the following result.*

Lemma 2. *Consider the complex Banach space* $L^p(\mu, L^q(\nu))$, $q \le p \le 2m$, $2 \le q \le 2m$. *Then for every* $L \in \mathscr{L}^s_{2m}(L^p(\mu, L^q(\nu)), \mathbb{C})$ *we have*

$$|L(x^m y^m)| \le \|\hat{L}\| \tag{10}$$

where x, y are unit vectors in $L^p(\mu, L^q(\nu))$.

Proof. We define $f(z) = \hat{L}(x + zy)$, for $z \in \mathbb{C}$. Then we can easily check that

$$f^{(m)}(0) = \frac{(2m)!}{m!} L(x^m y^m).$$

Since f is a polynomial we have

$$f^{(m)}(0) = \frac{m!}{2\pi i} \int_\gamma \frac{f(z)}{z^{m+1}} dz$$

where $\gamma(\vartheta) = e^{i\vartheta}$ $(0 \le \vartheta \le 2\pi)$. So

$$|L(x^m y^m)| \le \|\hat{L}\| \frac{m!\, m!}{(2m)!} \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\vartheta} y\|^{2m}_{q,p}\, d\vartheta.$$

Hence to prove (10), it suffices to show that

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\vartheta} y\|^{2m}_{q,p}\, d\vartheta \le \frac{1}{2} \binom{2m}{m} (\|x\|^{2m}_{q,p} + \|y\|^{2m}_{q,p}) \tag{11}$$

for all x, y in $L^p(\mu, L^q(\nu))$. The following inequality

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\vartheta} y\|^{2m}_{2}\, d\vartheta \le \frac{1}{2} \binom{2m}{m} (\|x\|^{2m}_{2} + \|y\|^{2m}_{2}) \tag{a}$$

is due to Harris (see [7], p. 160) and holds for all x, y in $L^2(\mu)$. Using Harris' inequality, we also have

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\vartheta} y\|^{2m}_{2m}\, d\vartheta \le \frac{1}{2} \binom{2m}{m} (\|x\|^{2m}_{2m} + \|y\|^{2m}_{2m}) \tag{b}$$

for all x, y in $L^{2m}(\mu)$. The result of applying the extension of Riesz-Thorin interpolation theorem to (a), (b) is the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \|x+e^{i\vartheta}y\|_q^{2m}\,d\vartheta \le \frac{1}{2}\binom{2m}{m}\left(\|x\|_q^{2m}+\|y\|_q^{2m}\right) \tag{c}$$

for all x, y in $L^q(\mu)$, $2 \le q \le 2m$. We deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \|x+e^{i\vartheta}y\|_{q,q}^{2m}\,d\vartheta \le \frac{1}{2}\binom{2m}{m}\left(\|x\|_{q,q}^{2m}+\|y\|_{q,q}^{2m}\right) \tag{d}$$

holds for all x, y in $L^q(\mu, L^q(\nu))$, $2 \le q \le 2m$.

Using inequality (c) we see that

$$\frac{1}{2\pi}\int_0^{2\pi} \|x+e^{i\vartheta}y\|_{q,2m}^{2m}\,d\vartheta \le \frac{1}{2}\binom{2m}{m}\left(\|x\|_{q,2m}^{2m}+\|y\|_{q,2m}^{2m}\right) \tag{e}$$

for all x, y in $L^{2m}(\mu, L^q(\nu))$. Now interpolating again between (d) and (e) the result is (11).

Proposition 7. *If* $m = 2^n, n \ge 2$, *then*

$$\mathbb{C}(2^n; p, q) \le \begin{cases} \mathbb{C}^2(2^{n-1}; \ p, q) \ , & 2^{n-1} \le q \le p \le 2^n \\ \mathbb{C}^{2^2}(2^{n-2}; p, q) \ , & 2^{n-2} \le q \le p \le 2^{n-1} \\ \vdots & \\ \mathbb{C}^{2^{n-1}}(2; p, q) \ , & 2 \le q \le p \le 2^2. \end{cases}$$

Proof. If $n = 2$, a simple argument, as in the proof of theorem 3 in [12], shows that

$$\mathbb{C}(4; p, q) \le \mathbb{C}^2(2; p, q), \quad 2 \le q \le p \le 4.$$

The proof now of the proposition follows easily by induction on n.

Note that if $n = 2$ then $\mathbb{C}(4; p, q) \le 2^{2-4/p}$, for $2 \le q \le p \le 4$.

Finally we consider the c_p spaces. We recall that by c_p we denote the Banach space of compact opereators T on a Hilbert space H such that

$$\|T\|_p = (\mathrm{tr}(T^*T)^{p/2})^{1/p} < \infty.$$

Since the generalized Clarkson type inequalities hold also in the c_p spaces (see [6]) we have, as in the case of $L^p(\mu)$ spaces, that

$$K(m; c_p) \le \frac{m^{m/p}}{m!}, \quad \text{for} \quad 1 \le p \le m'.$$

Now using the fact that there exists an isometric embedding of ℓ^p into c_p (see [10], p. 267), and lemma 0 in [12], we have

$$K(m; \ell^p) \leq K(m; c_p).$$

Hence, for $1 \leq p \leq m'$, we have

$$K(m, c_p) = \frac{m^{m/p}}{m!}.$$

REFERENCES

1. S. Banach, Über homogene polynome in (L^2). Studia Math. 7 (1938), 36-44.

2. A. Benedek and R. Panzone, The spaces L^P, with mixed norm. Duke Math. J. 28 (1961), 301-324.

3. J. Bochnak and J. Siciak, Polynomials and multilinear mappings in topological vector spaces. Studia Math. 39 (1971), 59-76.

4. J.G. van der Corput und G. Schaake, Ungleichungen für Polynome und Trigonometrishe Polynome, Compositio Math. 2 (1935), 321-361; Berichtigung ibid. 3, 1936, p. 128.

5. S.B. Chae, Holomorphy and calculus in normed spaces, Marcel Dekker, New York, 1985.

6. I.C. Gohberg and M. Krein, Introduction to the theory of linear non self-adjoint operators, AMS (Translations), vol. 18 (1969).

7. L.A. Harris, Bounds on the derivatives of holomorphic functions of vectors. Colloque d' analyse, Rio de Janeiro, 1972, ed. L. Nachbin, Act. Sc. et Ind. no. 1367 (Hermann, 1975), 145-163.

8. O.D. Kellogg, On bounded polynomials in several variables, Math Zeit. 27 (1928), 55-64.

9. R.S. Martin, Thesis, California Institute of Technology (1932).

10. C.A. Mc Carthy, c_p. Israel J. of Math. 5 (1967), 249-271.

11. The Scottish Book (Mathematics from the Scottish Café), ed. R.D. Mauldin (Birkhäuser, 1981).

12. I. Sarantopoulos, Estimates for polynomial norms on $L^p(\mu)$ spaces, Math. Proc. Cambridge Philos. Soc. 99 (1986), 263-271.

13. V.I. Smirnov — N.A. Lebedev, Functions of a Complex Variable, Illife Books Ltd. (1968).

14. A.M. Tonge, The failure of Bernestein's theorem for polynomials on C(K)-spaces. J. of Approximation Theory (to appear).

15. L. Williams and J. Wells, L^p inequalities, J. Math. Anal. Appl. 64 (1978), 518-529.

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Department of Mathematics, National Technical University, Athens, Greece.