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SOME RESIDUAL PROPERTIES OF CERTAIN HNN EXTENSIONS

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Introduction

If $\mathscr{X}$ is a class or a property of groups, then $r\mathscr{X}$ denotes the class of those groups which are residually in $\mathscr{X}$. We recall that $G \in r\mathscr{X}$, if for each $g \in G$, $g \neq 1$ there exists a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathscr{X}$ or G/N has the property $\mathscr{X}$.

In the rest $r\mathscr{F}$, $r\mathscr{P}$ and $r\mathscr{N}$ will mean residually finite, residually free and residually nilpotent groups respectively.

In this paper we deal with some residually properties of certain HNN extensions with base group a finitely generated (in short f.g.) abelian group.

The residually finiteness of HNN extensions with base group a f.g. abelian group and associated subgroups of finite index in the base group was studied in [1].

Here in the first section we study the same problem, in some cases, when the associated subgroups are of infinite index in the base group.

In the second section we study the residually freeness of HNN extensions with base group a f.g. free abelian group.

The residually nilpotency of some HNN extensions with base group a f.g. free abelian group is studied in the last section of this paper.

1. Residually finite HNN extensions

Proposition 1. *Let K be a f.g. abelian group, $A, B \leq K$ and $\phi : A \to B$ an isomorphism, let $G = \langle t, K \mid t^{-1}At = B, \phi\rangle$ be the corresponding HNN extension of K. If $A \cap B$ is finite, then G is $r\mathscr{F}$.*

Proof. We can assume that A, B are infinite, since if A, B are finte G is rF ([3]). Also A, B are of infinite index in K, since $A \cap B$ is finite.

Let $A = A_1 \times A_2$, $B = B_1 \times B_2$, where $A_1, B_1 = A_1\phi$ are the torsion parts of A and B and $A_2, B_2 = A_2\phi$ are torsion free. Obviously

$A \cap B = A_1 \cap B_1$ and $A_2 \cap B_2 = 1$. Let $K = K_1 \times K_2$, where K_1 is the torsion part of K and K_2 is torsion free such that

$$<A_2, B_2> = A_2 \times B_2 \leq K_2.$$

We can find $C_2 \leq K_2$ such that $(A_2 \times B_2) \cap C_2 = 1$ and $|K_2 : A_2B_2C_2| < \infty$. We consider the family $N_\nu = (A_2B_2C_2)^\nu$ $\nu \in \mathbb{N}$ of subgroups of K, where by $(A_2B_2C_2)^\nu$ we denote the subgroup of $A_2B_2C_2$ generated by all the ν-th powers of elements of $A_2B_2C_2$. It is easy to see that $AN_\nu = AB_2^\nu C_2^\nu$, $BN_\nu = A_2^\nu BC_2^\nu$ and $(A \cap N_\nu)\phi = (A_2)\phi = B_2 = B \cap N_\nu$. So ϕ induces, in a natural way, an isomorphism

$$\phi_\nu : AN_\nu / N_\nu \to BN_\nu / BN_\nu \quad \nu \in \mathbb{N}$$

with $(aN_\nu)\phi_\nu = a\phi N_\nu$ $a \in A$, $\nu \in \mathbb{N}$. Thus we can define the HNN extension

$$G_\nu = <\tau_\nu, K / N_\nu \mid \tau_\nu^{-1} AN_\nu / N_\nu \tau_\nu = BN_\nu / N_\nu, \phi_\nu>$$

Each G_ν, $\nu \in \mathbb{N}$ is $r\mathcal{F}$ as an HNN extension of a finite group (K / N_ν is finite because of $|K_2 : A_2B_2C_2| < \infty$. Now as in Th. 1. of [1] we have that G is $r\mathcal{F}$.

Proposition 2. *Let K be a f.g. abelian group, $<u>$, $<w>$ cyclic isomorphic subgroups of K. Let $G = <t, K \mid t^{-1}ut = w>$ be the HNN extension of K with associated subgroups the cyclic subgroups $<u>$ and $<w>$. Then G is $r\mathcal{F}$* if and only if $<u>$ and $<w>$ are finite or $<u>$, $<w>$ are infinite but $<u> \cap <w> = 1$ or there exists a promitive torsion free element s of K such that $u = s^p = w^{\pm 1}$.

Proof. Let G be $r\mathcal{F}$. Suppose that $<u>$, $<w>$ are infinite and $<u> \cap <w> \neq 1$. Then there exists a primitive torsion free element s of K such that $u = s^p$ and $w = s^q$. Now as in Th. 2 of [1] have that $p = |q|$.

Conversely, if $<u>$, $<w>$ are finite or $<u> \cap <w> = 1$, then from [3] and from Proposotion 1 above we have that G is $r\mathcal{F}$. Let $<u>$, $<w>$ be infinite and $u = s^p = w^{\pm 1}$, where s is a primitive torsion free element of K, then again as in Th. 2 of [1] we have that G is $r\mathcal{F}$.

The residually finitness of an HNN extension with base group a. f.g. abelian group and associated subgroups of finite index in the base group was studied in [1] (Th. 1).

In the case where the associated subgroups are of infinite index in the base group we have.

Proposition 3. *Let K be a f.g. abelian group, A, B isomorphic subgroups of K of infinite index in K and $A \cap B$ of finite index in A, B then the HNN extension $G = <t, K \mid t^{-1}At = B, \phi>$ is $r\mathcal{F}$* is if and only if there exists $H \lhd_f A \cap B$ such that $H \lhd G$, where in $H \lhd_f A \cap B$ the symbol f means that H is of finite index in $A \cap B$.

Before giving the proof of the Proposition above we shall need two Lemmas.

Lemma A. *Let $G = <t, K \mid t^{-1} At = B, \phi>$ be the HNN extension of the group, K with associated subgroups A and B. Let $K_1 \leq K$, $A_1 \leq A$, $B_1 \leq B$ be such that the subgroup $G_1 = <t, K_1>$ has the presentation $G_1 = <t, K_1 \mid t^{-1}A_1t = B_1, \phi_1 = \phi_{|A_1}>$. If H is a normal subgroup of G_1, such that $H \lhd K$, then $H \lhd G$.*

Proof. The proof is obvious because $H \lhd G_1 \Rightarrow t^{-1}Ht = H$, but $H \lhd K$. So we have that $H \lhd G$.

Lemma B. *Let $K = <x_1, ..., x_r>$ be the free abelian group of rank r. Let $\phi : <x_1, ..., x_\lambda> \to <x_1, ..., x_\lambda>$ $\lambda < r$ be a proper monomorphism. Then the group $G = <t, x_1, ..., x_r \mid t^{-1}x_it = x_i\phi$, $i = 1, ..., \lambda$, $[x_i, x_j] = 1$, $i \neq j>$ is non Horfian.*

Proof. The map $\vartheta : G \to G$ with $t \to t$, $x_i \to x_i\phi$, $i = 1, ... \lambda$, $x_j \to x_j$, $j = \lambda + 1, ... r$ is obviously an ependomorphism. But, because ϕ is a proper monomorphism, there exists x_i such that $x_i \notin (<x_1, ..., x_\lambda>)\phi$. So the non trivial element $[tx_it^{-1}, x_r]$ belongs to $\mathrm{Ker}\vartheta$.

Proof of the Proposotion 3.

We assume that there axists $H \lhd_f A \cap B$ such that $H \lhd G$. Since $H\phi = H$ we can define the HNN extension $\overline{G} < \tau, K / H \mid \tau^{-1} (A / H) t = B / H, \overline{\phi}>$, where $(kH) \overline{\phi} = k\phi H$. It is obvious that $G / H \simeq \overline{G}$. But the $\overline{G}$ is $r\mathcal{F}$ from the Proposition 1, since $|A / H \cap B / H| < \infty$. So as in proof of Th.1 of [1] we have that G is $r\mathcal{F}$.

Suppose that G is $r\mathcal{F}$, then if $A \cap B$ is finite, there exists $H = 1$ such that $H\phi = H$ and $H \lhd_f A \cap B$. For this we can suppose that

$A \cap B$ is infinite. Let $A = A_1 \times A_2$, $B = B_1 \times B_2$, where $A_1, B_1 = A_1\phi$ are the torsion parts of A and B. We choose $\{x_1, ..., x_r\}$ a subset of a minimal generating set of K such that $<x_1, ..., x_r>$ is torsion free and such that $A_2 = <x^{k_1}, ... x^{k_\lambda}>, \lambda \leq r$ and $B_2 = <x^{k_1}\phi, ..., \chi^{k_\lambda}\phi>$.

We define the following set of subgroups of K, $\mathcal{N} = \{N \mid N \lhd_f K$ and $(A \cap N)\phi = B \cap N\}$. Since G is $r\mathcal{F}$ we have that $\bigcap_{N\in\mathcal{N}} N = 1$ ([4]). Also it easy to see that $\bigcap_{N\in\mathcal{N}} AN = A$ and $\bigcap_{N\in\mathcal{N}} BN = B$, because if we suppose that there exists $h \in \bigcap_{N\in\mathcal{N}} AN \setminus A$, then the element $1 \neq [t^{-1}ht, b]$, $b \notin B$ belongs in $\bigcap_{N\in\mathcal{N}} N$, a contradiction (see Prop. 2 of [1]).

The subgroup B_2 is subgroup of $K_2 = <x_1, ..., x_\lambda>$ because the $A \cap B$ is of finite index in A, B. If $A_2 = K_2 = B_2$, then there exists $H = A_2 = K_2 = B_2$ with $H \lhd_f A \cap B$. So we can suppose that both A_2 and B_2 are proper subgroups of K_2. Since in the case where $A_2 = K_2$ and $B_2 \neq K_2$, then the subgroup $G_2 = <t, x_1, ..., x_r | t^{-1}A_2t = B_2, \phi_{|A_2}>$ of G is non Hopfian (Lem. B), a contradiction, because G is $r\mathcal{F}$. Let $\{1, a_1, ..., a_{n-1}\}$ be a transversal of A_2 in K_2 and $\{1, b_1, ..., b_{m-1}\}$ be a trasversal of B_2 in K_2. Then, because A_2, B_2 are proper subgroups of K_2, as in the proof of the Theoten 1 in [1] we have that there exists $H \lhd \overline{G} = <t, x_1, ..., x_\lambda | t^{-1}A_2t = B_2, \phi_{|A_2}>$ such that $H \lhd_f K_2$. But from Lem. A we have that $H \lhd G$ and the proof of the proposition is complete.

Corollary 4. *Let A, B, K, ϕ, G be as in the Proposition 3. If G is $r\mathcal{F}$, then $|K_2 : A_2| = |K_2 : B_2|$, where K_2, A_2, B_2 are the torsion free parts of K, A and B respectively.*

Proof. From the Proposition 3 we have that there exists $H \lhd_f A \cap B$ such that $H \lhd G$. Let $H_2 = K_2 \cap H$. Then $\phi : A_2 \to B_2$ implies an isomorphism $\bar{\phi} : A_2 / H_2 \to B_2 / H_2$. But $A_2 / H_2, B_2 / H_2$ are finite. So $|K_2 : A_2| = |K_2 : B_2|$.

Corollary 5. *The group $G = <t, a_1, ..., a_n \mid t^{-1}a_i^{p_i}t = a_i^{q_i}, i = 1, ..., r, r < n, [a_i, a_j] = 1 \; 1 \neq j>$ is rF if and only if $|p_i| = q_{ij}, i = 1, ..., r$.*

Proof. The proof follows easily from Lemma B and the Proposition 3.

2. Residyally free HNN extensions

Proposition 6. *Let K be a f.g. abelian group, $A, B \leq K$ and $\phi : A \to B$ an isomorphism. Let $G = <t, K \mid t^{-1} At = B, \phi>$ be the corresponding HNN extension of K. If the subgroups A and B are of finite index in K, then G is $r\mathscr{P}$* if and only if K is free avelian of rank k and G is free abelian of rank $k+1$.

Prof. If G is free avelian, then G is $r\mathscr{P}$ and $r(K) = r(G) - 1$. We assume that G is $r\mathscr{P}$. Then G has no elements of finite rank. So K is free avelian. Also G is $r\mathscr{F}$, therefore from Th. 1 in [1] we have that $A = K$ or $B = K$ or there exists $H \lhd_f K$ such that $H\phi = H$. In the two first cases we have that K^G is abelian and in third case we have that H is abelian. Therefore from Lem. 1 in [2] we have that $K^G \leq Z(G)$ or $H \leq Z(G)$. So G has the presentation

$$G = <t, a_1, ..., a_k \mid t^{-1}a_i\, t = a_i,\ \ i = 1, ..., k,\ [a_1, a_j] = 1,\ 1 \neq j>,$$

namely G is free avelian of rank $r(G) = r(K) + 1$, or

$$G = <t, \dot{a}_1, ..., a_k \mid t^{-1}a_i^{p_i}t = a_i^{p_i}, i = 1, ..., k, [a_1, a_j] = 1, 1 \neq j>,$$

in the second case we have $G / Z(G) \simeq <t, a_1, ..., a_k \mid [a_1, a_j] = 1, 1 \neq j, a_i^{p_i} = 1, i = 1, ..., k>$, but from Lem. 4 in [2] $G / Z(G)$ is $r\mathscr{P}$. So $|p_i| = 1$ and G is free abelian.

Proposition 7. *Let K, A, B ϕ and G be as in Proposition 6. If $A \cap B$ is of finite incex in A, B, then G is $r\mathscr{P}$ if and only if G has the presentation*

$$G = <t, a_1, ..., a_k \mid t^{-1}a_it = a_i, i = 1, ..., r, r < k, [a_1, a_j] = 1, i \neq j>.$$

Proof. We assume that A, B are of infinite index in K, since in the case, where A, B are of finite index the result follows from the previous proposition with $r = k$.

If G is $r\mathscr{P}$, the G is $r\mathscr{F}$ and from the Proposition 3 we have that there exists $H \lhd G$ such that H is of finite index in $A \cap B$. So as in the previous proposition we have that

$$G = <t, a_1, ..., a_k \mid t^{-1}a_1t = a_1, ..., t^{-1}a_rt = a_r, r < k, [a_i, a_j] = 1, i \neq j>.$$

Conversely, let G have the presentation $G = <t, a_1, ..., a_k \mid t^{-1}a_it = a_i, i = 1, ..., r, r < k, [a_l, a_j] = 1, l \neq j>$. Then every element of G is of the form $g = zw(t, a_j)$, where $z \in Z(G) = <a_1, ..., a_r>$ and $w(t, a_j)$ is a word in $t, a_j, j = r+1, ..., k$. Let $g \neq 1$, then if $z = 1$, we have that $w(t, a_j) \neq 1$ and $w(t, a_j) \notin Z(G)$. But

$$G / Z(G) \simeq <t> * <a_{r+1}, ..., a_k \mid [a_i, a_j] = 1, i \neq j>,$$

therefore from Th.1. and Th.6 in [2] we have that $G / Z(G)$ is $r\mathcal{P}$, thus finally there exists a free quotient of G such that $g = w(t, a_j)$ is not trivial in it. If $g = zw(t, a_i)$ and $z \neq 1$ then $g \notin <t, a_{r+1}, ..., a_n>^G = N$ and $G / N \simeq <a_1, ..., a_r>$ is $r\mathcal{P}$. So for every $1 \neq g \in G$ there exists a free quotient of G such that g is not trivial in it.

3. Residually nilpotent HNN extensions

Proposition 8. *Let*

$$G = <t, a_1, ..., a_r \mid t^{-1}a_it = a_i^{p_i}, i = 1, ..., r, [a_l, a_j] = 1, l \neq j$$

be the HNN extension of a f.g. free abelian group, where one of the associated subgroups is the whole base group. If $\gamma_n(G)$ *is the n-term of the lower central series, then* $\gamma_{n+1}(G) = <a_i^{(p_i-1)^n}, i = 1, ..., r>^G$ *for every* $n > 1$.

Proof. Let $\Lambda = <a_i^{p_i-1}, i = 1, ..., r>^G$. Then G / Λ is abelian that is $\gamma_2(G) < \Lambda$, but $a_i^{p_i-1} = [a_i, t]$. So $\Lambda = \gamma_2(G)$. Suppose that $\gamma_{n+1}(G) = <a_i^{(p_i-1)^n}, i = 1, ..., r>^G$ for $n > 1$. Then $[t, a_i^{(p_i-1)^n}] = a_i^{-(p_i-1)^{n+1}}$ and finally $\gamma_{n+2}(G) = <a_i^{(p_i-1)^{n+1}}. i = 1, ..., r>^G$. Since every element of G has the form $g = t^{\lambda}kt^{-\mu}$, $\lambda, \mu > 0$, $k \in <a_1, ..., a_r>$.

Corollary 9. *Let* G *be as above. Then* G *is nilpotent if and only if* $p_i = 1, i = 1, ..., r$. *Namely* G *is free abelian.*

Proof. It is obvious that $\gamma_{n+1}(G) = <a_i^{(p_i-1)^n}, i = 1, ..., r>^G$ is trivial for some $n \in \mathbb{N}$ if and only if $p_i = 1 \ \forall i = 1, ..., r$.

Corollary 10. *Let* G *be as above. Then* G *is* $r\mathcal{N}$ if and only if $p_i \neq 2 \ \forall i = 1, ... r$.

Proof. It is obvious that $\bigcap_{n=1}^{\infty} \gamma_n(G)$ is trivial if and only if

$$p_i \neq 2 \ \forall i = 1, ..., r.$$

In the corollary 10 we give a characterization for the residually nilpotence of an HNN extension with base group a f.g. free abelian group, where one of the associated subgoups is the whole base group.

In the case, where the associated subgroups are both proper subgroups, we do not have a complete answer concerning the residual nilpotence. In view of Th.1 in [1] and of Propositions 2 and 3 above and since a finitely generated $r\mathcal{N}$ group is $r\mathcal{F}$ we can only exhibit some cases for which these groups are not $r\mathcal{N}$.

REFERENCES

1 S. Andreadakis, E. Partis and D. Varsos. "Residual finitenesess and Horficity of certain HNN extensions" (To appear).

2. B. Baumslag. "Residually free groups". Proc. Lond. Math. Soc. (3) 17 (1967), 402-418.

3. B. Baumslac and M. Tretkoff. "Residually finite HNN extensions". Comm. Algebra 6, 179-194 (1978).

4. M. Shirvani. "On residually finite HNN extensions", Arc. Math. 44, 110-115 (1985).

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