

NUMERICAL RANGE AND GATEAUX DIFFERENTIAL IN TOPOLOGICAL ALGEBRAS

BY

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Studying the numerical range of the elements of a l.m.c. algebra we have information concerning commutativity of E and in the case of l.m.c. *-algebras (we have information concerning) the second dual space and the C^* -property (see [1] and [7]). In this respect the *strongly hermitian* elements of E (the elements of E with real numerical range) play an important role. In case of unital l.m.c. algebras Vidav-Palem's famous theorem says that: A complete unital l.m.c. algebra E may be a (complete) l.m.c. C^* -algebra if, and only if, $E = H(E) + iH(E)$ where $H(E)$ are the strongly hermitian elements of E .

A further information concerning the C^* -property is the following:

Theorem. *Let $(E, \Gamma \equiv \{p\})$ be a unital l.m.c. *-algebra. Then, the following statements are equivalent*

- (i) E is a l.m.c. C^* -algebra
- (ii) The normalized states of E coincide with normalized continuous positive linear forms of E
- (iii) Every self-adjoint element of E has real numerical range

$$(:S(E) \subseteq H(E)).$$

Proof. See [7].

In the sequel we denote the normalized states of E by $D(E; 1_E)$ that is

$$D(E, 1_E) \equiv \bigcup_{p \in \Gamma} \{f \in (U_p(1))^0 : f(1_E) = 1\}$$

where $(U_p(1))^0$ the polar of $U_p(1) \equiv \{x \in E : p(x) < 1\}$. On the other hand, if E is a unital l.m.c. algebra and $a \in E$ then, we called numerical range of a the set

$$V_E(a) := \hat{a}(D(E, 1_E))$$

where $\hat{\alpha}(f) = f(\alpha)$, $f \in D(E, 1_E)$

The number

$$V_E(\alpha) := \sup_{\lambda \in V_E(\alpha)} |\lambda|$$

is the numerical radius of α .

On the other hand, $f \in (U_p(1))^0$ iff $|f(x)| \leq p(x)$, $x \in E$ (cf. [4; Chapter I, Lemma 1.2.]).

Now, in the present work, given a unital l.m.c. algebra E we give some information concerning the strongly hermitian elements of E , by studying a real-valued function on E (: the Gateaux differential corresponding to the unit of E). More precisely (let $(E, \Gamma \equiv \{p\})$ be a unital l.m.c. algebra with $p(1_E) = 1$, $p \in \Gamma$ (see [2; Th. 3.2.]) and $\alpha \in E$. Then, for every semi-norm $p \in \Gamma$ the real-valued function

$$f_{\alpha p} : \mathbb{R} \rightarrow \mathbb{R} : t \rightarrow f_{\alpha p}(t) := p(1_E + t\alpha)$$

is convex, hence the left and the right hand derivatives exist and satisfy $f'_{\alpha p^-}(0) \leq f'_{\alpha p^+}(0)$. So, for every $p \in \Gamma$ we may define the real-valued function on E

$$(1) \quad \phi_p : E \rightarrow \mathbb{R} : x \rightarrow \phi_p(x) := f'_{x p^+}(0) \equiv \lim_{t \rightarrow 0^+} \frac{p(1_E + tx) - 1}{t}$$

which is called Gateaux differential corresponding to the unit of E , and besides satisfies the relation

$$(2) \quad -\phi_p(-x) \leq \phi_p(x), \quad x \in E$$

The equality in (2) is valid iff there exists the derivative of zero of the function $f_{x p}$.

In this respect we have the next

Theorem 1. *Let $(E, \Gamma \equiv \{p\})$ be a unital l.m.c. algebra and $\alpha \in E$. Then, for every complex number $\lambda \in V_E(\alpha)$ there is a seminorm $p \in \Gamma$ such that*

$$(3) \quad -\phi_p(-\alpha) \leq \operatorname{Re}\lambda \leq \phi_p(\alpha)$$

$$(4) \quad -\phi_p(i\alpha) \leq \operatorname{Im}\lambda \leq \phi_p(-i\alpha)$$

Proof. Let $\lambda \in V_E(\alpha)$. By the definition of $V_E(\alpha)$ there is $p \in \Gamma$ and $f \in (\cup_p (1))^0$ such that $\lambda = f(\alpha)$. Moreover

$$\begin{aligned} 1 + t \cdot \operatorname{Re}\lambda &= f(1_E) + t \cdot \operatorname{Re}f(\alpha) = \operatorname{Re}f(1_E + t\alpha) \leq |f(1_E + t\alpha)| \\ &\leq p(1_E + t\alpha), \quad t > 0 \quad \text{hence} \quad \operatorname{Re}\lambda \leq \frac{p(1_E + t\alpha) - 1}{t}, \quad t > 0 \end{aligned}$$

and passing to the limit we have $\operatorname{Re}\lambda \leq \phi_p(\alpha)$. On the other hand

$$-\lambda = f(-\alpha) \in V_E(-\alpha) \quad \text{so that} \quad \operatorname{Re}(-\lambda) \leq \phi_p(-\alpha)$$

which yields

$$-\phi_p(-\alpha) \leq \operatorname{Re}\lambda \leq \phi_p(\alpha)$$

$$\begin{aligned} \text{For the relation (4) we have that} \quad 1 + t \cdot \operatorname{Im}\lambda &= 1 + t \cdot \operatorname{Re}(-i\lambda) = \\ &= \operatorname{Re}f(1_E + t(-i\alpha)) \leq |f(1_E + t(-i\alpha))| \leq p(1_E + t(-i\alpha)), \quad t > 0 \end{aligned}$$

from which we take (4) by the same way, as above.

Scholium 2. *Since in every complete l.m.c. algebra the spectrum of an element is contained in its numerical range (see [1] and [3]) it is clear that Theorem 1 is valid if one substitutes the numerical range of α by the spectrum of it. In this case the relation (3) extends in the case of l.m.c. algebras a result of J. Nieto for Banach algebras (cf. [6], Lemma 1).*

Corollary 3. *Let $(E, \Gamma \equiv \{p\})$ be as in Theorem 1 and $x \in E$. We consider the assertions*

$$i) \quad \phi_p(ix) = -\phi_p(-ix) \equiv k \in \mathbb{R}, \quad p \in \Gamma$$

$$ii) \quad x + ik \in H(E)$$

Then, $i) \Rightarrow ii)$.

Proof. We have that

$$\begin{aligned} V_E(x + ik \cdot 1_E) &= \widehat{x + ik1_E} (D(E, 1_E)) = \{f(x + ik1_E) : f \in D(E, 1_E)\} \\ &= \{\text{Ref}(x) + i(k + \text{Imf}(x)) : f \in D(E, 1_E)\} \end{aligned}$$

On the other hand, for every $f \in D(E, 1_E)$, $f(x) \in V_E(x)$, so that, by the above Theorem 1, there is $p \in \Gamma$ such that:

$$- \phi_p(ix) \cong \text{Imf}(x) \cong \phi_p(-ix)$$

so that, by hypothesis $\text{Imf}(x) = -k \quad \forall f \in D(E, 1_E)$ hence $x + ik \in H(E)$.

Corollary 4. Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a unital l.m.c. algebra and $h \in E$. The following statements are equivalent

i) $\phi_\alpha(ih) = -\phi_\alpha(-ih) = 0, \quad \alpha \in I$

ii) $h \in H(E)$

where $\phi_\alpha = \phi_{p_\alpha}$ (see relation (1)).

Proof. i) \Rightarrow ii) Implies by Corollary 3, for $k = 0$.

ii) \Rightarrow i) Since $V_E(h) = \bigcup_\alpha V_{E_\alpha}(h_\alpha) = \bigcup_\alpha V_{\tilde{E}_\alpha}(h_\alpha)$ where $E_\alpha \equiv E/\ker p_\alpha$ and \tilde{E}_α , $\alpha \in I$ are the normed and Banach algebras corresponding to the Arens-Michael analysis of E (see [2] and [3]) we have, by hypothesis, that $h_\alpha \in H(\tilde{E}_\alpha)$ hence (cf. [1; Lemma 5.2])

$$\lim_{t \rightarrow 0} \frac{\|1_\alpha + ith_\alpha\|_\alpha - 1}{t} = 0 \text{ for every } \alpha \in I.$$

On the other hand,

$$\phi_\alpha(ih) \equiv \lim_{t \rightarrow 0^+} \frac{p_\alpha(1_E + ith) - 1}{t} = \lim_{t \rightarrow 0^+} \frac{\|(1_E + ith) + \ker p_\alpha\|_\alpha - 1}{t}$$

$$\lim_{t \rightarrow 0} \frac{\|1_\alpha + ith_\alpha\|_\alpha - 1}{t} = 0 \text{ and } \phi_\mu(-ih) = \lim_{t \rightarrow 0} \frac{\|1_\alpha - ith_\alpha\|_\alpha - 1}{t}$$

$$= - \lim_{(-t) \rightarrow 0} \frac{\|1_\alpha + i(-t)h_\alpha\|_\alpha - 1}{(-t)} = 0$$

which yields the assertion.

The above Corollary 4 extends is the case of l.m.c. algebras a result which is valid for Banach algebras (cf. [1 ; Lemma 5.2] and [6; comments before Theorem 1]).

Now, let $(E, \Gamma \equiv \{p\})$ be a unital l.m.c. algebra. We consider the sets

$$X_p^1 \equiv \{x \in E: -\phi_p(-x) = \phi_p(x)\}$$

$$X_p^2 \equiv \{x \in E -\phi_p(-ix) = \phi_p(ix)\}$$

$$X_p \equiv X_p^1 \cap X_p^2, \quad E_0 \equiv \bigcap_{p \in \Gamma} X_p$$

It is easy to verify that these sets are real linear subspaces of E and ϕ_p is a real linear function on those.

We will formulate now, two propositions one of which is referred to the relation between the numerical radius and the Gateaux diferential and the other to the relation between the spectral radius and the Gateaux differential. The later extends to the case of l.m.c. algebras a result of J. Nieto for Banach algebras (cf. [6; Lemma 2 and Theorem 1).

Proposition 5. *Let $(E, \Gamma \equiv \{p\})$ be a unital l.m.c. algebra and $\alpha \in E$. Then,*

$$(5) \quad \begin{aligned} v_E(\alpha) &\leq \sup_{p \in \Gamma} (\text{Max} \{\phi_p(\alpha)^2, \phi_p(-\alpha)^2\} + \text{Max} \{\phi_p(i\alpha)^2, \phi_p(-i\alpha)^2\})^{1/2} \\ &\leq \sqrt{2} \cdot v_E(\alpha) \end{aligned}$$

Particularly, if $\alpha \in E_0$ then

$$(6) \quad v_E(\alpha) \leq \sup_{p \in \Gamma} (\phi_p(\alpha)^2 + \phi_p(i\alpha)^2)^{1/2} \leq \sqrt{2} \cdot v_E(\alpha)$$

Proof. Let $\lambda \in V_E(\alpha)$. By the relations (3) and (4) of Theorem 1 we have that

$$(\operatorname{Re}\lambda)^2 \leq \operatorname{Max} \{ \phi_p(\alpha)^2, \phi_p(-\alpha)^2 \} \text{ and}$$

$$(\operatorname{Im}\lambda)^2 \leq \operatorname{Max} \{ \phi_p(i\alpha)^2, \phi_p(-i\alpha)^2 \}$$

Adding these relations and taking the square roots and suprema we have the first inequality to the relations (5). For the second inequality we have that

$$\phi_p(\alpha) = \lim_{t \rightarrow 0^+} \frac{p(1_E + t\alpha) - 1}{t} \leq$$

$$\lim_{t \rightarrow 0^+} \frac{\operatorname{Supp}(1_E + t\alpha) - 1}{t} = (\text{by [3; Lemma 2]})$$

$$\operatorname{Sup} \operatorname{Re}V_E(\alpha) \leq \operatorname{Sup} |V_E(\alpha)| = v_E(\alpha)$$

that is

$$(7) \quad \phi_p(\alpha) \leq v_E(\alpha).$$

On the other hand, the numerical radius $v_E(\alpha)$ is a linear norm (see [7]) so that

$$-v_E(\alpha) = -v_E(-\alpha) \stackrel{(7)}{\leq} -\phi_p(-\alpha) \stackrel{(2)}{\leq} \phi_p(\alpha) \stackrel{(7)}{\leq} v_E(\alpha)$$

$$-v_E(\alpha) = -v_E(-i\alpha) \leq -\phi_p(-i\alpha) \leq \phi_p(i\alpha) \leq v_E(i\alpha) = v_E(\alpha)$$

hence

$$\operatorname{Max} \{ \phi_p(\alpha)^2, \phi_p(-\alpha)^2 \} \leq v_E(\alpha)^2 \text{ and}$$

$$\operatorname{Max} \{ \phi_p(i\alpha)^2, \phi_p(-i\alpha)^2 \} \leq v_E(\alpha)^2$$

Adding these relations and taking the square roots and suprema we have the second inequality in (5).

Now (6) is clear by the definition of E_0 and (5).

Proposition 6. *Let $(E, \Gamma \equiv \{p\})$ be a complete unital l.m.c. algebra and $\alpha \in E$. If the family $\Gamma \equiv \{p\}$ is upper directes then,*

$$(8) \quad \rho_E(\alpha) \leq \sup_{p \in \Gamma} [2 \cdot \text{Max} \{ \phi_p(\alpha)^2, \phi_p(-\alpha)^2 \} + \phi_p(-\alpha^2)]^{1/2}$$

Particularly, for $\alpha \in E_1 \equiv \bigcap_{p \in \Gamma} X_p^1$ we have

$$(9) \quad \rho_E(\alpha) \leq \sup_{p \in \Gamma} (2 \cdot \phi_p(\alpha)^2 + \phi_p(-\alpha^2))^{1/2}$$

where $\rho_E(\alpha)$ is the spectral radius of α .

Proof. For $\lambda = \mu + \nu i \in \text{Sp}_E(\alpha)$ there is $q \in \Gamma$ such that (see Scholium 2).

$$-\phi_q(-\alpha) \leq \mu \leq \phi_q(\alpha)$$

On the other hand, $\lambda^2 = \mu^2 - \nu^2 + 2\mu\nu i \in \text{Sp}_E(\alpha^2)$ (see [4]) so that, there is $r \in \Gamma$ with:

$$-\phi_r(-\alpha^2) \leq \text{Re}(\lambda^2) = \mu^2 - \nu^2$$

i.e.,
$$\nu^2 - \mu^2 \leq \phi_r(-\alpha^2)$$

Now, since the family $\Gamma \equiv \{p\}$ is upper directed there is $p \in \Gamma$ with $\phi_q(x) \leq \phi_p(x)$ and $\phi_r(x) \leq \phi_p(x)$, $x \in E$ so that

$$-\phi_p(-\alpha) \leq \mu \leq \phi_p(\alpha) \text{ and}$$

$$\nu^2 - \mu^2 \leq \phi_p(-\alpha^2)$$

hence

$$|\lambda|^2 = \mu^2 + \nu^2 = 2\mu^2 + (\nu^2 - \mu^2) \leq 2 \cdot \text{Max} \{ \phi_p(\alpha)^2, \phi_p(-\alpha)^2 \} + \phi_p(-\alpha^2).$$

Taking the square roots and the suprema we have (8).

By (8) and definition of E_1 we get (9).

REFERENCES

1. F. Bonsall and J. Duncan (1971), Numerical Ranges of Operators on Normed Spaces and of Elements of Normal Algebras, London Math. Soc. Lect. Note Ser 2.
2. R. Brooks, (1967), On locally m-convex algebras, Pacific J. Math. 25, 5-23.

3. J. Gilles and D. Koehler (1973), On numerical ranges of elements of locally m -convex algebras. *Pac. J. Math.* 49, 79-81.
4. A. Mallios, *General Theory of Topological Algebras: Selected Topics*.
5. E. Michael, (1952), Locally multiplicatively-convex topological algebras, *Amer. Math Soc.* No 11.
6. J. Nieto, (1974), Gateaux differentials in Banach algebras, *Math. z.*, 139, 23-34.
7. Th. Chryssakis, Numerical ranges in locally m -convex algebras I, *J. Austral. Math. Soc. (Series A)* 41 (1986) 304-316.

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