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ON THE STRONG LIFTING PROPERTY FOR PRODUCTS

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ABSTRACT. *The class of compact measure spaces that have the property: their product with any compact measure space with the strong lifting property, admits also a strong lifting, is much larger of those of metric spaces. At the begining, it contains every group equipped with the Haar measure. This fact, in conjunction with Losert's example and Kyrka's arguments, yields invariant measures on transformations groups which fail to have a lifting commuting with the translations. Moreover, the above class contains any product of metric spaces and every totally ordered measure space that admits only strong liftings.*

1. Introduction. J. Kypka [6,3.3. QUESTION] put the question: "If μ, ν are regular, Borel measures on compact spaces, and if μ and ν both have the strong property, then does the measure $\mu \otimes \nu$ have the strong lifting property?".

In view of [8, th. 4, p. 115] in the bounded case, the answer to this question is always positive, if one, at least, space is metrizable. We can say, in less precise language, that the metric spaces have the property "to preserve the strong lifting property on products". It is therefore obvious that the determination of the class of all compact measure spaces that "preserve the strong lifting property on products" gives a full answer to the above question.

In this paper we introduce and study this (product-strong lifting) property. Theorem 2.3 asserts that a given Haar measure on an arbitrary compact group has the above cited property. Theorem 2.' presents another class of measure spaces with this property. A special case of such spaces is a product of metrizable spaces. Furthermore, every totally ordered measure space (which as khown-cf. [7]-admits a strong lifting) has the product-strong lifting property, if the set of its limit points is measurable.

1.1. A compact measure space is a pair (T, μ), where T is a compact topological space and μ a (positive, Radon) measure on T. If there exists a strong (resp. almost strong) lifting for (T, μ), then (T, μ) (or just μ) is said to have the strong (resp. almost strong) lifting property (in brief: SLP, resp. ASLP) (*). It will be convenient to suppress the term compact and we will say, simply, a measure space.

1.2. Let (T, μ), (X, λ) be measure spaces and $p : X \to T$ a continuous surjection such that $p(\lambda) = \mu$. Then a family $\{\lambda_t, t \in T\}$ of measures on T will be said to constitue a strict p-disintegration of λ if for every function $f \geqslant 0$, on X, λ-measurable the following conditions are satisfied:

(1) f is l_t-measurable a.e. (μ).

(2) The (μ-almost everywhere defined) function $t \to \lambda_t(f)$ is μ-measurable and for every subset A of T, μ-measurable we have:

$$\int_{p^{-1}(A)} f d\lambda = \int_A \lambda_t(f)\, d\mu(t)$$

(3) $\operatorname{supp}(\lambda_t) \subset p^{-1}\{t\}$ a.e. (μ)

1.3. Let (H, X) be a (right) compact transformation group, where X is a compact space and H a compact topological group which acts on X by (the action): $H \times X \to X: (\xi, x) \to x\xi$.

Let X/H denote the compact space of (right) cosets of X and

$$\pi: X \to X/H : x \to \pi(x) = x$$

the canonical mapping. Let β be a (left) Haar measure on H and λ a measure on X/H. Then (cf. [1]) there is a unique measure λ^* on X (the so called Haar lift of λ) with thw property:

$$\lambda^*(f) = \int_{X/H} d(\dot{x}) \int_H f(x\xi) d\beta(\xi),\ f \in C(X).$$

We remind that a transformation group is called free, if $\xi \in H$, $\xi \neq e$ then $x\xi \neq x$, $x \in X$.

1.4. A totally ordered measure space is a measure space (T, μ) where T is a (compact) totally ordered topological space.

In section 2 we give the main definition of "product-strong lifting property" and presents some results concerning this property.

(*) For lifting theory we rely on [8] and for Measure Theory on [1]

2. THE PRODUCT-STRONG LIFTING PROPERTY

We introduce the following concept.

2.1. Definition. *Let us say a measure space (X, λ) (or just λ) has the product-strong lifting property (in brief: $\otimes$ SLP), if for every measure space (T, μ) with SLP, the product measure $\mu \otimes \lambda'$ has the SLP.*

2.2. Remark. In [2] K. Bichteler has noticed the fact that the set of Radon measures λ on a (locally compact) space X such that $(X, |\lambda|)$ admits an almost strong lifting is a band of the space $M(X)$ of all measures on X. The problem as to whether or not every measure space (with full Support) has the $\otimes$ SLP os open. However, by elementary arguments, we can also show that the set of all measures λ on a compact space X such that $|\lambda|$ "preserves the ASLP on products" is a band of $M(X)$.

2.3. Theorem: *Every (compact) group equipped with the Haar measure has the $\otimes$ SLP.*

Proof. The proof essentially repeats arguments due to R.A. Johnson [4].

Let (B, λ) be a measure space with Supp $(\lambda) = B$, H compact group β a (left). Haar measureon H. We define the following action of H on the product $X : B \times H$, by:

(A) $$H \times (B \times H) \to B \times H: (\eta, (b, \xi)) \to (b, \xi\eta).$$

Then, for this action, it is easy to see that $X/H \simeq B$ and that the Haar lift λ^* of λ is the product measure $\lambda \otimes \beta$.

Now, with the above notations, assume that (B, λ) has the strong lifting property. We will verify that $(B \times H, \lambda \otimes \beta)$ admits also a strong lifting. To this end we immediately verify that H acts freely by the action defined in (A). Consequently, from the main theorem of [4] we see that there will exist a strong lifting for $\lambda^* = \lambda \otimes \beta$, commuting with this action. Hence the proof of the theorem is complete.

2.4. Discussion. *[concerning the examples of [6]].*

It is known (cf. [6]) that the strong lifting property "is preserved by pro-

jection". Consequently, for the transformation group $(H, X_H := B \times H)$ *of 2.3 we have that*

$$\text{“}(H/X, \lambda^*) \text{ has the SLP} \Rightarrow (B, \lambda) \text{ has the SLP”}.$$

This model yields many examples of spaces without the SLP under the form of a (free) transformation group with respect to an invariant measure. For instance, if we take a measure λ on a compact space B without the strong lifting property then, for the corresponding (X_H, λ^*), where H denotes any compact group, there is not a (strong) lifting commuting with the translations.

We note that, in addition, we can take, for X_H, an arbitrary $(\geqq \aleph_2)$ topological weight.

The next theorem extends a classical proposition by A. and C. Ionescu Tulcea in the case that the one mostly factor is not metric (cf. [8. EXAMPLE 1, p. 119]).

2.5. Theorem. *Let* (T, μ) *be a measure space which satisfies the conditions:*

1) (T, μ) *has the strong lifting property.*

2) There exist: a family $\{(U_i, \lambda_i), i \in J\}$, *where* U_i *is metrizable,* λ_i *a probability measure on* U_i *and a continuous mapping p from* $\prod_{i \in J} U_i$ *onto* T *such that* $p(\bigotimes_{i \in J} \lambda_i) = \mu$.

Then (T, μ) *has the* $\otimes$ *SLP.*

2.6. Lemma. *Let* $X = S \times (\prod_{i \in J} U_i)$, *where* $\{U_i, i \in J\}$ *is a family of compact metrizable spaces and* $S = U_{i_0}$ *a given compact space. Let* λ *be a positive measure on* $\prod_{i \in J} U_i$ *with* $Supp\lambda = \prod_{i \in J} U_i$ *and* ν *a positive measure on* S *with full Support. Suppose that:*

(#) If $I \subset J \cup \{i_O\}$, $i \in J \cup \{i_O\} - I$, $B \subset \prod_{i \in J \cup \{i_O\}} U_i$ *is* $P_I\,(\nu \otimes \lambda)$*-measurable and* $V \subset U_i$ *is open, then the relation* $P_{I \cup \{i\}}\,(\nu \otimes \lambda)\,(B \times V) = 0$ *implies that* $P_I(\nu \otimes \lambda)\,(B)\,P_{\{i\}}\,(V) = 0$.

Then $(X, \nu \otimes \lambda)$ *has the strong lifting property.*

Proof. The measure space $(S \times \prod_{i \in J} U_i, \ \nu \otimes \lambda)$ is of the form: $(X, \mu) = (\prod_{k \in k} X_k, \nu \otimes \lambda)$. We propose to show that (X, μ) admits a strong lifting. Reproducing, step by step, the proof of [8, theorem 5, p. 118] without any innovation, we establish the fact that this space has the SLP.

We sketch the steps of the proof:

1. We verify that μ satisfies the condition 5.2 of [8, th. 5, p. 118].

2. Using the notations and the content of [8, p. 116-118] we see that the set Γ of all pairs (I, r^I), where r^I is a strong lifting for $(P_I(X), P_I(\mu))$, $I \subset K$, is inductive with respect to the order:

$(I, r^I) \leqq (I_1, r^{I_1})$ iff I is contained in I_1 and r^{I_1} extends r^I. Therefore there will exist a maximal element of Γ.

3. We take a maximal element (I_o, r^{I_o}) of Γ such that $i_o \varepsilon I_o$.

4. We easily check that $\prod_{i \varepsilon I_o} X_i = X$ and consequently that r^{I_o} is a strong lifting for (X, μ).

2.7. Theorem. *A product measure on an arbitrary product of metric spaces has the* $\otimes$ *SLP.*

Proof. Let $(U := \prod_{i \varepsilon J} U_i, \lambda := \otimes_{i \varepsilon J} \lambda_i)$ be a product of measure spaces, where U_i is metrizable and λ_i a probability measure with full Support on U_i, $i \varepsilon J$. Let us assume that (S, ν) is any measure space with the strong lifting property. Then, since λ is a product measure, we easily verify that the space $(S \times U, \nu \otimes \lambda)$ satisfies the relation (#) of 2.8 lemma, so we have the conclusion.

Property $\otimes$ SLP is not an exclusive privilege of the Haar measure on a (compact) group. In Fact, we have the following.

2.8. Example. On $\{0, 1\}^\alpha$, where α is an arbitrary set, we can construct a measure λ with $\text{Supp}\lambda = \{0, 1\}^\alpha$ which is not, generally, equivalent to a product measure (*):

For each $p \varepsilon [0, 1]$, let λ_p be the measure on $\{0, 1\}$ defined by:

$$\lambda_p(\{0\}) = p \quad \text{and} \quad \lambda_p(\{1\}) = 1 - p.$$

For each $i \varepsilon \alpha$ let $\lambda_p^i := \lambda_p$. For every subset β of a define the me-

(*) We use the construction of [8, EXAMPLE 2, p. 118].

asure λ_p^β on $\{0, 1\}^\beta$ by: $\lambda_p^\beta := \bigotimes_{i \varepsilon \beta} \lambda_p^i$. We note that the mapping $p \to \lambda_p^\alpha$ of [0, 1] into M(X) is vaguely continuous.

Taking now a probability measure m on [0, 1] such that $m(\{0\}) = m(\{1\}) = 0$, define the measure λ on $\{0, 1\}^\alpha$ by:

$$\lambda := \int_{[0,1]} \lambda_p^\alpha dm(p).$$

Now we take a measure space (S, ν) with the strong lifting property. Then we immediately verify that the following relation is valid:

$$(\cdot) \qquad \nu \otimes P_\beta(\lambda) = \int_{[0,1]} \lambda_p^\beta dm(p), \quad \text{for} \quad \beta \subset \alpha.$$

Using (·) we easily check that the measure space $(S \times \{0, 1\}^\alpha, \nu \otimes \lambda)$ satisfies the condition (#) of 2.8 lemma and so $(\{0, 1\}^\alpha, \lambda)$ has the ⊗ SLP.

Proof. of 2.5 theorem

We will reduce the proof to 2.7 theorem. Let (B, k) be an arbitrary measure space with the strong lifting property, (S, ν) the hyperstonian space associated with (B, k) and w: S → B the canonical map. We define the mapping

$$q : S \times U \to B \times T : (s, u) \to (w(s), p(u)).$$

Then, since $w(\nu) = k$ and $p(\lambda) = \mu$, we may easily verify that

$$k \otimes \mu = q\,(\nu \otimes \lambda).$$

Moreover the following are valid:

(I) The measrure $\nu \otimes \lambda$ on $S \times U$ is completion regular.

In fact, since ν is completion regular and λ a product measure on a product of metrizable spaces, this is evidently clear by the arguments of [3, th. 3].

(II) There is a family $\Lambda = \{\lambda_{b,t},\ b \varepsilon B,\ \tau \varepsilon T\}$ of measures on $S \times U$ with the properties:

(II₁) $\nu \otimes \lambda = \int_{B \times T} \lambda_{b,t} d\,(k \otimes \mu)(b, t)$ and

(II₂) Supp $\lambda_{b,t} \subset q^{-1}\{(b, t)\},\ (b, t) \varepsilon B \times T$.

In fact, since (B, k) and (T, m) have both the strong lifting property, there will exist (cf. [8]):

a strict w-disintegration $(\lambda_b)_{b \varepsilon B}$ of ν and

a strict p-disintegration $(\lambda_t)_{t \varepsilon T}$ of λ.

If we put $\lambda_{b,t} := \lambda_b \otimes \lambda_t$, $b \varepsilon B$, $t \varepsilon T$, we easily verify that Λ is the desired one.

(III) The family Λ is in addition a strict q-disintegration of $\nu \otimes \lambda$.

Indeed, this fact is an immediate consequence of (I), (II) and the completion regularity of the measure $\nu \otimes \lambda$ (sf. [5, prop. 1.5]). However, from 2.8 lemma, the space $(S \times U, \nu \otimes \lambda)$ has the strong lifting property. It remains to show that there is a strong lifting for $(B \times T, \kappa \otimes \mu)$, but this derives from the key projection theorem [6, theorem 2.7].

Let now (X, λ) be a meausre space with $\mathrm{Supp}\lambda = X$, satisfying the following condition:

(##) There is a (countable) family $\{U_n, n \varepsilon N\}$ of compact, metrizable subsets of X such that $\lambda(X - \bigcup_{n \varepsilon N} U_n) = 0$.

Then it is easy to see that (X, λ) will have the $\otimes$ SLP. There are many examples of such spaces. The following is of special interest.

2.9. Example. Let (X, λ) be a totally ordered measure space, with $\mathrm{Supp}\lambda = X$. Suppose that the set of all limit points of X is λ-measurable of full measure. Then, if λ is non atomic, (X, λ) satisfies (##).

In fact, since X is compact, by arguments similar to that of [7], we can verify the existence of a family as in (##).

2.10. Remarks. (1) As it is mentioned in [7], the totally ordered measure spaces share a number of properties with measures on metric spaces. Therefore, in view of 2.9 example, it is natural to ask, if every totally ordered measure space has the $\otimes$ SLP.

(2) In 2.11 example, for the space (X, λ) there are only almost strong liftings, so we can ask, finally, if a space which admits only almost strong liftings has always the $\otimes$ SLP.

(3) The proof of 2.3 theorem apply unchanged for Haar measures on locally compact groups, if we define the property $\otimes$ SLP in the class of locally compact spaces.

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