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ON SUPER LIE GROUPS

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1. Introduction.

The concept of "superspace" was introduced by Salam and Strathdee [14] in the context of supersymmetry as a space parametrised by eight coordinates with the first four taking their values in the even part of a Grassmann algebra and second four in the odd part.

In attempting to provide superspace with a global topology, A. Rogers [11] defined the category of superanalytic supermanifolds analogously to that of analytic manifolds, but locally isomorphic to that Banach space which is the Cartesian product of a certain number of even and odd parts of a Grassmann algebra.

Subsequently, A. Rogers [12] introduced the concept of "super Lie group" as an abstract group, which is also a superanalytic supermanifold with a superanalytic group operation.

Earlier the concept of "supermatrix", a block form matrix over a Grassmann algebra with the diagonal blocks even and the others odd, had been introduced by several authors together with that of a "supergroup" of supermatrices satisfying a certain algebraic constraint [10]. R. Picken [9], in attempting to investigate the global nature of supergroups, proved that they not exhibit new topological features beyond the homotopy groups of the underlying Lie groups and raised the question whether these supergroups are super Lie Groups in the rigorous sence of Rogers [12].

It is the main purpose of this paper to answer this question in the case of the general linear supergroup GL(m, n) as well as for the special linear supergroup SPL(m, n). We prove directly that GL(m, n) satisfies the requirements of the definition of super Lie group, whereas for the special linear group we use a suitable theorem of Rogers [12].

The contents of the paper are as follows. In Secs 2-4 we review the elementary theory of Grassmann algebras as well as the basic definitions on supermatrices, supergroups and Lie superlgebras. Section 5 contains a review of the theory of "differentiation" extended to Grassmann algebras, as it has introduced by Rogers [11]. In Sec. 6 we review the basic definition of supermanifolds and super Lie groups while in Secs 7 and 8 we prove explicitly that GL(m, n) and SPL(m, n), respectively, are really super Lie groups in the sense of the definition of Rogers [12].

2. Grassmann algebras

We denote by $B_p(F)$ the Grassmann algebra, over the field F ($F = \mathbb{R}$ or $\mathbb{C}$), generated by the identity 1 and the p independent generators $\theta_1, \theta_2, \ldots, \theta_p$. Then $B_p(F)$ has as basis the 2^p independent generators

$$\begin{array}{ll} 1 & \\ \theta_i \, , & 1 \leq i \leq p \\ \theta_i\theta_j \, , & 1 \leq i \leq j \leq p \\ \vdots & \\ \theta_1\theta_2 \ldots \theta_p \, , & \end{array} \tag{1}$$

The product in $B_p(F)$ is associative and is subject to the identity

$$\theta_i\theta_j + \theta_j\theta_i = 0, \tag{2}$$

for all $i, j = 1, 2, \ldots, p$.

The general element of $B_p(F)$ can be written as

$$x = \alpha_0 1 + \sum_{i=1}^{p} \alpha_i\theta_i + \sum_{i<j}^{p} \alpha_{ij}\theta_i\theta_j + \ldots + \alpha_{12\ldots p}\theta_1\theta_2 \ldots \theta_p, \tag{3}$$

where $\alpha_0, \alpha_i, \alpha_{ij}, \ldots$ belong to F.

The summand

$$x_k = \sum_{i_1<i_2<\ldots<i_k} \alpha_{i_1 i_2 \ldots i_k} \theta_{i_1}\theta_{i_2} \ldots u_{i_k}, \tag{4}$$

which is a homogeneous element of degree k, is called the kth part of x, for $k = 1, 2, \ldots, p$. Then

$$x_N := \sum_{k=1}^{p} x_k, \quad x_0 = x - x_N \tag{5}$$

define the nilpotent and numeric parts of x, respectively.

B_p is a Z-graded algebra, that is,

$$B_p = \bigoplus_{k=0} B_{p,k} \tag{6}$$

with

$$B_{p,k} \cdot B_{p,\ell} = \begin{cases} B_{p,k+\ell} & \text{if } k+\ell \le p \\ \{0\} & \text{if } k+\ell > p \end{cases} \tag{7}$$

For every element in B_p we can write

$$x = x_0 + x_1, \tag{8}$$

where x_0 in $B_{p,0}$ is the even part of x and x_1 in $B_{p,1}$ is the odd part of x. Thus there is a secondary grading on B_p by Z_2

$$B_p = B_{p,0} \oplus B_{p,1} \tag{9}$$

We note that with respect to multiplication the inverse x^{-1} of x in B_p exists and is unique, if and only if $x_0 \neq 0$. The set of all invertible elements in B_p is a multiplicative group, which is called the Grassmann group and is denoted by B_p^*. (See [2]).

Finally we recall that a sign function is defined on B_p as follows,

$$|x| = \begin{cases} 0 & \text{if } x \in B_{p,0} \\ 1 & \text{if } x \in B_{p,1} \end{cases}$$

A such function is defined on every Z_2-graded vector space.

3. Supermatrices and Supergroups

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{10}$$

be an $(m+n) \times (m+n)$ matrix, where A and D are $m \times m$ and $n \times n$ matrices, respectively, with entries in the even part of a Grassmann algebra $B_p(F)$, and B, C are $m \times n$, $n \times m$ matrices, respectively, with entries in the odd part of the Grassmann algebra $B_p(F)$. Any such matrix is said to be an (m, n)-supermatrix or simply supermatrix.

A supermatrix M can be written as

$$M = M_0 + M_N \tag{11}$$

where M_0 (resp. M_N) contain only the mumeric (resp. nilpotent) parts of the matrix elements of M. Also, M can be written as

$$M = M_{\bar{0}} + M_{\bar{1}}, \tag{12}$$

where $M_{\bar{0}}$ (resp. $M_{\bar{1}}$) contain the even (resp. odd) parts of the elements of M.

A supermatrix M is invertible if and only if M_0 is invertible ([3] or [8]), or equivalently, if and only if $\det M_0 \neq 0$. The last relation is equivalent to that $\det M_{\bar{0}}$ is an invertible element in $B_{p,\bar{0}}$, where the determinant of $M_{\bar{0}}$ makes sense, because $B_{p,\bar{0}}$ is a commutative algebra.

The explicit form of the inverse supermatrix of the supermatrix M, given by (10), is as follows (see [1])

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{13}$$

The invertible supermatrices of the form (10) define through the usual matrix multiplication a group, called "the general linear supergroup" and denoted by $GL(m, n; F)$, $F = \mathbb{R}$ or $\mathbb{C}$, or simply $GL(m, n)$.

The surtrace of a supermatrix M of the form (10), denoted by strM, is defined by

$$\mathrm{str}M := \mathrm{tr}A\text{-}\mathrm{tr}D, \tag{14}$$

while the superdeterminant of an invertible supermatrix M of the form (10), denoted by sdetM, is defined by

$$\mathrm{sdet}M = \det A \det(D\text{-}CA^{-1}B)^{-1} \tag{15}$$

and is a multiplicative function [1], that is, for all invertible (m, n)-supermatrices M and N we have

$$\mathrm{sdet}(MN) = \mathrm{sdet}M \quad \mathrm{sdet}N. \tag{16}$$

Finally, we note that one of the classical supergroups is the "special linear supergroup", denoted by $SPL(m, n; F)$ or simply $SPL(m,n)$ and defined by (see [10])

$$SPL(m, n; F) = \{M \in GL(m, n; F) : \mathrm{sdet}M = 1\} \tag{17}$$

4. Lie superalgebras

Let B de an algebra over F (F = $\mathbb{R}$ or C). We recall that 'it is said to be Z_2-graded, $Z_2 = Z / 2Z = \{\bar{0}, \bar{1}\}$, if its underlying vector space is the direct sum of $B_{\bar{0}}$, the even part, and $B_{\bar{1}}$, the odd part and if

i) $1 \in B_{\bar{0}}$ (if 1 is a unit in B).

ii) $B_iB_j \subset B_{i+j(\text{mod}2)}$ for all i, j in Z_2.

A Z_2-graded algebra is called a superalgebra.

Let now $\underline{g} = \underline{g}_{\bar{0}} \oplus \underline{g}_{\bar{1}}$ be a superalgebra over F whose multiplication is denoted by a bracket [,]. Then g is called a Lie superalgebra if the multiplication satisfies the following properties:

i) $[X, Y] = -(-1)^{|X|\,|Y|}[Y, X]$, for all homogeneous X, Y in g. (graded skew-symmetry)

ii) $(-1)^{|X|\,|Z|}[[X, Y], Z] + (-1)^{|Y|\,|X|}[[Y, Z], X] + (-1)^{|Z|\,|Y|}[[Z, X], Y] = 0$

for all homogeneous X, Y, Z in $\underline{g}$ (graded Jacobi identity).

Let now $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a graded vector space, over the field F, with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. The set Hom(V) of all F-linear mappings of V into itself becomes an associative superalgebra, if we define a Z_2-grading by

$$\text{Hom}(V)_i = \{f \in \text{Hom}(V) : f(V_j) \text{ c } V_{i+j},\ j \in Z_2\},\quad i = 1, 2$$

The general linear Lie superalgebra $p\ell(V)$ or $p\ell(m, n)$ of V is the Lie superalgebra associated with the associative superalgebra Hom(V). We have the isomorphism

$$p\ell(m, n)_{\bar{0}} \cong g\ell(m) \times g\ell(n),$$

where $g\ell(m)$ is the general linear algebra over F. (see [15]).

5. Real analysis extended to Grassmann algebras

Following Kostant's notation [6], M_p denotes the set of sequences $\mu = (\mu_1, \mu_2, ..., \mu_k)$, $1 \le \mu_1 < \mu_2 < ... < \mu_k \le p$, $\mu_i \in Z$, $1 \le i \le p$. If ω represents the empty sequence in M and (i) or simply i denotes the sequence with one element i, then the basis for B_p can be written as

$$\{\theta_\mu : \mu \in M_p\}, \tag{18}$$

where

$$\theta_\omega = 1$$
$$\theta_\mu = \theta_{\mu_1}\theta_{\mu_2} \dots \theta_{\mu_K}, \quad \text{for all} \quad \mu \quad \text{in} \quad M_p$$
$$\theta_i\theta_j + \theta_j\theta_i = 0, \quad \text{for all} \quad i, j = 1, 2, \dots, p.$$

Also, $M_{p,\bar{0}}$ denotes the set of sequences $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ in M_p with k even, including the empty sequence, while $M_{p,\bar{1}}$ denotes the set of sequences μ in M_p with k odd.

A complete norm on $B_p(\mathbb{R})$ has been defined by Rogers [11] as follows:

Definition 1: *Let* $x = \sum_{\mu \in M_p} x_\mu \theta_\mu$ *(19)*

be an arbitrary element in $B_p(\mathbb{R})$*, where* x *belongs to* $\mathbb{R}$*. Then a norm is defined on* $B_p(\mathbb{R})$ *by*

$$\|x\| = \sum_{\mu \in M_p} |x_\mu| \tag{20}$$

Endowed with the norm defined by (20), $B_p(\mathbb{R})$ becomes a Banach algebra. The usual Euclidean norm is equivalent to the norm given by (20), but does not make $B_p(\mathbb{R})$ into a Banach algebra. Next we suppress mention of the field $\mathbb{R}$, writting simply B_p instead of $B_p(\mathbb{R})$.

Let now m, n be finite positive integers. Then we denote by

$$B_p^{m,n} := B_{p,\bar{0}}^m \times B_{p,\bar{1}}^n, \tag{21}$$

the Cartesian product of m copies of the even part of B_p and n copies of the odd part. The arbitrary element of this set will be written

$$(\underline{a}, \underline{b}) = (a_1, \dots, a_m; b_1, \dots, b_n)$$

or

$$(\underline{c}) = (c_1, c_2, \dots, c_{m+n}).$$

The set $B_p^{m,n}$ can be considered as a $2^{p-1}(m + n)$-dimensional real vector space and a norm on it is defined by

$$\|(\underline{c})\| := \|c_1\| + \|c_2\| + \dots + \|c_{m+n}\| \tag{22}$$

The topology on $B_p^{m,n}$ is the topology induced by the norm (22), which is also the product topology. It is a Hausdorff topology and, since p is finite, it is the usual topology on $B_p^{m,n}$ regarded as a $2^{p-1}(m+n)$-dimensional real vector space.

The definition of "differentiation" on Grassmann algebras has been introduced by Rogers [11]. It is a generalization of the definition of differentiation of Banach spaces (see [5] or [7]). However, the use of multiplication in the Grassmann algebra instead of the multiplication in the real numbers makes this definition more restrictive.

Definition 2: *Let U be an open set in $B_p^{m,n}$ and $f: U \to B_p$. Then*

i) f is said to be G^0 on U if f is continuous on U.

ii) f is said to be G^1 on U if there exist $m+n$ functions

$$G_k f : U \to B_p, \quad k = 1, 2, ..., m+n$$

and a function $h: B_p^{m+n} \to B_p$, such that, for $(\underline{a}, \underline{b})$ and $(\underline{a}+\underline{h}, \underline{b}+\underline{k})$ in U, we have

$$f(\underline{a}+\underline{h}, \underline{b}+\underline{k}) = f(\underline{a}, \underline{b}) + \sum_{i=1}^{m} h_i(G_i f)(\underline{a}, \underline{b})$$

$$+ \sum_{j=1}^{n} k_j(G_{j+m} f)\ (\underline{a}, \underline{b}) + ||\underline{h}, \underline{k})||\ \eta(\underline{h}, \underline{k})$$

and $||\eta(\underline{h}, \underline{k})|| \to 0$ as $||(\underline{h}, \underline{k})|| \to 0$. We write $f \in G^1(U)$.

iii) If r is a finite positive integer, f is said to be G^r on U if f is G^1 and we can found $m+n$ function

$$G_k f : U \to B_p, \quad k = 1, 2, ..., m+n$$

which are G^{r-1} on U.

Also, f is said to be G^∞ on U if f is G^r on U for every positive integer r.

iv) f is said to be superanalytic or G^ω on U, if for given $(\underline{c})$ in U, there exists a neighbourthood U_c of $(\underline{c})$, such that for all $(\underline{x})$ in U, $f(\underline{x})$ is equal to the sum of an absolutely convergent power series in $(\underline{x} - \underline{c})$ of the form

$$f(x) = \sum_{k_1=0,\dots \kappa_{\mu+n}=0} a_{k_1\dots k_{m+n}} (x_1 - c_1)^{k_1} \dots (x_{m+n} - c_{m+n})^{k_{m+n}}$$

where $a_{k_1\dots k_{m+n}}$ *belong to* B_p *and* $(\underline{x}) = (x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n})$.

6. Supermanifolds and super Lie groups

The definition of a superanalytic supermanifold has been modelled by Rogers [11] on the standard definition of an analytic manifold. So as an m-dimensional analytic manifold looks like $\mathbb{R}^m$ locally and has local coordinates $(x_1(q), \dots, x_m(q))$ in $\mathbb{R}^m$, a superanalytic supermanifold over B_p looks like $B_p^{m,n}$ locally and has local coordinates $(u_1(q), \dots, u_m(q), \upsilon_1(q), \dots, \upsilon_n(q))$ in $B_p^{m,n}$, where q is an arbitrary element of the supermanifold.

Let now M be a Hausdorff topological space. An (m, n) open chart on M over B_p is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to B_p^{m,n}$ is a homeomorphism of U onto an open subset of $B_p^{m,n}$.

The definition of an (m, n)-superanalytic structure on M over B_p is similar to the usual definition of an analytic structure and thus the definition of a superanalytic supermanifold is given as follows:

Definition 3: *An (m, n)-dimensional superanalytic supermanifold over* B_p *is a Hausdorff topological space M endowed with an (m, n)-superanalytic structure over* B_p*, that is, there exists a collection* $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ *of open charts on M such that:*

i) $M = \bigcup_{\alpha\in A} U_\alpha$,

ii) for each pair α, β *in A the map* $\phi_\beta o \phi_\alpha^{-1}$ *is a* G^ω *map of* $\phi_\alpha(U_\alpha \cap U_\beta)$ *onto* $\phi_\beta(U_\alpha \cap U_\beta)$

ii) the collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ *is a maximal collection of open charts which satisfies i) and ii).*

The definition of a vector field on an (m, n)-dimensional supermanifold is analogous to the definition of a vector field on an m-dimensional analytic manifold. Also the set

$$G^\omega(U) := \{f \mid f : U \to B_p \text{ with } fo\phi^{-1} \text{ in } G^\omega(\phi(U \cap V), (V, \phi) \text{ chart in } M\} \tag{24}$$

is a graded commutative algebra over $\mathbb{R}$, as well as, a graded (left) B_p-module.

By End $(G^\omega(U))$ we denote the set of vector space endomorphism of $G^\omega(U)$. A vector field on U is an element X of $End(G^\omega(U))$ such that

i) $X(fg) = (Xf)g + (-1)^{|f||X|} fXg$, for all f, g in $G^\omega(U)$.

ii) $X(af) = (-1)^{|X||\alpha|} aXf$, for all f in $G^\omega(U)$, a in B_p.

We denote the set of vector fields on U by $D^1(U)$. This set is a graded Lie (left) B_p-module, which is defined as follows:

Definition 4: *A graded Lie (left) B_p-module W over F ($F = \mathbb{R}$ or C) is a Lie superalgebra over F which is also a graded (left) B_p-module such that*

$$[aX, Y] = a\,[X, Y], \tag{25}$$

for all a in B_p and X, Y in W.

Next, we give the definition of a super Lie group and two theorems ([12]), which we will use later. We have:

Definition 5: *An (m, n)-dimensional super Lie group is a set H such that*

i) H is an abstract group.

ii) H is an (m, n)-dimensional superanalytic supermanifold

iii) the function $(x, y) \to xy^{-1}$ of the product supermanifold $H \times H$ into H is superanalytic.

The left translations and left invariants vector fields are defined in the same way as for the standard Lie groups, while the set of all left invariant vector fields, say $L(H)$, on an (m, n)-dimensional super Lie group H is an (m, n)-dimensional graded Lie (left) B_p-module with bracket operation

$$[X, Y] = XY - (-1)^{|X||Y|}YX, \tag{26}$$

for all X, Y in $L(H)$. We note that saying $L(H)$ is (m, n)-dimensional we mean that it is a free module with a basis consisting of m even and n odd elements.

Thus to every super Lie group corresponds a graded Lie left B_p-module which is isomorphic to the tangent module T_eH.

When p is a finite positive integer, an (m, n)-dimensional super Lie group over B_p is also a $2^{p-1}(m+n)$-dimensional real Lie group. The next theorems are referred to relationship between the graded Lie (left) B_p-module of a super Lie group and the Lie algebra of H regarded as a real Lie group. ([12]) These will be useful to us later.

Theorem 1: *Let H be an (m, n)-dimensional super Lie group over B_p, $n < p < \infty$, and let W be its graded Lie left B_p-module. Moreover, let $\underline{h}$ be the $2^{p-1}(m+n)$-dimensional Lie algebra of H regarded as a $2^{p-1}(m+n)$-dimensional real Lie group. Then the even part W_0 of W, regarded as a $2^{p-1}(m+n)$-dimensional Lie algebra, is isomorphic to $\underline{h}$*

Theorem 2: *Let W be an (m, n)-dimensional graded Lie left B_p-module and let $\underline{h}$ be the $2^{p-1}(m+n)$-dimensional real Lie algebra derived from the even part of W. Let also H be any $2^{p-1}(m+n)$-dimensional real Lie group with Lie algebra $\underline{h}$. Then H can be given the structure of an (m, n)-dimensional super Lie group over B_p with graded Lie left B_p-module W.*

7. GL(m, n; F) is a super Lie group

Let us denote by $\widetilde{\mathbb{M}}(m, n; F)$ the set of all (m, n)-supermatrices. We consider the map

$$f : \widetilde{\mathbb{M}}(m, n; F) \to B_p^{m^2+n^2, 2mn}(F)$$

defined by

$$F(M) := (\underline{A}, \underline{D}, \underline{B}, \underline{C}) \tag{27}$$

where M is an (m, n)-supermatrix of the form (10) and

$$\underline{A} = (a_{11}, \ldots, a_{1m}, a_{21}, \ldots, a_{mm}) \quad \text{when} \quad A = (a_{ij})_{m\times m},$$

and similarly are defined $\underline{D}$, $\underline{B}$, and $\underline{C}$.

The map f defines the topology in $\widetilde{\mathbb{M}}(m, n; F)$ and therefore it can be considered as a homeomorphism. Then we can write

$$\widetilde{\mathbb{M}}(m, n; F) \cong B_p^{m^2+n^2; 2mn}(F) \tag{28}$$

Thus, the set $\widetilde{\mathbb{M}}(m, n; F)$ can be given the structure of a superanalytic supermanifold by means of a global chart $(\widetilde{\mathbb{M}}(m, n; F), f)$, where f is the map defined by (27).

Let now $GL(m, n; F)$ be the general linear supergroup, which is an (abstract) group with respect to the usual matrix multiplication. We shall show that $GL(m, n; F)$ is an $(m^2 + n^2, 2mn)$-dimensional open superanalytic sub-supermanifold of $\widetilde{\mathbb{M}}(m, n; F)$.

First we establish a necessary lemma.

Lemma 1: *The Grassmann group $B^*_{p,0}(F)$ is an open subset of $B_{p,0}(F)$ with respect to the topology on $B_{p,0}(F)$ induced by the norm (20).*

Proof. If $x_0 = k1 + \sum_{j \in M_{p,0}} \lambda_j \theta_j$, $k \neq 0$ is an arbitrary element in $B^*_{p,0}(F)$ and $y = \sum_{j \in M_{p,0}} \mu_j \theta_j$ is an arbitrary not invertible element in $B_{p,0}(F)$, then we have

$$d(y, x_0) = \|y - x_0\| = |k| + \sum_{j \in M_{p,0}} |\lambda_j - \mu_j| > |k|.$$

Thus, choosing r such that $0 < r < |k|$ we can find an open non-empty neighbourhood

$$B(x_0, r) : \{x \in B_{p,0}(F): \|x - x_0\| < r\}$$

contained in $B^*_{p,0}$.

Theorem 3: *The set $GL(m, n; F)$ in an $(m^2 + n^2, 2mm)$-dimensional open superanalytic sub-supermanifold of $\widetilde{\mathbb{M}}(m, n; F)$.*

Proof. If M is an arbitrary (m, n)-supermatrix of the form (10) we consider two composition maps f_A and f_D, defined by

$$f_A(M) := \det A, \quad f_D(M) := \det D. \tag{29}$$

Then we have

$$f_A = \det \circ pr_A, \quad f_D = \det \circ pr_D, \tag{30}$$

where

$$pr_A : \widetilde{\mathbb{M}}(m, n; F) \to \widetilde{\mathbb{M}}(m, B_{p,0}(F)): M \to pr_A(M) = A \tag{31}$$

$$pr_D : \widetilde{\mathbb{M}}(m, n; F) \to \widetilde{\mathbb{M}}(n, B_{p,0}(F)): M \to pr_D(M) = D, \tag{32}$$

while the map

$$\det : \widetilde{\mathbb{M}}(m; B_{p,\bar{0}}(F)) \to B_{p,\bar{0}}(F) \tag{33}$$

is defined in the usual way, because $B_{p,\bar{0}}(F)$ is commutative.

We note that $\widetilde{\mathbb{M}}(m, B_{p,\bar{0}}(F))$ (resp. $GL(m, B_{p,\bar{0}}(F))$ denotes the set of all (resp. invertible) $m \times m$ Grassmann matrices over $B_{p,\bar{0}}(F)$. The map det is a polynomial with respect to its variables and clearly it is superanalytic and therefore continuous. Also, the maps pr_A, pr_D are projection maps and clearly they are continuous. Therefore the composition maps f_A and f_D are continuous.

Bearing in mind that an (m, n)-supermatrix of the form (10) belongs to $GL(m, n; F)$ if and only if detA and detD belong to $B^*_{p,\bar{0}}(F)$, we get the equality

$$GL(m, n; F) = f_A^{-1}(B^*_{p,\bar{0}}(F)) \cap f_D^{-1}(B^*_{p,\bar{0}}, (F)) \tag{34}$$

and since, by Lemma 1, $B^*_{p,\bar{0}}(F)$ is an open subset of $B_{p,\bar{0}}(F)$, from the continuity of f_A and f_D, we conclude that $GL(m, n; F)$ is open in $\widetilde{\mathbb{M}}(m, n; F)$.

Therefore, $GL(m, n; F)$ can be considered as open superanalytic sub-supermanifold of $\widetilde{\mathbb{M}}(m, n; F)$ by means of a global chart $(GL(m, n; F), f)$, where f is defined as in (27). We have

$$\dim GL(m, n; F) = (m^2 + n^2, 2mn). \tag{35}$$

We continue with a number of preliminary jemmas.

Lemma 2: *The map*

$$h : GL(m, B_{p,\bar{0}}(F)) \to GL(m, B_{p,\bar{0}}(F))$$

defined by

$$h(A) := A^{-1} \tag{36}$$

is syperanalytic.

Proof. We note that the inverse matrix of A in $GL(m, B_{p,\bar{0}}(F))$ is the usual one, since $B_{p,\bar{0}}(F)$ is cummutative. Thus, if $A = (a_{ij})_{m\times m}$, then we have

$$A^{-1} = (\underline{a}_{ij})_{m\times m},$$

where

$$\underline{a}_{ij} = (\det A)^{-1} A^{ij} \tag{37}$$

with A^{ij} polynomials with respect to the entries of A.

Therefore all A_{ij} are superanalytic maps and thus the map h is superanalytic if and only if the map

$$g : GL(m, B_{p,\bar{0}}(F)) \to B^*_{p,\bar{0}}(F),\ g(A) := (\det A)^{-1} \tag{38}$$

is superanalytic. Using the superanalytic map

$$g_1 : B^*_{p,\bar{0}}(F) \to B^*_{p,\bar{0}}(F)$$

defined by

$$g_1(x) := x^{-1}, \tag{39}$$

for all x in $B_{p,\bar{0}}(F)$, we can write $g = g_1 \circ \det$.

Since det is a polynomial with respect to its variables and g_1 is superanalytic, we conclude that the map g is superanalytic.

Lemma 3: *The map*

$$w : GL(m, n; F) \times GL(m, n; F) \to GL(m, n; F)$$

defined by

$$w(MN) := MN^{-1} \tag{40}$$

for all M, N in GL(m, n; F) is superanalytic.

Proof. We write $w = w_2 o w_1$, where w_1 is defined on the Cartesian product of two copies of GL(m, n; F) into itself by

$$w_1(M, N) := (M, N^{-1}) \tag{41}$$

and w_2 is the map defining the supermatrix multiplication on GL(m, n; F), that is

$$w_2(M, N) = MN, \tag{42}$$

for all M, N in GL(m, n; F).

The map w_2 is superanalytic, because every of its component maps is a polynomial with respect to its variables.

Also, bearing in mind the explicit form of the inverse of a supermatrix

M, given by (13), and lemma 2, we conclude that the map w_1 is superanalytic.

We close this section with two theorems which are consequences of the previous lemmas.

Theorem 4: *The general linear supergroup $GL(m, n; F)$ over $B_p(F)$ is a super Lie group.*

Proof. By theorem 3, the set $GL(m, n; F)$ is a superanalytic supermanifold.

Also, since the inverse supermatrix for every member of $GL(m, n; F)$ exists, it is a routine matter to check that this set is an abstract group.

Finally, from lemma 3, it is clear that the map

$$w : (M, N) \to MN^{-1}$$

is superanalytic for all (M, N) in $(GL(m, n; F) \times GL(m, n; F)$. Hence the set $GL(m, n; F)$ is a super Lie group.

Theorem 5: *The map*

$$sdet : GL\,(m, n; F) \to B^*_{p,0}(F)$$

defined by

$$sdetM := detA \quad det(D - CA^{-1}B)^{-1} \tag{43}$$

for all M in $GL(m, n; F)$ of the form (10), is superanalytic.

Proof. It is clear from lemma 2.

8. The SPL(m, n) is a super Lie group.

We note that everything we prove is valid for both fields, real and complex, unless otherwise stated.

Also, we recall from section 5 that the topology on $B_p^{m,n}$ induced by the norm (22), which is also the procuct topology, is a Hausdorff topology and, for p finite, it is the usual topology on $B_p^{m,n}$ regarded as a $2^{p-1}(m + n)$ or $2^p(m + n)^2$-dimensional real vector space, according to as $F = \mathbb{R}$ or C, respectively.

Since $GL(m, n)$ can be considered homeomorphic to $B_p^{m^2+n^2,2mn}$ in order to prove that a subset of $GL(m, n)$, regarded as a real Lie group, is closed, it is sufficient to prove that it is a closed subset of $GL(m, n)$, regarded as a super Lie group over B_p. Concerning with that we state the following theorem.

Theorem 6: *The special linear supergroup SPL(m, n) is a closed subset of the super Lie group GL(m, n). Moreover, SPL(m, n) can be considered as a closed subset of GL(m, n), regarded as a real Lie group.*

Proof. We consider the map sdet, defined by (43), which by theorem 5 is superanalytic and therefore continuous. From the definition of the special linear supergroup we have

$$SPL(m, n) = sdet^{-1}\{1\}, \tag{44}$$

where $\{1\}$ is a closed subset in $B^*_{p,0}$.

Therefore SPL(m, n) is a closed subset of GL(m, n), regarded as a super Lie group and by the discussion at the beginning of this section, it is a closed subset of GL(m, n), regarded as a real Lie group.

From all the above and a well-known theorem on real Lie groups (see [16], page 99), we conclude that SPL(m, n) can be considered as a real Lie group.

Next we consider SPL(m, n) as a real Lie group and using the classical method involving one-parameters subgroups we will find its Lie algebra. We work over the reals, that is $F = \mathbb{R}$.

Let us now consider the one-parameter subgroup

$$\gamma : \mathbb{R} \to SPL(m, n)$$

such that $\gamma(0) = I$ and $sdet\gamma(t) = 1$. In the usual (m, n)-supermatrix form we can write

$$\gamma(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}. \tag{45}$$

and we have

$$detA(t)\ det(D(t) - C(t)A^{-1}(t)\ B(t))^{-1} = 1 \tag{46}$$

Considering the supermatrix $\gamma(t)$ as an element with $2^{p-1}(m + n)$ real components, equation (45) represents the arbitrary element of the real Lie group $SPL(m, n; \mathbb{R})$, provided that equation (46) is satisfied.

Then the corresponding tangent map of γ given by

$$\gamma_{*,0}\left(\frac{\vartheta}{\vartheta t}\right) = \dot{\gamma}(0) = \begin{pmatrix} \dot{A}(0) & \dot{B}(0) \\ \dot{C}(0) & \dot{D}(0) \end{pmatrix} \tag{47}$$

where $\dot{A}(0) = \dfrac{dA(t)}{dt}$, etc.

Writing equation (46) in the form

$$\det A(t) = \det(D(t) - C(t)A^{-1}(t)B(t)) \tag{48}$$

and differentiating with respect to t both sides of (48), we obtain at $t = 0$,

$$\mathrm{tr}\dot{A}(0) = \mathrm{tr}\{\dot{D}(0) - \dot{C}(0)A^{-1}(0)B(0) - C(0)\dot{A}^{-1}(0)B(0) - \\ + C(0)A^{-1}(0)\dot{B}(0)\}$$

or equivalently

$$\mathrm{str}\dot{\gamma}(0) = \mathrm{tr}\dot{A}(0) - \mathrm{tr}\dot{D}(0) = 0. \tag{49}$$

Conversely, if M is an arbitrary element of the real Lie group $GL(m, n; \mathbb{R})$ such that $\mathrm{str}M = 0$, we consider the one-parameter subgroup

$$\rho : U \to GL(m, n; \mathbb{R})$$

defined by

$$\rho(t) = e^{tM}, \tag{50}$$

where U is a suitable neighboorhood of 0 in $\mathbb{R}$ such that ρ is a diffeomorphism.

It is perhaps worthwhile noting here that the exponential and the logarithim of a supermatrix are defined as for the real matrices. ([4]).

Then we find

$$\dot{\rho}(0) = M \tag{51}$$

and
$$\mathrm{sdet}\rho(t) = \mathrm{sdet}\, e^{tM} = e^{\mathrm{str}tM} = 1 \tag{52}$$

Hence the Lie algebra of $SPL(m, n; \mathbb{R})$ is given by

$$L(SPL(m, n; \mathbb{R})) = \{M \in \widetilde{\mathbb{M}}(m, n; \mathbb{R}) : \mathrm{str}M = 0\} \tag{53}$$

Thus we have proved the following theorem:

Theorem 7: *The Lie algebra of $SPL(m, n; \mathbb{R})$, regarded as a real lie group, is given by (53).*

Finally, we state the following theorem:

Theorem 8: *The special linear group $SPL(m, n; \mathbb{R})$ is a super Lie group.*

Proof. First we consider the real Lie superalgebra $spl(m, n)$, defined by

$$spl(m, n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{M}(m, n; \mathbb{R}) : trA = trD \right\}, \qquad (54)$$

where $\mathbb{M}(m, n; \mathbb{R})$ denotes the set of all (m, n)-block form real matrices.

It is a standard result that the tensor product $B_p(\mathbb{R}) \otimes spl(m, n)$ is an $(m^2 + n^2 - 1, 2mn)$-dimensional graded Lie left B_p-module (see [12]), while its even part $(B_p(\mathbb{R}) \otimes spl(m, n))_{\bar{0}}$ is a $2^{p-1}[(m + n)^2 - 1]$-dimensional real Lie algebra.

It is also trivial to check that the Lie algebra of the real Lie group $SLP(m, n; \mathbb{R})$ is isomorphic to the even part of the tensor rpoduct $B_p(\mathbb{R}) \otimes spl(m, n)$. Thus we have

$$L(SPL(m, n; \mathbb{R}) \cong (B_p(\mathbb{R}) \otimes spl(m, n))_{\bar{0}} \qquad (55)$$

Therefore we have proved that $SPL(m, n; \mathbb{R})$ is a real Lie group with corresponding Lie algebra isomorphic to the even part of the graded Lie left B_p-module $B_p(\mathbb{R}) \otimes spl(m, n)$. Thus, according to theorem 2, $SPL(m, n; \mathbb{R})$ can be given the structure of a super Lie group with corresponding graded Lie left B_p-module $B_p(\mathbb{R}) \otimes spl(m, n)$.

Remark. A similar situation occurs when the Grassmann algebra B_p is considered over C. Then the arbitrary Grassmann number depends on 2^p real parameters. Then we have

$$\dim L(SPL(m, n; C)) = 2^p[(m + n)^2 - 1] \qquad (56)$$

and

$$L(SPL(m, n; C)) \cong (B_p(C) \otimes spl(m, n))_{\bar{0}} \qquad (57)$$

Similarly, the supergroup $SPL(m, n; C)$ can be given the structure of a super Lie group.

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