

## A USEFUL APPLICATION OF GAUSS THEOREM

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People, working in applied mathematical problems, are aware of the need to evaluate specific classes of integrals. Recently, working on a scattering theory problem [1], we were confronted with the problem of evaluating the integrals, over the surface of the unit sphere, of the monomials in the variables  $0_1, 0_2, 0_3$  which are the directional cosines of the direction  $\hat{r} = (0_1, 0_2, 0_3)$ . In other words we need to know the values of the integral

$$I_{n_1, n_2, n_3} = \int_0^{2\pi} \int_0^{\pi} 0_1^{n_1} 0_2^{n_2} 0_3^{n_3} \sin\theta d\theta d\phi \quad (1)$$

where,  $n_1, n_2, n_3$  are nonnegative integers and

$$\left. \begin{aligned} 0_1 &= \sin\theta \cos\phi \\ 0_2 &= \sin\theta \sin\phi \\ 0_3 &= \cos\theta \end{aligned} \right\} \quad (2)$$

Straightforward evaluation of the integrals in (1) is possible, but the task of actually going through the calculations, even for small values of the degree  $n = n_1 + n_2 + n_3$  of the monomial is almost formidable. Nevertheless, Gauss theorem can be used to evaluate the integrals  $I_{n_1, n_2, n_3}$  for every value of  $n_1, n_2$  and  $n_3$  without going through the standard calculations. Along this line we first state and prove the following lemma.

**Lemma.** For every integer  $n \geq 2$  the following identity holds

$$\int_S \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_n dw$$

$$= \frac{1}{n+1} \sum_{k=0}^{n-2} \sum_{i=1}^3 \int_S \underbrace{\hat{e}_i \otimes \hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r} \otimes \hat{e}_i \otimes \hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_k dw \quad (3)$$

where  $S$  denotes the surface of the unit sphere,  $\hat{e}_i$ ,  $i = 1, 2, 3$  are the orthonormal vectors in  $\mathbb{R}^3$ ,  $\hat{r}$  is the variable radius of the unit sphere and  $\otimes$  stands for the tensor product.

**Proof.** By induction. For  $n = 2$ , Gauss theorem implies

$$\int_S \hat{r} \otimes \hat{r} dw = \int_B \nabla \hat{r} dv = \int_B \nabla \left( \frac{\mathbf{r}}{r} \right) dv$$

$$= - \int_B \frac{1}{r} \hat{r} \otimes \hat{r} dv + \int_B \frac{1}{r} \bar{\Pi} dv = - \frac{1}{2} \int_S \hat{r} \otimes \hat{r} dw + \frac{1}{2} \int_S \bar{\Pi} dw, \quad (4)$$

where  $B$  is the unit ball and

$$\bar{\Pi} = \sum_{i=1}^3 \hat{e}_i \otimes \hat{e}_i \quad (5)$$

is the identity second rank tensor. Solving (4) with respect to the surface integral we obtain

$$\int_S \hat{r} \otimes \hat{r} dw = \frac{1}{3} \int_S \bar{\Pi} dw \quad (6)$$

which, in view of (5), coincides with (3) for  $n = 2$ .

Assume next that (3) holds and will prove the validity of (3) for  $n + 1$ . Working as in the case  $n = 2$  we obtain

$$\int_S \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_{v+1} dw = \int_B \nabla \left( \frac{1}{r^n} \underbrace{\mathbf{r} \otimes \mathbf{r} \otimes \dots \otimes \mathbf{r}}_n \right) dv$$

$v+1$   $n$

$$\begin{aligned}
 &= -n \int_B \frac{1}{r} \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_{n+1} du \\
 &+ \sum_{k=0}^{n-1} \sum_{i=1}^3 \int_B \frac{1}{r} \hat{e}_i \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_k \otimes \hat{e}_i \otimes \hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r} du \\
 &= -\frac{n}{2} \int_S \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_{n+1} dw \\
 &+ \frac{1}{2} \sum_{k=0}^{n-1} \sum_{i=1}^3 \int_S \hat{e}_i \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_k \otimes \hat{e}_i \otimes \hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r} dw. \quad (7)
 \end{aligned}$$

By solving (7) with respect to the first integral in its left end we obtain (3) for  $n + 1$ . This proves the lemma.

From the Lemma we observe that the integral over  $S$ , of the tensor product of  $n$  vectors equal to  $\hat{r}$ , is expressed with respect to  $n - 1$  similar surface integrals, where in each one of them only  $n - 2$  vectors are equal to  $\hat{r}$  while the other 2 are constant base vectors.

By repeatedly applying Formula (3) it follows that every surface integral of the tensor product of an even number of  $\hat{r}$ 's is expressed with respect to the integral  $\int_S dw = 4\pi$ . Similarly, the corresponding integrals involving the tensor product of an odd number of  $\hat{r}$ 's is expressed with respect to the intergral  $\int_S \hat{r} dw$  which vanishes, by Gauss theorem. Therefore, the following conclusion holds.

**Conclusion 1.** Whenever  $n$  is a nonnegative odd integer the integral

$$\int_S \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_n d\omega$$

vanishes. In other words,  $I_{n_1 n_2 n_3} = 0$  whenever  $n_1 + n_2 + n_3 = n$  is a nonnegative odd integer.

Every application of (3) replaces two of the  $\hat{r}$ 's by one and the same base vector. Therefore,  $n$  successive applications of (3) to a surface integral containing  $2n$   $\hat{r}$ 's will replace all the  $\hat{r}$ 's by base vectors which will appear in pairs. This will in turn imply that all the powers  $n_1$ ,  $n_2$ , and  $n_3$  in (1) are even, and our second conclusion follows.

**Conclusion 2.** A necessary and sufficient condition for the integrals  $I_{n_1 n_2 n_3}$ , in (1) to be nonzero is that  $n_1$ ,  $n_2$  and  $n_3$  are even, in which case

$$I_{n_1 n_2 n_3} = \frac{4\pi}{(n_1 + n_2 + n_3 + 1)!!}, \quad n_1, n_2, n_3 \text{ even} \quad (8)$$

where the double factorial indicates the product of all the odd numbers from 1 to  $n_1 + n_2 + n_3 + 1$ .

Formula (8) is the result of successive applications of Formula (3).

The value of the surface integral of the tensor product is given by the following formula

$$\int_S \underbrace{\hat{r} \otimes \hat{r} \otimes \dots \otimes \hat{r}}_{2n} d\omega = \frac{4\pi}{(2n + 1)!!} \sum \hat{e}_{i_1} \otimes \hat{e}_{i_2} \otimes \dots \otimes \hat{e}_{i_{2n}} \quad (9)$$

where the summation is taken over all the arrangements of the indices  $i_1, i_2, \dots, i_{2n}$ , which appear in pairs of coinciding values of 1, 2, or 3.

It is worthwhile to notice, that all the nonvanishing components of the  $2n$ -th rank tensor equation (9) have the same value  $\frac{4\pi}{(2n + 1)!!}$

From combinatorial analysis we obtain that there are  $\binom{n-1+3}{n}$  ways to put  $n$  spheres in 3 boxes. Therefore, there are  $(n+1)(2n+1)$  integrals of the form  $I_{n_1 n_2 n_3}$  with  $2n = n_1 + n_2 + n_3$  and among these,

only the  $\frac{1}{2} (n + 1)(n + 2)$ , which have the form  $I_{(2n_1)(2n_2)(2n_3)}$ , are nonvanishing and equal to  $\frac{4\pi}{(2n + 1)!!}$ .

In conclusion, Gauss theorem can be appropriately used to obtain the value of every integral of the form (1) without any calculations.

Further more, particular definite integrals involving higher powers of sines and cosines can also be evaluated by inspection!

### REFERENCES

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